COMPLEX ANALYSIS - Solved Problems 1

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PROBLEMS

Problem 1

Compute:

1a $(2+3\mathrm{i})^2+(\overline{1-\mathrm{i}})+\frac{2+\mathrm{i}}{1+\mathrm{i}}+\mathrm{i}^{10}+|3-4\mathrm{i}|$.

1b $\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}+\mathrm{e}^{3+\pi\mathrm{i}}+\cos(3\mathrm{i})$.

1c $(z^3+\frac{1}{z}-\sin z+z^2\mathrm{e}^{\mathrm{i}z})'$.

1d $\log(1+\sqrt{3}\,\mathrm{i})$.

Problem 2

Expand each of the functions

2a $f:\mathbb{C}\to\mathbb{C},\quad f(z)=\mathrm{e}^{\mathrm{i}z}$

2b $g:\mathbb{C}\backslash\{\mathrm{i}\}\to\mathbb{C},\quad g(z)=\frac{1}{z-\mathrm{i}}$

in a power series around $z_0=0$ (series of powers of z), and around $z_1=-\mathrm{i}$ (series of powers of $(z+\mathrm{i})$).

Problem 3

Compute the residues of the functions

3a $f:\mathbb{C}\backslash\{0,3\mathrm{i},-3\mathrm{i}\}\to\mathbb{C},\quad f(z)=\frac{1}{z^2(z^2+9)}$

3b $g:\mathbb{C}\backslash\{-\mathrm{i}\}\to\mathbb{C},\quad g(z)=(z^2+1)\mathrm{e}^{\frac{1}{z+\mathrm{i}}}$

in the corresponding singular points.

Problem 4

Compute the integrals

4a $I_0=\int_{\gamma_0} z^2dz$, where $\gamma_0:[0,1]\to\mathbb{C}$, $\gamma_0(t)=t+t\mathrm{i}$.

4b $I_1=\int_{\gamma_1}\frac{1}{z^2(z^2+9)}dz$, where $\gamma_1:[0,2\pi]\to\mathbb{C}$, $\gamma_1(t)=\mathrm{i}+3\mathrm{e}^{\mathrm{i}t}$.

Problem 5

Compute by using the theorem of residues

5 $I=\int_0^{2\pi}\frac{1}{3+\sin t}dt$.

SOME DEFINITIONS AND THEOREMS

D1 Definition

$(x_1+y_1\mathrm{i})(x_2+y_2\mathrm{i})=(x_1x_2-y_1y_2)+(x_1y_2+x_2y_1)\mathrm{i}$

$\overline{x+y\mathrm{i}}=x-y\mathrm{i}$,

$|x+y\mathrm{i}|=\sqrt{x^2+y^2}$,

$|z_1-z_2|$ = distance between z_1 and z_2,

$|z|$ = distance between z and 0,

$\mathrm{e}^{\mathrm{i}t}\overset{\mathrm{def}}{=}\cos t+\mathrm{i}\sin t$ (Euler's formula),

$\mathrm{e}^{x+y\mathrm{i}}=\mathrm{e}^x\cos y+\mathrm{i}\,\mathrm{e}^x\sin y$,

$z=|z|\,\mathrm{e}^{\mathrm{i}\arg z}$, where $-\pi<\arg z\le\pi$,

$\cos z=\frac{1}{2}(\mathrm{e}^{\mathrm{i}z}+\mathrm{e}^{-\mathrm{i}z})$,

$\sin z=\frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}z}-\mathrm{e}^{-\mathrm{i}z})$,

$\log z=\ln|z|+\mathrm{i}\arg z$, $\quad\log_k z=\ln|z|+\mathrm{i}(\arg z+2k\pi)$.

D2 Definition Let $D\subseteq\mathbb{C}$ be an open set, and $z_0\in D$.

A function $f:D\longrightarrow\mathbb{C}$ is **complex-differentiable** ($\mathbb{C}$-differentiable) at z_0 $\overset{\mathrm{def}}{\Longleftrightarrow}$ there exists and is finite $f'(z_0)\overset{\mathrm{def}}{=}\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$.

D3 Definition

$f:D\to\mathbb{C}$ defined on an open set D is called $\mathbb{C}$-*differentiable* if (or *holomorphic function*) f is $\mathbb{C}$-differentiable at any point $z_0\in D$.

T1 Theorem

$(f\pm g)'=f'\pm g'$ $\qquad (z^n)'=nz^{n-1}$

$(fg)'=f'g+fg'$ $\qquad (\mathrm{e}^z)'=\mathrm{e}^z$

$\left(\frac{f}{g}\right)'=\frac{f'g-fg'}{g^2}$ $\qquad (\sin z)'=\cos z$

$(f(\varphi(z))'=f'(\varphi(z))\,\varphi'(z)$ $\qquad (\cos z)'=-\sin z$.

T2 Theorem

$\frac{1}{1-z}=1+z+z^2+z^3+\dots$ for $|z|<1$

$\mathrm{e}^z=1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots$ for any z

D4 Definition

Let $D\subseteq\mathbb{C}$ be an open set,

$f:D\longrightarrow\mathbb{C}$ be a continuous function,

$\gamma:[a,b]\longrightarrow D$ be a path of class C^1.

The complex line integral of f along γ is $\int_\gamma f(z)\,dz=\int_a^b f(\gamma(t))\,\gamma'(t)dt$

T3 Theorem

$g:D\to\mathbb{C}$ is a primitive of f (that is $g'=f$) $\Rightarrow$ $\int_\gamma f(z)\,dz=g(\gamma(b))-g(\gamma(a))$

If $f:D\to\mathbb{C}$ admits a primitive and γ is closed, then $\int_\gamma f(z)\,dz=0$.

D5 Definition Let D be an open set and $f:D\to\mathbb{C}$ holomorphic.

$z_0\in\mathbb{C}\backslash D$ is an *isolated singular point* if there exists $r>0$ such that $\{\,z\mid 0<|z-z_0|<r\,\}\subset D$.

$z_0\in D$ is a *zero of multiplicity k* if $f(z_0)=f'(z_0)=\dots=f^{(k-1)}(z_0)=0$ and $f^{(k)}(z_0)\neq0$.

$z_0\in\mathbb{C}\backslash D$ is *a pole of order k of f* if z_0 is a zero of multiplicity k of $\frac{1}{f}$.

T4 Theorem Let D be an open set and $f:D\longrightarrow\mathbb{C}$ holomorphic.

z_0 is a an isolated singular point and $\{\,z\mid 0<|z-z_0|<r\,\}\subset D$ $\Rightarrow$ there exists a unique Laurent series such that $f(z)=\sum_{n=-\infty}^{\infty}a_n\,(z-z_0)^n$, for z satisfying $0<|z-z_0|<r$.

The number $\mathbf{Res}_{z_0}f\overset{\mathrm{def}}{=}a_{-1}$ is the **residue** of f in z_0.

T5 Theorem

If z_0 pole of order k, then around z_0 the function f admits an expansion of the form

$$f(z)=\frac{a_{-k}}{(z-z_0)^k}+\dots+\frac{a_{-1}}{(z-z_0)}+a_0+a_1(z-z_0)+\dots$$

and

$$\mathbf{Res}_{z_0}f=\frac{1}{(k-1)!}\lim_{z\to z_0}\left((z-z_0)^k\,f(z)\right)^{(k-1)}$$

D6 Definition (*Index* of a point z_0 with respect to a path).

For a closed path γ not passing through z_0,

$n(z_0,\gamma)\overset{\mathrm{def}}{=}\frac{1}{2\pi\mathrm{i}}\int_\gamma\frac{1}{z-z_0}dz$ shows how many turns around z_0, γ makes.

D7 Definition. Let D be an open set .

A closed path $\gamma:[a,b]\to D$ is called *homotopic to zero in D* if γ can be continuosly deformed inside D up to a constant path (a path having as image just a point).

T6 Theorem of Residues. For an open set $D\subseteq\mathbb{C}$:

$f:D\longrightarrow\mathbb{C}$ holomorphic function, S set of isolated singular points, γ path homotopic to zero in $D\cup S$ $\Rightarrow$ $\int_\gamma f(z)dz=2\pi\mathrm{i}\sum_{z\in S}n(z,\gamma)\mathbf{Res}_z f$

SOLUTIONS

Problem 1

$\boxed{1a}$ By using D1, we get

$$(2+3\mathrm{i})^2=4+12\mathrm{i}-9,$$
$$\overline{(1-\mathrm{i})}=1+\mathrm{i},$$
$$\frac{2+\mathrm{i}}{1+\mathrm{i}}=\frac{(1-\mathrm{i})(2+\mathrm{i})}{(1-\mathrm{i})(1+\mathrm{i})}=\frac{3-\mathrm{i}}{2},$$
$$\mathrm{i}^{10}=\mathrm{i}^8\,\mathrm{i}^2=(\mathrm{i}^4)^2\,(-1)=-1,$$
$$|3-4\mathrm{i}|=\sqrt{3^2+4^2}=5,$$

whence

$$(2+3\mathrm{i})^2+\overline{(1-\mathrm{i})}+\frac{2+\mathrm{i}}{1+\mathrm{i}}+\mathrm{i}^{10}+|3-4\mathrm{i}|=\frac{3+25\mathrm{i}}{2}.$$

$\boxed{1b}$ By using D1, we get

$$\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}=\cos\frac{\pi}{6}+\mathrm{i}\,\sin\frac{\pi}{6}=\frac{\sqrt{3}}{2}+\frac{\mathrm{i}}{2},$$
$$\mathrm{e}^{3+\pi\mathrm{i}}=\mathrm{e}^3\,\mathrm{e}^{\pi\mathrm{i}}=\mathrm{e}^3\,(\cos\pi+\mathrm{i}\sin\pi)=-\mathrm{e}^3,$$
$$\cos(3\mathrm{i})=\frac{\mathrm{e}^{-3}+\mathrm{e}^3}{2},$$

whence

$$\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}+\mathrm{e}^{3+\pi\mathrm{i}}+\cos(3\mathrm{i})=\frac{\sqrt{3}+\mathrm{e}^{-3}-\mathrm{e}^3}{2}+\frac{\mathrm{i}}{2}.$$

$\boxed{1c}$ By using D1, we obtain

$$(z^3+\tfrac{1}{z}-\sin z+z^2\mathrm{e}^{\mathrm{i}z})'=3z^2-\tfrac{1}{z^2}-\cos z+2z\mathrm{e}^{\mathrm{i}z}+\mathrm{i}z^2\mathrm{e}^{\mathrm{i}z}.$$

$\boxed{1d}$ By using D1, we get

$$\log(1+\sqrt{3}\,\mathrm{i})=\ln|1+\sqrt{3}\,\mathrm{i}|+\mathrm{i}\arg(1+\sqrt{3}\,\mathrm{i})=\ln 2+\frac{\pi\mathrm{i}}{3}.$$

Problem 2

$\boxed{2a}$ By using T2, around 0, we get

$$f(z)=\mathrm{e}^{\mathrm{i}z}=1+\frac{\mathrm{i}z}{1!}+\frac{(\mathrm{i}z)^2}{2!}+\frac{(\mathrm{i}z)^3}{3!}+\frac{(\mathrm{i}z)^4}{4!}+\dots$$
$$=1+\frac{\mathrm{i}z}{1!}-\frac{z^2}{2!}-\frac{\mathrm{i}z^3}{3!}+\frac{z^4}{4!}+\dots \text{ for any } z.$$

By using T2, around $-\mathrm{i}$, we get

$$f(z)=\mathrm{e}^{\mathrm{i}z}=\mathrm{e}^{\mathrm{i}(z+\mathrm{i}-\mathrm{i})}=\mathrm{e}\,\mathrm{e}^{\mathrm{i}(z+\mathrm{i})}$$
$$=\mathrm{e}+\mathrm{e}\,\frac{\mathrm{i}(z+\mathrm{i})}{1!}+\mathrm{e}\,\frac{(\mathrm{i}(z+\mathrm{i}))^2}{2!}+\mathrm{e}\,\frac{(\mathrm{i}(z+\mathrm{i}))^3}{3!}+\mathrm{e}\,\frac{(\mathrm{i}(z+\mathrm{i}))^4}{4!}+\dots$$
$$=\mathrm{e}+\mathrm{e}\,\frac{\mathrm{i}(z+\mathrm{i})}{1!}-\mathrm{e}\,\frac{(z+\mathrm{i})^2}{2!}-\mathrm{e}\,\frac{\mathrm{i}(z+\mathrm{i})^3}{3!}+\mathrm{e}\,\frac{(z+\mathrm{i})^4}{4!}+\dots \text{ for any } z.$$

$\boxed{2b}$ By using T2, around 0, we get

$$g(z)=\frac{1}{z-\mathrm{i}}=\frac{1}{-\mathrm{i}}\,\frac{1}{1-(-\mathrm{i}z)}=\mathrm{i}\,\frac{1}{1-(-\mathrm{i}z)}$$
$$=\mathrm{i}\,(1-\mathrm{i}z+(-\mathrm{i}z)^2+(-\mathrm{i}z)^3+(-\mathrm{i}z)^4+\dots)\quad\text{for } |-\mathrm{i}z|<1$$
$$=\mathrm{i}+z-\mathrm{i}z^2-z^3+\mathrm{i}z^4+\dots\quad\text{for } |z|<1.$$

By using T2, around $-\mathrm{i}$, we get

$$g(z)=\frac{1}{z-\mathrm{i}}=\frac{1}{(z+\mathrm{i})-2\mathrm{i}}=\frac{1}{-2\mathrm{i}}\,\frac{1}{1-\frac{z+\mathrm{i}}{2\mathrm{i}}}=\frac{\mathrm{i}}{2}\,\frac{1}{1-\frac{z+\mathrm{i}}{2\mathrm{i}}}$$
$$=\frac{\mathrm{i}}{2}\left[1+\left(\frac{z+\mathrm{i}}{2\mathrm{i}}\right)+\left(\frac{z+\mathrm{i}}{2\mathrm{i}}\right)^2+\left(\frac{z+\mathrm{i}}{2\mathrm{i}}\right)^3+\left(\frac{z+\mathrm{i}}{2\mathrm{i}}\right)^4+\dots\right]\quad\text{for } \left|\frac{z+\mathrm{i}}{2\mathrm{i}}\right|<1$$
$$=\frac{\mathrm{i}}{2}+\frac{1}{4}(z+\mathrm{i})-\frac{\mathrm{i}}{8}(z+\mathrm{i})^2+\dots\quad\text{for } |z+\mathrm{i}|<2.$$

Problem 3

$\boxed{3a}$ We use T5 and D1. The singular points are 0, $3\mathrm{i}$ and $-3\mathrm{i}$.

$z_0=0$ is a pole of order 2, and consequently

$$\mathbf{Res}_0 f=\frac{1}{1!}\lim_{z\to 0}\left(z^2f(z)\right)'=\lim_{z\to 0}\left(z^2\frac{1}{z^2(z^2+9)}\right)'$$
$$=\lim_{z\to 0}\left(\frac{1}{z^2+9}\right)'=\lim_{z\to 0}\frac{-2z}{(z^2+9)^2}=0.$$

$z_1=3\mathrm{i}$ is a pole of order 1, and consequently

$$\mathbf{Res}_{3\mathrm{i}} f=\frac{1}{0!}\lim_{z\to 3\mathrm{i}}(z-3\mathrm{i})f(z)$$
$$=\lim_{z\to 3\mathrm{i}}(z-3\mathrm{i})\frac{1}{z^2(z-3\mathrm{i})(z+3\mathrm{i})}=\lim_{z\to 3\mathrm{i}}\frac{1}{z^2(z+3\mathrm{i})}=\frac{\mathrm{i}}{54}.$$

$z_1=-3\mathrm{i}$ is a pole of order 1, and consequently

$$\mathbf{Res}_{-3\mathrm{i}} f=\frac{1}{0!}\lim_{z\to -3\mathrm{i}}(z+3\mathrm{i})f(z)$$
$$=\lim_{z\to -3\mathrm{i}}(z+3\mathrm{i})\frac{1}{z^2(z-3\mathrm{i})(z+3\mathrm{i})}=\lim_{z\to 3\mathrm{i}}\frac{1}{z^2(z-3\mathrm{i})}=-\frac{\mathrm{i}}{54}.$$

$\boxed{3b}$ We use T2 and T4. The only singular point is $-\mathrm{i}$.
It is not a pole. So, we have to use the Laurent series of g.
By looking for a representation of z^2+1 of the form

$$z^2+1=\alpha(z+\mathrm{i})^2+\beta(z+\mathrm{i})+\gamma$$

we get

$$z^2+1=(z+\mathrm{i})^2-2\mathrm{i}(z+\mathrm{i}).$$

$\mathbf{Res}_{-\mathrm{i}} g$ is the coefficient of $\frac{1}{z+\mathrm{i}}$ from the Laurent expansion of g around the point $-\mathrm{i}$.

By direct computation, we get

$$g(z)=(z^2+1)\mathrm{e}^{\frac{1}{z+\mathrm{i}}}=\left[(z+\mathrm{i})^2-2\mathrm{i}(z+\mathrm{i})\right]$$
$$\left[1+\frac{1}{1!}\frac{1}{z+\mathrm{i}}+\frac{1}{2!}\frac{1}{(z+\mathrm{i})^2}+\frac{1}{3!}\frac{1}{(z+\mathrm{i})^3}+\frac{1}{4!}\frac{1}{(z+\mathrm{i})^4}+\dots\right]$$
$$=\dots+\left(\frac{1}{3!}-\frac{2\mathrm{i}}{2!}\right)\frac{1}{z+\mathrm{i}}+\dots$$

Consequently, $\mathbf{Res}_{-\mathrm{i}} g=\frac{1}{6}-\mathrm{i}$.

Problem 4

$\boxed{4a}$ The function $h:\mathbb{C}\to\mathbb{C},\ h(z)=\frac{z^3}{3}$ is a primitive of z^2.
Therefore, by using T3, we get

$$I_0=\int_{\gamma_0} z^2dz=h(\gamma(1))-h(\gamma(0))=\frac{(1+\mathrm{i})^3}{3}=\mathrm{i}-1.$$

$\boxed{4b}$ We use the theorem of residues T6.
The singular points are 0, $3\mathrm{i}$ and $-3\mathrm{i}$, but only 0, $3\mathrm{i}$ are inside the circular path γ of center i and radius 3.
Therefore, by using the solution of 3a, we obtain

$$I_1=\int_{\gamma_1}\frac{1}{z^2(z^2+9)}dz=2\pi\mathrm{i}\,(\mathbf{Res}_0 f+\mathbf{Res}_{3\mathrm{i}} f)=-\frac{\pi}{27}.$$

Problem 5

By using the definition D1, the expression

$$I=\int_0^{2\pi}\frac{1}{3+\sin t}dt$$

can be written as

$$I=\int_0^{2\pi}\frac{1}{3+\frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}t}-\mathrm{e}^{-\mathrm{i}t})}dt$$

or

$$I=\int_0^{2\pi}\frac{2\mathrm{i}}{6\mathrm{i}+\mathrm{e}^{\mathrm{i}t}-\mathrm{e}^{-\mathrm{i}t}}dt$$

or

$$I=\int_0^{2\pi}\frac{1}{\mathrm{e}^{\mathrm{i}t}}\,\frac{2}{6\mathrm{i}+\mathrm{e}^{\mathrm{i}t}-\mathrm{e}^{-\mathrm{i}t}}\,(\mathrm{e}^{\mathrm{i}t})'dt.$$

This formula, written as

$$I=\int_0^{2\pi}\frac{2}{\mathrm{e}^{2\mathrm{i}t}+6\mathrm{i}\mathrm{e}^{\mathrm{i}t}-1}\,(\mathrm{e}^{\mathrm{i}t})'dt$$

is the complex integral

$$\int_\gamma\frac{2}{z^2+6\mathrm{i}z-1}dz$$

along the circular path $\gamma:[0,2\pi]\to\mathbb{C},\quad \gamma(t)=\mathrm{e}^{\mathrm{i}t}$.
It can be computed by using the theorem T6. Since

$$z^2+6\mathrm{i}z-1=(z+3\mathrm{i})^2+9-1=(z+3\mathrm{i})^2+8$$
$$=(z+3\mathrm{i})^2-(\mathrm{i}2\sqrt{2})^2=(z-z_1)(z-z_2),$$

where $z_1=(2\sqrt{2}-3)\mathrm{i}$ and $z_2=(-2\sqrt{2}-3)\mathrm{i}$,
the singular points are z_1 and z_2,
but only z_1 is inside the domain with the frontier γ.
The point z_1 is a pole of order 1, and

$$\mathbf{Res}_{z_1}\frac{2}{z^2+6\mathrm{i}z-1}=\lim_{z\to z_1}(z-z_1)\frac{2}{(z-z_1)(z-z_2)}$$
$$=\lim_{z\to z_1}\frac{2}{z-z_2}=\frac{2}{z_1-z_2}=\frac{2}{4\sqrt{2}\,\mathrm{i}}=\frac{-\mathrm{i}}{2\sqrt{2}}.$$

Therefore,

$$I=2\pi\mathrm{i}\,\mathbf{Res}_{z_1}\frac{2}{z^2+6\mathrm{i}z-1}=2\pi\mathrm{i}\,\frac{-\mathrm{i}}{2\sqrt{2}}=\frac{\pi}{\sqrt{2}}.$$