Representable functions in Moisil logic

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Łukasiewicz logic

In the 1920s, Jan Łukasiewicz was one of the first to propose a series of many-valued logical systems. The $(n+1)$-valued incarnation, on which we will focus, has the set of "truth values"

$$L_{n+1} := \left\{0, \frac{1}{n}, ..., \frac{n-1}{n}, 1\right\}$$

and the basic logical operations defined, for any $x, y \in L_{n+1}$, by:

$$\neg x := 1 - x$$

$$x \to y := \min(1, 1 - x + y)$$

Its (Hilbert-style) axiomatizability remained an open problem for some time. Solutions include those of Rosser and Turquette (1952) and Grigolia (1977).

Moisil algebras

The problem of finding an *algebraic* axiomatization was first tackled by Moisil in 1941. He produced the following definition of what we now call *Moisil algebras*.

Definition

An *n-nuanced* or $(n+1)$-*valued Moisil algebra* is a tuple $\mathcal{L} = (L, \vee, \wedge, N, 0, 1, \Delta_1, \ldots, \Delta_n)$ such that $(L, \vee, \wedge, N, 0, 1)$ is a De Morgan algebra and $\Delta_1, \ldots, \Delta_n$ are unary operations on L (called *nuances*) such that, for any $i, j \in [n]$ and $x, y \in L$,

1. $\Delta_i(x \vee y) = \Delta_i(x) \vee \Delta_i(y)$,
2. $\Delta_i(x) \vee N\Delta_i(x) = 1$,
3. $\Delta_i \Delta_j(x) = \Delta_j(x)$,
4. $\Delta_i Nx = N\Delta_{n+1-i}(x)$,
5. if $i \leq j$, then $\Delta_i(x) \leq \Delta_j(x)$,
6. if $\Delta_i(x) = \Delta_i(y)$, for all $i \in [n]$, then $x = y$.

Concrete Moisil algebras

It may be shown that the set L_{n+1} from earlier, together with naturally defined operations, e.g. we set, for all suitable i, j,

$$\Delta_i\left(\frac{j}{n}\right) = 1 \Leftrightarrow i + j > n.$$

Moisil's representation theorem (an extension of Stone's) states that every n-nuanced Moisil algebra is isomorphic to a subdirect power of L_{n+1}.

It makes sense, then, to ask whether this class of algebras is adequate for Łukasiewicz logic.

Moisil algebras are not adequate

The answer was given in the negative by Alan Rose in 1956 for $n \geq 4$.
Consider, e.g. for $n = 4$, the following subset of L_5:

$$\left\{0, \frac{1}{4}, \frac{3}{4}, 1\right\}.$$

We may see that it is closed under all Moisil operations (i.e. it is a Moisil subalgebra), but not under the Łukasiewicz implication, since

$$\frac{3}{4} \to \frac{1}{4} = \frac{2}{4}.$$

(A suitable structure to model Łukasiewicz logic is for example the MV_n-algebra.)

Therefore, Moisil algebras define a separate logic, commonly dubbed *Moisil logic*, whose formulas are constructed by the Moisil operators.

We may now ask: which functions $f : L_{n+1}^r \to L_{n+1}$ are represented by Moisil formulas (since the Łukasiewicz implication does not qualify)? Obviously the necessary condition is that such an f preserves subalgebras, i.e., for all $x_1, ..., x_r \in L_{n+1}$,

$$f(x_1, ..., x_r) \in \langle x_1, ..., x_r \rangle = \{0, 1, x_1, ..., x_r, 1 - x_1, ..., 1 - x_r\}.$$

Our goal will be to show that this condition is also sufficient.

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The classical case

Let us remember how this problem looks for classical propositional logic, i.e. $n = 1$.

x_1	x_2	x_3	???
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	1
0	1	1	0
0	1	0	0
0	0	1	1
0	0	0	0

The answer there is that whatever we choose as the function's output, the function arises from a formula, using a construction that yields a formula in *disjunctive normal form*.

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The disjunctive normal form

The formula in disjunctive normal form for the table above is:

$$(x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge \neg x_2 \wedge x_3) \vee (x_1 \wedge \neg x_2 \wedge \neg x_3) \vee (\neg x_1 \wedge \neg x_2 \wedge x_3).$$

It may also be written more elaborately as:

$$(x_1 \wedge x_2 \wedge x_3 \wedge 1) \vee (x_1 \wedge x_2 \wedge \neg x_3 \wedge 0) \vee (x_1 \wedge \neg x_2 \wedge x_3 \wedge 1) \vee$$
$$(x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge 1) \vee (\neg x_1 \wedge x_2 \wedge x_3 \wedge 0) \vee (\neg x_1 \wedge x_2 \wedge \neg x_3 \wedge 0) \vee$$
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$$(\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge 1) \vee (\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge 0).$$

Strategy

This "extended" DNF has two essential ingredients that make it tick:

1. A way to pick out lines of a table (e.g. 1-0-0 becomes $x_1 \wedge \neg x_2 \wedge \neg x_3$).
2. A way to point at the value of the cell in the result column.

Remember that we only want to show that functions that *preserve subalgebras* are representable, therefore the second condition is automatically fulfilled. We will now focus on the first.

Enter the J's

What we need, actually, is to pick out lines that might look like this:

$\frac{1}{17}$	$\frac{5}{17}$	$\frac{3}{17}$	$\frac{14}{17}$

We will need then a separate function to pick each of the $n+1$ values, i.e. for each i, j to have

$$J_i\left(\frac{j}{n}\right) = \delta_{ij},$$

where the δ is the Kronecker delta. By using these "mutually exclusive" nuances, the term corresponding to the line above would be

$$J_1(x_1) \wedge J_5(x_2) \wedge J_3(x_3) \wedge Nx_3.$$

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The interesting part is that these J's actually exist! That is, they were shown by Cignoli in 1982 to arise from the formulas

$$J_i(x) := \Delta_{n-i+1}(x) \wedge N\Delta_{n-i}(x),$$

for $i \in \{1, ..., n-1\}$, together with $J_0(x) := N\Delta_n(x)$ and $J_n := \Delta_1$.

Therefore, our characterization result is now completely proven. In addition, we note that Diaconescu and Leuştean have shown in 2015 that Moisil algebras can be alternately equationally axiomatized using the J's instead of the Δ's.

The J's at a glance

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Free algebras

Given Moisil's representation theorem, one may immediately see that the set of all functions $f : L_{n+1}^r \to L_{n+1}$ is actually the free n-nuanced Moisil algebra with r generators.
Therefore, the result above helps in describing these free algebras: because of the joint independence of the lines, the free algebra is transparently isomorphic to a product:

$$F_n(r) \cong \prod_A A^{\alpha(r,A)},$$

where the product is taken over all subalgebras A of L_{n+1} and $\alpha(r, A)$ is the number of all tuples $(a_1, ..., a_r)$ that generate A.

Computations

We may refine this result a little. Consider, for simplicity, that n is odd. As per Leuștean (2008), a subalgebra of L_{n+1} is completely determined by a choice of a subset of $\left\{\frac{1}{n}, \dots, \frac{\frac{n-1}{2}}{n}\right\}$. For each $k \in \{1, \dots, \frac{n+1}{2}\}$ the $(k-1)$-element subsets are therefore $\binom{\frac{n-1}{2}}{k-1}$ in number, and we denote the corresponding $2k$-element subalgebras of $\mathcal{L}_n$, for each $j \in \left\{1, \dots, \binom{\frac{n-1}{2}}{k-1}\right\}$, by $A_{k,j}$. The relation above becomes:

$$F_n(r) \cong \prod_{k=1}^{\frac{n+1}{2}} \prod_{j=1}^{\binom{\frac{n-1}{2}}{k-1}} A_{k,j}^{\alpha(r,k)},$$

where $\alpha(r, k)$ is the number of all tuples $(a_1, \dots, a_r)$ that generate a $2k$-element subalgebra, since that number does not depend on the specific such subalgebra (and not even on n).

Connections

Cignoli arrived at this result in 1970, though without giving an explicit description of the free algebra, but rather using a detour through a structure theorem for finite Moisil algebras.
He also derived the formula:

$$\alpha(r,k) = 2^r \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} (k-i)^r,$$

by an immediate application of the inclusion-exclusion principle.

Connections II

Since Łukasiewicz logic, modelled by MV_n-algebras, is strictly more powerful than Moisil logic, the J's are also available there, so it is immediate that in that context, too, representable functions coincide with subalgebra-preserving ones – a fact given as Corollary 8.6.2 in Cignoli, d'Ottaviano, Mundici (2000), where it is proven in a more roundabout way.

In addition, that book's assertion that "the 5-valued Moisil-Łukasiewicz algebra over one free generator has 192 elements" may be easily verified using our characterization theorem.

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Thank you for your attention.