ON THE MATHEMATICAL BASES OF QUTRIT QUANTUM MECHANICS

Nicolae Cotfas, version 01 Dec 2019 (for future updates see “Documente” at https://unibuc.ro/user/nicolae.cotfas/)

• Hilbert space $\mathbb{C}^3$

$$\mathbb{C}^3 \equiv \{\psi : \{-1,0,1\} \longrightarrow \mathbb{C}\} \equiv \left\{ |\psi\rangle = \begin{pmatrix} \psi(-1) \\ \psi(0) \\ \psi(1) \end{pmatrix} \,\middle|\, \psi(k) \in \mathbb{C} \right\}, \qquad \langle \varphi, \psi \rangle = \sum_{n=-1}^{1} \overline{\varphi(n)}\, \psi(n) = \langle \varphi | \psi \rangle, \quad ||\psi|| = \sqrt{\langle \psi, \psi \rangle},$$

$$\left\{ |\delta_{-1}\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |\delta_0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |\delta_1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ is an orthonormal basis: } \begin{array}{c} |\delta_{-1}\rangle\langle\delta_{-1}| + |\delta_0\rangle\langle\delta_0| + |\delta_1\rangle\langle\delta_1| = \mathbf{I}, \\ \langle \delta_n | \delta_m \rangle = \delta_{nm} \end{array}$$

• Discrete Fourier transform $\mathbf{F} : \mathbb{C}^3 \longrightarrow \mathbb{C}^3$

$$\mathbf{F} = \frac{1}{\sqrt{3}} \sum_{n,k=-1}^{1} \mathrm{e}^{-\frac{2\pi \mathrm{i}}{3} kn} |\delta_k\rangle\langle\delta_n| = \frac{1}{\sqrt{3}} \begin{pmatrix} \mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}} & 1 & \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}} \\ 1 & 1 & 1 \\ \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}} & 1 & \mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}} \end{pmatrix}, \qquad \boxed{\begin{array}{l} \mathbf{F}[\psi](k) = \frac{1}{\sqrt{3}} \sum_{n=-1}^{1} \mathrm{e}^{-\frac{2\pi\mathrm{i}}{3} kn} \psi(n) \\ \mathbf{F}^\dagger[\psi](k) = \frac{1}{\sqrt{3}} \sum_{n=-1}^{1} \mathrm{e}^{\frac{2\pi\mathrm{i}}{3} kn} \psi(n) \end{array}} \qquad \begin{array}{l} \mathbf{F}\mathbf{F}^\dagger = \mathbf{I} \\ \mathbf{F}^\dagger\mathbf{F} = \mathbf{I}. \end{array}$$

$$\boxed{|\tilde{\delta}_n\rangle = \mathbf{F}^\dagger |\delta_n\rangle} \qquad \left\{ |\tilde{\delta}_{-1}\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}} \\ 1 \\ \mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}} \end{pmatrix}, |\tilde{\delta}_0\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, |\tilde{\delta}_1\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} \mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}} \\ 1 \\ \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}} \end{pmatrix} \right\} \text{ is an orthonormal basis.}$$

$\mathbf{F}$ has the distinct eigenvalues 1, $-\mathrm{i}$, -1, and the spectral decomposition

$$\boxed{\mathbf{F} = |\mathfrak{F}_0\rangle\langle\mathfrak{F}_0| - \mathrm{i}|\mathfrak{F}_1\rangle\langle\mathfrak{F}_1| - |\mathfrak{F}_2\rangle\langle\mathfrak{F}_2|}$$

where

$$|\mathfrak{F}_0\rangle = \begin{pmatrix} \frac{1}{2}\sqrt{1 - \frac{1}{\sqrt{3}}} \\ \frac{1}{\sqrt{2}}\sqrt{1 + \frac{1}{\sqrt{3}}} \\ \frac{1}{2}\sqrt{1 - \frac{1}{\sqrt{3}}} \end{pmatrix}, \quad |\mathfrak{F}_1\rangle = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad |\mathfrak{F}_2\rangle = \begin{pmatrix} \frac{1}{2}\sqrt{1 + \frac{1}{\sqrt{3}}} \\ -\frac{1}{\sqrt{2}}\sqrt{1 - \frac{1}{\sqrt{3}}} \\ \frac{1}{2}\sqrt{1 + \frac{1}{\sqrt{3}}} \end{pmatrix}$$

• Discrete Gaussian functions in $\mathbb{C}^3$.

For any $\kappa \in (0, \infty)$, we define $g_\kappa : \{-1,0,1\} \longrightarrow \mathbb{R}$,

$$\boxed{g_\kappa(n) = \sum_{m=-\infty}^{\infty} \mathrm{e}^{-\frac{\kappa\pi}{3}(n+3m)^2}} \qquad \boxed{|\mathbf{g}\rangle = \frac{|g_1\rangle}{||g_1||} = \begin{pmatrix} \frac{1}{2}\sqrt{1 - \frac{1}{\sqrt{3}}} \\ \frac{1}{\sqrt{2}}\sqrt{1 + \frac{1}{\sqrt{3}}} \\ \frac{1}{2}\sqrt{1 - \frac{1}{\sqrt{3}}} \end{pmatrix} = |\mathfrak{F}_1\rangle} \qquad \boxed{\begin{array}{l} \mathbf{F}[g_k] = \frac{1}{\sqrt{\kappa}}\, g_{\frac{1}{\kappa}} \\ \mathbf{F}[\mathbf{g}] = \mathbf{g} \end{array}}$$

• Position and momentum operators in $\mathbb{C}^3$

$$\boxed{\begin{array}{ll} \mathbf{Q} : \mathbb{C}^3 \longrightarrow \mathbb{C}^3, & (\mathbf{Q}\psi)(n) = n\,\psi(n) \\ \mathbf{P} : \mathbb{C}^3 \longrightarrow \mathbb{C}^3, & \mathbf{P} = \mathbf{F}^\dagger\mathbf{Q}\mathbf{F} \end{array}}$$

$$\mathbf{P}\psi(n) = \frac{1}{3} \sum_{k,m=-1}^{1} k\mathrm{e}^{\frac{2\pi\mathrm{i}}{3} km} \psi(n-m)$$

$$[\mathbf{Q}, \mathbf{P}]\psi(n) = \frac{1}{3} \sum_{m,k=-1}^{1} m\, k\, \mathrm{e}^{\frac{2\pi\mathrm{i}}{3} mk}\, \psi(n-m)$$

$$\mathbf{Q} = \sum_{n=-1}^{1} n|\delta_n\rangle\langle\delta_n| = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{P} = \sum_{k=-1}^{1} k|\tilde{\delta}_k\rangle\langle\tilde{\delta}_k| = \frac{\mathrm{i}}{\sqrt{3}} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}, \quad [\mathbf{Q}, \mathbf{P}] = \frac{\mathrm{i}}{\sqrt{3}} \begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix}.$$

• Translation operators

$$\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}\mathbf{Q}} = \begin{pmatrix} \mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}} \end{pmatrix}, \qquad \mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}\mathbf{P}} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

• Displacement operators in $\mathbb{C}^3$

For $(n,k)\in\{-1,0,1\}\times\{-1,0,1\}$, we define the unitary operators

$$\boxed{\begin{aligned}\mathbf{D}(n,k) &= \mathrm{e}^{-\frac{\pi\mathrm{i}}{3}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k\mathbf{Q}}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}n\mathbf{P}}\\ &= \mathrm{e}^{\frac{\pi\mathrm{i}}{3}nk}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}n\mathbf{P}}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}k\mathbf{Q}}\\ \mathbf{D}(n,k)\psi(m) &= \mathrm{e}^{-\frac{\pi\mathrm{i}}{3}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}km}\,\psi(m-n)\end{aligned}}$$

$$\mathbf{D}^\dagger(n,k)=\mathbf{D}(-n,-k)=\mathbf{D}^{-1}(n,k)$$

$$\boxed{\langle\mathbf{D}(n,k),\mathbf{D}(m,\ell)\rangle=3\,\delta_{nm}\,\delta_{k\ell}}$$

$$\mathbf{D}(n,k)\,\mathbf{D}(m,\ell)=\mathrm{e}^{-\frac{\pi\mathrm{i}}{3}(n\ell-km)}\mathbf{D}(n+m,k+\ell)$$

but (!!!) $\quad \mathbf{D}(n+3,k)=(-1)^k\,\mathbf{D}(n,k)$

$$\mathbf{D}(n,k+3)=(-1)^n\,\mathbf{D}(n,k)$$

The displacement operators

$$\mathbf{D}(-1,-1)=\begin{pmatrix}0 & \mathrm{e}^{\frac{\pi\mathrm{i}}{3}} & 0\\ 0 & 0 & \mathrm{e}^{-\frac{\pi\mathrm{i}}{3}}\\ -1 & 0 & 0\end{pmatrix},\quad \mathbf{D}(-1,0)=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},\quad \mathbf{D}(-1,1)=\begin{pmatrix}0 & \mathrm{e}^{-\frac{\pi\mathrm{i}}{3}} & 0\\ 0 & 0 & \mathrm{e}^{\frac{\pi\mathrm{i}}{3}}\\ -1 & 0 & 0\end{pmatrix},$$

$$\mathbf{D}(0,-1)=\begin{pmatrix}\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}}\end{pmatrix},\quad \mathbf{D}(0,0)=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad \mathbf{D}(0,1)=\begin{pmatrix}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\end{pmatrix},$$

$$\mathbf{D}(1,-1)=\begin{pmatrix}0 & 0 & -1\\ \mathrm{e}^{\frac{\pi\mathrm{i}}{3}} & 0 & 0\\ 0 & \mathrm{e}^{-\frac{\pi\mathrm{i}}{3}} & 0\end{pmatrix},\quad \mathbf{D}(1,0)=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix},\quad \mathbf{D}(1,1)=\begin{pmatrix}0 & 0 & -1\\ \mathrm{e}^{-\frac{\pi\mathrm{i}}{3}} & 0 & 0\\ 0 & \mathrm{e}^{\frac{\pi\mathrm{i}}{3}} & 0\end{pmatrix}$$

form an orthogonal basis in the complex Hilbert space of all the linear operators

$$\{A:\mathbb{C}^3\longrightarrow\mathbb{C}^3\ |\ A \text{ is a linear operator}\},\qquad \boxed{\langle A,B\rangle=\mathrm{tr}(A^\dagger B)}$$

and define a projective representation of the discrete Weyl-Heisenberg group.

• Discrete coherent states in $\mathbb{C}^3$

$$\boxed{|n,k\rangle=\mathbf{D}(n,k)|\mathbf{g}\rangle}\ \text{ where }\ |\mathbf{g}\rangle=\begin{pmatrix}\mathbf{g}(-1)\\ \mathbf{g}(0)\\ \mathbf{g}(1)\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\sqrt{1-\frac{1}{\sqrt{3}}}\\ \frac{1}{\sqrt{2}}\sqrt{1+\frac{1}{\sqrt{3}}}\\ \frac{1}{2}\sqrt{1-\frac{1}{\sqrt{3}}}\end{pmatrix}\qquad \boxed{\frac{1}{3}\sum_{n,k=-1}^{1}|n,k\rangle\langle n,k|=\mathbf{I}}\quad \boxed{\mathbf{F}|n,k\rangle=|k,-n\rangle}$$

$$|-1,-1\rangle=\begin{pmatrix}\mathrm{e}^{\frac{\pi\mathrm{i}}{3}}\mathbf{g}(0)\\ \mathrm{e}^{-\frac{\pi\mathrm{i}}{3}}\mathbf{g}(1)\\ -\mathbf{g}(-1)\end{pmatrix},\quad |-1,0\rangle=\begin{pmatrix}\mathbf{g}(0)\\ \mathbf{g}(1)\\ \mathbf{g}(-1)\end{pmatrix},\quad |-1,1\rangle=\begin{pmatrix}\mathrm{e}^{-\frac{\pi\mathrm{i}}{3}}\mathbf{g}(0)\\ \mathrm{e}^{\frac{\pi\mathrm{i}}{3}}\mathbf{g}(1)\\ -\mathbf{g}(-1)\end{pmatrix},$$

$$|0,-1\rangle=\begin{pmatrix}\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\mathbf{g}(-1)\\ \mathbf{g}(0)\\ \mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}}\mathbf{g}(1)\end{pmatrix},\quad |0,0\rangle=\begin{pmatrix}\mathbf{g}(-1)\\ \mathbf{g}(0)\\ \mathbf{g}(1)\end{pmatrix},\quad |0,1\rangle=\begin{pmatrix}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}}\mathbf{g}(-1)\\ \mathbf{g}(0)\\ \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\mathbf{g}(1)\end{pmatrix},$$

$$|1,-1\rangle=\begin{pmatrix}-\mathbf{g}(1)\\ \mathrm{e}^{\frac{\pi\mathrm{i}}{3}}\mathbf{g}(-1)\\ \mathrm{e}^{\frac{\pi\mathrm{i}}{3}}\mathbf{g}(0)\end{pmatrix},\quad |1,0\rangle=\begin{pmatrix}\mathbf{g}(1)\\ \mathbf{g}(-1)\\ \mathbf{g}(0)\end{pmatrix},\quad |1,1\rangle=\begin{pmatrix}-\mathbf{g}(1)\\ \mathrm{e}^{-\frac{\pi\mathrm{i}}{3}}\mathbf{g}(-1)\\ \mathrm{e}^{\frac{\pi\mathrm{i}}{3}}\mathbf{g}(0)\end{pmatrix}$$

• Parity operators in $\mathbb{C}^3$

$$\boxed{\Pi(n,k)=\mathbf{D}(n,k)\,\Pi\,\mathbf{D}(-n,-k),\ \text{ where }\ \Pi=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}}$$

$$\Pi\psi(n)=\psi(-n),$$
$$\Pi(n,k)\psi(m)=\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}2k(n-m)}\,\psi(2n-m),$$
$$\boxed{\langle\Pi(n,k),\,\Pi(m,\ell)\rangle=3\,\delta_{nm}\,\delta_{k\ell}}$$

The parity operators

$$\Pi(-1,-1)=\begin{pmatrix}1&0&0\\0&0&-\mathrm{e}^{-\frac{\pi\mathrm{i}}{3}}\\0&-\mathrm{e}^{\frac{\pi\mathrm{i}}{3}}&0\end{pmatrix},\quad \Pi(-1,0)=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix},\quad \Pi(-1,1)=\begin{pmatrix}1&0&0\\0&0&-\mathrm{e}^{\frac{\pi\mathrm{i}}{3}}\\0&-\mathrm{e}^{-\frac{\pi\mathrm{i}}{3}}&0\end{pmatrix},$$

$$\Pi(0,-1)=\begin{pmatrix}0&0&\mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}}\\0&1&0\\\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}&0&0\end{pmatrix},\quad \Pi(0,0)=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix},\quad \Pi(0,1)=\begin{pmatrix}0&0&\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\\0&1&0\\\mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}}&0&0\end{pmatrix},$$

$$\Pi(1,-1)=\begin{pmatrix}0&-\mathrm{e}^{-\frac{\pi\mathrm{i}}{3}}&0\\-\mathrm{e}^{\frac{\pi\mathrm{i}}{3}}&0&0\\0&0&1\end{pmatrix},\quad \Pi(1,0)=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix},\quad \Pi(1,1)=\begin{pmatrix}0&-\mathrm{e}^{\frac{\pi\mathrm{i}}{3}}&0\\-\mathrm{e}^{-\frac{\pi\mathrm{i}}{3}}&0&0\\0&0&1\end{pmatrix},$$

satisfy the relations

$$\boxed{\frac{1}{3}\sum_{n,k=-1}^{1}\Pi(n,k)=\mathbf{I}},\qquad \operatorname{tr}\Pi(n,k)=1,\qquad \Pi(n,k)^2=\mathbf{I}.$$

and form an orthogonal basis in the real Hilbert space of all the self-adjoint operators

$$\boxed{\begin{array}{c}\mathcal{A}(\mathbb{C}^3)=\{A:\mathbb{C}^3\longrightarrow\mathbb{C}^3\ |\ A^\dagger=A\},\\ \langle A,B\rangle=\operatorname{tr}(A\,B)\end{array}}\quad \text{that is}\quad \begin{array}{l}\langle\Pi(n,k),\,\Pi(m,\ell)\rangle=3\,\delta_{nm}\,\delta_{k\ell}\\ \frac{1}{3}\displaystyle\sum_{n,k=-1}^{1}|\Pi(n,k)\rangle\langle\Pi(n,k)|=\mathbf{I}_{\mathcal{A}(\mathbb{C}^3)}.\end{array}$$

• Convex set of all the **quantum states** $\equiv\boxed{\{\varrho:\mathbb{C}^3\longrightarrow\mathbb{C}^3\ |\ \varrho^\dagger=\varrho,\ \ \operatorname{tr}\varrho=1,\ \ \varrho\geq 0\}}$

$$\varrho^\dagger=\varrho\qquad\Rightarrow\qquad\boxed{\varrho=\sum_{n,k=-1}^{1}W_\varrho(n,k)\,\Pi(n,k)}\quad\text{where}\ \ W_\varrho(n,k)=\tfrac{1}{3}\langle\Pi(n,k),\varrho\rangle\in\mathbb{R}.$$

$$\boxed{\sum_{n,k=-1}^{1}W_\varrho(n,k)=1}\ \Rightarrow$$ quantum states can be described by using 8 real parameters $W_\varrho(n,k)$.

The allowed values of the parameters $W_\varrho(n,k)$ are only those for which $\varrho\geq 0$.

Discrete Wigner function of ϱ is

$$\boxed{\begin{array}{l}W_\varrho:\{-1,0,1\}\times\{-1,0,1\}\longrightarrow\mathbb{R},\\ W_\varrho(n,k)=\frac{1}{3}\langle\Pi(n,k),\varrho\rangle=\frac{1}{3}\operatorname{tr}(\varrho\,\Pi(n,k))\\ \qquad\qquad=\frac{1}{3}\displaystyle\sum_{m=-1}^{1}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{3}km}\langle\delta_{n+m}|\varrho|\delta_{n-m}\rangle.\end{array}}$$

$$W_\varrho(-1,-1)=\tfrac{1}{3}(\varrho_{-1-1}-\mathrm{e}^{\frac{\pi\mathrm{i}}{3}}\varrho_{01}-\mathrm{e}^{-\frac{\pi\mathrm{i}}{3}}\varrho_{10}),\quad W_\varrho(-1,0)=\tfrac{1}{3}(\varrho_{-1-1}+\varrho_{01}+\varrho_{10}),\quad W_\varrho(-1,1)=\tfrac{1}{3}(\varrho_{-1-1}-\mathrm{e}^{-\frac{\pi\mathrm{i}}{3}}\varrho_{01}-\mathrm{e}^{\frac{\pi\mathrm{i}}{3}}\varrho_{10}),$$

$$W_\varrho(0,-1)=\tfrac{1}{3}(\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\varrho_{-11}+\varrho_{00}+\mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}}\varrho_{1-1}),\quad W_\varrho(0,0)=\tfrac{1}{3}(\varrho_{-11}+\varrho_{00}+\varrho_{1-1}),\quad W_\varrho(0,1)=\tfrac{1}{3}(\mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}}\varrho_{-11}+\varrho_{00}+\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\varrho_{1-1}),$$

$$W_\varrho(1,-1)=\tfrac{1}{3}(-\mathrm{e}^{\frac{\pi\mathrm{i}}{3}}\varrho_{-10}-\mathrm{e}^{-\frac{\pi\mathrm{i}}{3}}\varrho_{0-1}+\varrho_{11}),\quad W_\varrho(1,0)=\tfrac{1}{3}(\varrho_{-10}+\varrho_{0-1}+\varrho_{11}),\quad W_\varrho(1,1)=\tfrac{1}{3}(-\mathrm{e}^{-\frac{\pi\mathrm{i}}{3}}\varrho_{-10}-\mathrm{e}^{\frac{\pi\mathrm{i}}{3}}\varrho_{0-1}+\varrho_{11}).$$

• Fractional Fourier transform in $\mathbb{C}^3$

$$\boxed{\mathbf{F}^\alpha = |\mathfrak{F}_0\rangle\langle\mathfrak{F}_0| + (-\mathrm{i})^\alpha |\mathfrak{F}_1\rangle\langle\mathfrak{F}_1| + (-1)^\alpha |\mathfrak{F}_2\rangle\langle\mathfrak{F}_2|}$$

$$|\mathfrak{F}_0\rangle = \begin{pmatrix} \frac{1}{2}\sqrt{1-\frac{1}{\sqrt{3}}} \\ \frac{1}{\sqrt{2}}\sqrt{1+\frac{1}{\sqrt{3}}} \\ \frac{1}{2}\sqrt{1-\frac{1}{\sqrt{3}}} \end{pmatrix}, \quad |\mathfrak{F}_1\rangle = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad |\mathfrak{F}_2\rangle = \begin{pmatrix} \frac{1}{2}\sqrt{1+\frac{1}{\sqrt{3}}} \\ -\frac{1}{\sqrt{2}}\sqrt{1-\frac{1}{\sqrt{3}}} \\ \frac{1}{2}\sqrt{1+\frac{1}{\sqrt{3}}} \end{pmatrix}$$

• Frame quantization

Based on the resolution of the identity $\boxed{\mathbf{I} = \frac{1}{3}\sum_{n,k=-1}^{1} |n,k\rangle\langle n,k|}$

we associate a linear operator
$$\boxed{\begin{array}{l} \mathbf{A}_f : \mathbb{C}^3 \longrightarrow \mathbb{C}^3 \\ \mathbf{A}_f = \frac{1}{3}\sum_{n,k=-1}^{1} f(n,k)\, |n,k\rangle\langle n,k| \end{array}}$$

to any function $f : \{-1,0,1\} \times \{-1,0,1\} \longrightarrow \mathbb{C}$.

One can remark that:

$$\begin{array}{l} f = 1 \quad \Rightarrow \quad \mathbf{A}_f = \mathbf{I} \\ \mathbf{A}_f^\dagger = \mathbf{A}_f \quad \text{in the case} \quad f : \{-1,0,1\} \times \{-1,0,1\} \longrightarrow \mathbb{R} \\ \mathbf{A}_f \geq 0 \quad \text{in the case} \quad f : \{-1,0,1\} \times \{-1,0,1\} \longrightarrow [0,\infty) \\ \mathrm{tr}\, \mathbf{A}_f = \frac{1}{3}\sum_{n,k=-1}^{1} f(n,k). \end{array}$$

• Discrete Hermite-Gauss functions

The self-adjoint operator corresponding to $f(n,k) = \frac{n^2+k^2}{2}$, namely

$$\boxed{\begin{array}{l} \mathbf{H} : \mathbb{C}^3 \longrightarrow \mathbb{C}^3 \\ \mathbf{H} = \frac{1}{3}\sum_{n,k=-1}^{1} \frac{n^2+k^2}{2}\, |n,k\rangle\langle n,k| \end{array}}$$

is Fourier invariant,

$$\mathbf{F}|n,k\rangle = |k,-n\rangle \quad \Rightarrow \quad \mathbf{FH} = \mathbf{HF},$$

and can be regarded as the Hamiltonian of a 3-dimensional oscillator.

The common eigenfunctions of $\mathbf{F}$ and $\mathbf{H}$

$$|0\rangle \equiv |\mathfrak{F}_0\rangle = \begin{pmatrix} \frac{1}{2}\sqrt{1-\frac{1}{\sqrt{3}}} \\ \frac{1}{\sqrt{2}}\sqrt{1+\frac{1}{\sqrt{3}}} \\ \frac{1}{2}\sqrt{1-\frac{1}{\sqrt{3}}} \end{pmatrix}, \quad |1\rangle \equiv |\mathfrak{F}_1\rangle = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad |2\rangle \equiv |\mathfrak{F}_2\rangle = \begin{pmatrix} \frac{1}{2}\sqrt{1+\frac{1}{\sqrt{3}}} \\ -\frac{1}{\sqrt{2}}\sqrt{1-\frac{1}{\sqrt{3}}} \\ \frac{1}{2}\sqrt{1+\frac{1}{\sqrt{3}}} \end{pmatrix}$$

can be regarded as a 3-dimensional version of the Hermite-Gauss functions.

• Discrete thermal states in $\mathbb{C}^3$

The self-adjoint operator corresponding to $f(n,k)=\mathrm{e}^{-\frac{n^2+k^2}{2\kappa}}$, namely

$$\boxed{\varrho=\frac{1}{C}\sum_{n,k=-1}^{1}\mathrm{e}^{-\frac{n^2+k^2}{2\kappa}}\,|n,k\rangle\langle n,k|}$$

where $\kappa\in(0,\infty)$ is a parameter and C a normalizing constant, is Fourier invariant,

$$\mathbf{F}|n,k\rangle=|k,-n\rangle\qquad\Rightarrow\qquad\mathbf{F}\varrho=\varrho\mathbf{F},$$

and can be regarded as the density operator of a discrete thermal state.

The common eigenfunctions of $\mathbf{F}$ and ϱ

$$|0\rangle\equiv|\mathfrak{F}_0\rangle=\begin{pmatrix}\frac{1}{2}\sqrt{1-\frac{1}{\sqrt{3}}}\\ \frac{1}{\sqrt{2}}\sqrt{1+\frac{1}{\sqrt{3}}}\\ \frac{1}{2}\sqrt{1-\frac{1}{\sqrt{3}}}\end{pmatrix},\quad |1\rangle\equiv|\mathfrak{F}_1\rangle=\begin{pmatrix}-\frac{1}{\sqrt{2}}\\ 0\\ \frac{1}{\sqrt{2}}\end{pmatrix},\quad |2\rangle\equiv|\mathfrak{F}_2\rangle=\begin{pmatrix}\frac{1}{2}\sqrt{1+\frac{1}{\sqrt{3}}}\\ -\frac{1}{\sqrt{2}}\sqrt{1-\frac{1}{\sqrt{3}}}\\ \frac{1}{2}\sqrt{1+\frac{1}{\sqrt{3}}}\end{pmatrix}$$

represent the pure states forming ϱ, namely $\varrho=\sum_{m=0}^{2}\langle m|\varrho|m\rangle\;|m\rangle\langle m|$.

• Four mutually unbiased bases in $\mathbb{C}^3$

Definition. Two orthonormal bases $\{|\varphi_0\rangle,|\varphi_1\rangle,|\varphi_2\rangle\}$, $\{|\psi_0\rangle,|\psi_1\rangle,|\psi_2\rangle\}$ are called *unbiased* if

$$|\langle\varphi_n|\psi_k\rangle|^2=\frac{1}{3},\qquad\text{for any}\quad n,k\in\{0,1,2\}.$$

The orthonormal bases

$$\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}\quad\text{eigenbasis of}\quad\mathbf{V}=\mathbf{D}(0,1)=\begin{pmatrix}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}}&0&0\\0&1&0\\0&0&\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\end{pmatrix},$$

$$\left\{\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix},\frac{1}{\sqrt{3}}\begin{pmatrix}1\\ \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\\ \mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}}\end{pmatrix},\frac{1}{\sqrt{3}}\begin{pmatrix}1\\ \mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}}\\ \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\end{pmatrix}\right\}\quad\text{eigenbasis of}\quad\mathbf{U}=\mathbf{D}(1,0)=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix},$$

$$\left\{\frac{1}{\sqrt{3}}\begin{pmatrix}1\\ \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\\ 1\end{pmatrix},\frac{1}{\sqrt{3}}\begin{pmatrix}1\\ \mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}}\\ \mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}}\end{pmatrix},\frac{1}{\sqrt{3}}\begin{pmatrix}1\\ 1\\ \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\end{pmatrix}\right\}\quad\text{eigenbasis of}\quad\mathbf{UV}=\begin{pmatrix}0&0&\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\\ \mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}}&0&0\\0&1&0\end{pmatrix},$$

$$\left\{\frac{1}{\sqrt{3}}\begin{pmatrix}1\\ \mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}}\\ 1\end{pmatrix},\frac{1}{\sqrt{3}}\begin{pmatrix}1\\ 1\\ \mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}}\end{pmatrix},\frac{1}{\sqrt{3}}\begin{pmatrix}1\\ \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\\ \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\end{pmatrix}\right\}\quad\text{eigenbasis of}\quad\mathbf{UV}^2=\begin{pmatrix}0&0&\mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}}\\ \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}&0&0\\0&1&0\end{pmatrix}$$

are mutually unbiased bases.

These bases can be regarded as the eigenbases of *four mutually complementary observables !*

• Kravchuk orthonormal basis of $\mathbb{C}^3$

$$|\mathfrak{K}_{-1}\rangle=\begin{pmatrix}\frac{1}{2}\\ \frac{1}{\sqrt{2}}\\ \frac{1}{2}\end{pmatrix},\quad |\mathfrak{K}_0\rangle=\begin{pmatrix}-\frac{1}{\sqrt{2}}\\ 0\\ \frac{1}{\sqrt{2}}\end{pmatrix},\quad |\mathfrak{K}_1\rangle=\begin{pmatrix}\frac{1}{2}\\ -\frac{1}{\sqrt{2}}\\ \frac{1}{2}\end{pmatrix}.$$

- **Remarkable unitary operators $\mathbb{C}^3 \longrightarrow \mathbb{C}^3$**

$$\boxed{\begin{array}{c} U \text{ unitary} \\ (UU^\dagger = \mathbf{I}) \end{array} \;\Rightarrow\; \begin{array}{c} \text{there exists} \\ A \text{ self-adjoint} \\ (A^\dagger = A) \end{array} \quad \text{such that} \quad U = \mathrm{e}^{\mathrm{i}A}}$$

$$\text{Fourier} \quad \mathbf{F} = \frac{1}{\sqrt{3}} \begin{pmatrix} \mathrm{e}^{-\frac{2\pi \mathrm{i}}{3}} & 1 & \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}} \\ 1 & 1 & 1 \\ \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}} & 1 & \mathrm{e}^{-\frac{2\pi \mathrm{i}}{3}} \end{pmatrix} \qquad \text{with} \quad \mathbf{F}^4 = \mathbf{I}$$

$$\text{Kravchuk} \quad \mathbf{K} = \begin{pmatrix} \frac{\mathrm{i}}{2} & \frac{1}{\sqrt{2}} & -\frac{\mathrm{i}}{2} \\ -\frac{\mathrm{i}}{\sqrt{2}} & 0 & -\frac{\mathrm{i}}{\sqrt{2}} \\ \frac{\mathrm{i}}{2} & -\frac{1}{\sqrt{2}} & -\frac{\mathrm{i}}{2} \end{pmatrix} \qquad \text{with} \quad \mathbf{K}^3 = \mathbf{I}$$

$$\mathbf{S} = \begin{pmatrix} -\frac{\mathrm{i}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -\mathrm{i} \\ \frac{\mathrm{i}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \qquad \text{with} \qquad \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{S}^\dagger \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{S}$$

$$\text{Translation operators} \quad \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}\mathbf{Q}} = \begin{pmatrix} \mathrm{e}^{-\frac{2\pi \mathrm{i}}{3}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}} \end{pmatrix}, \qquad \mathrm{e}^{-\frac{2\pi \mathrm{i}}{3}\mathbf{P}} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Displacement operators

$$\mathbf{D}(-1,-1) = \begin{pmatrix} 0 & \mathrm{e}^{\frac{\pi \mathrm{i}}{3}} & 0 \\ 0 & 0 & \mathrm{e}^{-\frac{\pi \mathrm{i}}{3}} \\ -1 & 0 & 0 \end{pmatrix}, \quad \mathbf{D}(-1,0) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{D}(-1,1) = \begin{pmatrix} 0 & \mathrm{e}^{-\frac{\pi \mathrm{i}}{3}} & 0 \\ 0 & 0 & \mathrm{e}^{\frac{\pi \mathrm{i}}{3}} \\ -1 & 0 & 0 \end{pmatrix},$$

$$\mathbf{D}(0,-1) = \begin{pmatrix} \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{e}^{-\frac{2\pi \mathrm{i}}{3}} \end{pmatrix}, \quad \mathbf{D}(0,0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{D}(0,1) = \begin{pmatrix} \mathrm{e}^{-\frac{2\pi \mathrm{i}}{3}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}} \end{pmatrix},$$

$$\mathbf{D}(1,-1) = \begin{pmatrix} 0 & 0 & -1 \\ \mathrm{e}^{\frac{\pi \mathrm{i}}{3}} & 0 & 0 \\ 0 & \mathrm{e}^{-\frac{\pi \mathrm{i}}{3}} & 0 \end{pmatrix}, \quad \mathbf{D}(1,0) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{D}(1,1) = \begin{pmatrix} 0 & 0 & -1 \\ \mathrm{e}^{-\frac{\pi \mathrm{i}}{3}} & 0 & 0 \\ 0 & \mathrm{e}^{\frac{\pi \mathrm{i}}{3}} & 0 \end{pmatrix}$$

Gell-Mann unitary operators $\mathrm{e}^{\mathrm{i}\lambda_n}$, where

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

- **Remarkable self-adjoint operators** $\mathbb{C}^3 \longrightarrow \mathbb{C}^3$

$$\boxed{\begin{array}{ccc} A \text{ self-adjoint} & \Longleftrightarrow & U = \mathrm{e}^{\mathrm{i}A} \text{ unitary} \\ (A^\dagger = A) & & (UU^\dagger = \mathbf{I}) \end{array}}$$

Position and momentum operators $\mathbf{Q} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\quad \mathbf{P} = \mathbf{F}^\dagger \mathbf{Q} \mathbf{F} = \frac{\mathrm{i}}{\sqrt{3}} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$

Hermitian generators of the $SU(2)$ irreducible representation

$$\mathbf{J}_z = \mathbf{Q} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{S} \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{S}^\dagger$$

$$\mathbf{J}_x = \mathbf{K}\mathbf{Q}\mathbf{K}^\dagger = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} = \mathbf{S} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix} \mathbf{S}^\dagger \qquad \text{where} \qquad \mathbf{K} = \begin{pmatrix} \frac{\mathrm{i}}{2} & \frac{1}{\sqrt{2}} & -\frac{\mathrm{i}}{2} \\ -\frac{\mathrm{i}}{\sqrt{2}} & 0 & -\frac{\mathrm{i}}{\sqrt{2}} \\ \frac{\mathrm{i}}{2} & -\frac{1}{\sqrt{2}} & -\frac{\mathrm{i}}{2} \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} -\frac{\mathrm{i}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -\mathrm{i} \\ \frac{\mathrm{i}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$\mathbf{J}_y = \mathbf{K}^\dagger \mathbf{Q} \mathbf{K} = \begin{pmatrix} 0 & \frac{\mathrm{i}}{\sqrt{2}} & 0 \\ -\frac{\mathrm{i}}{\sqrt{2}} & 0 & \frac{\mathrm{i}}{\sqrt{2}} \\ 0 & -\frac{\mathrm{i}}{\sqrt{2}} & 0 \end{pmatrix} = \mathbf{S} \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix} \mathbf{S}^\dagger$$

Parity operators $\Pi(n,k) = \mathbf{D}(n,k)\, \Pi\, \mathbf{D}(-n,-k)$, where $\Pi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$$\Pi(-1,-1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -\mathrm{e}^{-\frac{\pi \mathrm{i}}{3}} \\ 0 & -\mathrm{e}^{\frac{\pi \mathrm{i}}{3}} & 0 \end{pmatrix}, \quad \Pi(-1,0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Pi(-1,1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -\mathrm{e}^{\frac{\pi \mathrm{i}}{3}} \\ 0 & -\mathrm{e}^{-\frac{\pi \mathrm{i}}{3}} & 0 \end{pmatrix},$$

$$\Pi(0,-1) = \begin{pmatrix} 0 & 0 & \mathrm{e}^{-\frac{2\pi \mathrm{i}}{3}} \\ 0 & 1 & 0 \\ \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}} & 0 & 0 \end{pmatrix}, \quad \Pi(0,0) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \Pi(0,1) = \begin{pmatrix} 0 & 0 & \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}} \\ 0 & 1 & 0 \\ \mathrm{e}^{-\frac{2\pi \mathrm{i}}{3}} & 0 & 0 \end{pmatrix},$$

$$\Pi(1,-1) = \begin{pmatrix} 0 & -\mathrm{e}^{-\frac{\pi \mathrm{i}}{3}} & 0 \\ -\mathrm{e}^{\frac{\pi \mathrm{i}}{3}} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Pi(1,0) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Pi(1,1) = \begin{pmatrix} 0 & -\mathrm{e}^{\frac{\pi \mathrm{i}}{3}} & 0 \\ -\mathrm{e}^{-\frac{\pi \mathrm{i}}{3}} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Gell-Mann Hermitian generators of $SU(3) = \{U : \mathbb{C}^3 \longrightarrow \mathbb{C}^3 \mid U^{-1} = U^\dagger,\ \det U = 1\}$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Operators defined by using the finite frame quantization

$$A_f = \frac{1}{3} \sum_{n,k=-1}^{1} f(n,k)\, |n,k\rangle\langle n,k| \quad \text{defined by} \quad f : \{-1,0,1\} \times \{-1,0,1\} \longrightarrow \mathbb{R}.$$

- **Density operators**

A self-adjoint operator $\varrho : \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ is a *density operator*	$\Longleftrightarrow$	$\mathrm{tr}\,\varrho = 1$, $\mathrm{tr}\,\varrho^2 \leq 1$, $3\,\mathrm{tr}\,\varrho^2 - 2\,\mathrm{tr}\,\varrho^3 \leq 1.$
A self-adjoint operator $\varrho : \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ describes a *boundary state*	$\Longleftrightarrow$	$\mathrm{tr}\,\varrho = 1$, $\mathrm{tr}\,\varrho^2 \leq 1$, $3\,\mathrm{tr}\,\varrho^2 - 2\,\mathrm{tr}\,\varrho^3 = 1.$
A self-adjoint operator $\varrho : \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ describes a *pure state*	$\Longleftrightarrow$	$\mathrm{tr}\,\varrho = 1$, $\mathrm{tr}\,\varrho^2 = 1$, $3\,\mathrm{tr}\,\varrho^2 - 2\,\mathrm{tr}\,\varrho^3 = 1.$