Proof Complexity and Satisfiability Solving for several Combinatorial Principles

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1 Introduction

In this talk (based on papers [1, 2] and unpublished work performed with Cosmin Bonchiş and Adrian Crãciun, both affiliated with the West University of Timişoara) we discuss the use of statements from topological combinatorics as a source of interesting propositional formulas. Our main example is the Kneser-Lovász theorem and its strenghtening due to Schrijver. We first considered these formulas in conjunction with the open problem of separating the Frege and extended Frege propositional proof systems (see [3] for an in-depth presentation)

1.1 Theoretical Results

The Kneser-Lovász theorem is stated as follows:

Proposition 1. *Given $n \geq 2k \geq 1$ and a function $c : \binom{n}{k} \rightarrow [n-2k+1]$ there exist two disjoint sets A, B and a color $1 \leq l \leq n-2k+1$ with $c(A) = c(B) = l$.*

See [4, 5] for readable introductions. It was first established by Lovász, using methods rooted in Algebraic Topology.

We can map this statement to a class (denoted by Kneser_k^n) of propositional formulas in the obvious way, by encoding with a boolean variable $X_{A,i}$ the statement $c(A) = i$. For $k = 1$ we retrieve the wellknown class PHP_n of formulas encoding the pigeonhole principle.

The following results were proved in [1]:

Theorem 1. *For all $k \geq 1, n \geq 3$ there exists a variable substitution Φ_k, $\Phi_k : Var(\text{Kneser}_{k+1}^n) \longrightarrow Var(\text{Kneser}_k^{n-2})$ such that $\Phi_k(\text{Kneser}_{k+1}^n)$ is a formula consisting precisely of the clauses of Kneser_k^{n-2} (perhaps repeated and in a different order).*

Therefore, all known existing proof complexity lower bounds for the pigeonhole formulas extend to the Kneser_k^n formulas, and the class $\text{Kneser}_k^n + 1$ is "harder" than Kneser_k^n.

For $k = 2$ and $k = 3$, the cases where the Kneser-Lovász theorem has combinatorial proofs, we proved:

Theorem 2. *The following are true:*

- *(a) The class of formulas* Kneser_2^n *has polynomial size Frege proofs.*
- *(b) The class of formulas* Kneser_3^n *has polynomial size* extended *Frege proofs.*

We continued the investigation of Kneser formulae in [2], where the following surprising results were proved:

Theorem 3. *For fixed parameter* $k \geq 1$*, the propositional translations* Kneser_k^n *of the Kneser-Lovász theorem have polynomial size extended Frege proofs.*

Theorem 4. *For fixed parameter* $k \geq 1$*, the propositional translations* Kneser_k^n *of the Kneser-Lovász theorem have quasi-polynomial size Frege proofs.*

Remarkably, the proof of the previous two theorems **do not** use techniques from algebraic topology, but instead reduce each case of the Kneser-Lovász theorem with fixed $k \geq 1$ to the verification of a finite number of instances. Therefore each such fixed case has efficient combinatorial proofs.

2 Truncations of combinatorial results

A complementary perspective provided in paper [2] concerned the encoding of a combinatorial principle, the so-called *octahedral Tucker lemma*, a discrete version of the Borsuk-Ulam theorem that is strong enough to prove the Kneser-Lovász theorem. The propositional encoding of this principle is inefficient (leads to exponential-size formulas). However, in [2] we found a weaker but polynomial-size truncation. Therefore, our initial intuitions on the hardness of Kneser formulas may in fact hold for the harder truncated Tucker formulas.

In the talk we will present some complexity-theoretic results on the truncated Tucker formulae, as well as a new (unpublished) second example of the truncation of the Octahedral Tucker lemma that is strong enough to establish another combinatorial result, the so-called *necklace-splitting theorem* due to Noga Alon [6].

3 Experiments: solving Kneser formulae in practice

We will discuss experiments performed using the SAT solver *lingeling* and the Integer Programming solver *GUROBI* in solving various instances of the Schrijver formulas.

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