An overview on $\boxed{\textbf{FRAMES AND TIGHT FRAMES}}$

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Reference: [1] S F D Waldron, *An Introduction to Finite Tight Frames*, Springer 2017, and references therein.

■ FRAMES IN A HILBERT SPACE $\mathcal{H}$ $\quad (d=\dim\mathcal{H})$

■ **Definition**. $(|\varphi_k\rangle)_{k=1}^m$ is a **frame** in $\mathcal{H}$ if

$$\boxed{\begin{array}{l}\text{there exist } \alpha,\beta\in(0,\infty) \text{ such that}\\ \alpha||\psi||^2\leq\sum_{k=1}^m|\langle\varphi_k|\psi\rangle|^2\leq\beta||\psi||^2,\ \forall\,|\psi\rangle\in\mathcal{H}.\end{array}}$$

■ $\boxed{\mathrm{span}(\{|\varphi_k\rangle\}_{k=1}^m)=\mathcal{H}} \Rightarrow \boxed{(|\varphi_k\rangle)_{k=1}^m \text{is a frame in } \mathcal{H}}$

Proof. We have $\sum_{k=1}^m|\langle\varphi_k|\psi\rangle|^2\leq\sum_{k=1}^m||\varphi_k||^2\,||\psi||^2=\beta\,||\psi||^2.$

$\mathcal{H}\to\mathbb{C}:\psi\mapsto\sum_{k=1}^m|\langle\varphi_k|\psi\rangle|^2$ is continuous on the compact set $M=\{|\varphi\rangle\ |\ ||\varphi||=1\}\Rightarrow\exists|\psi_0\rangle\in M$ such that

$$\alpha=\inf_{\varphi\in M}\sum_{k=1}^m|\langle\varphi_k|\varphi\rangle|^2=\sum_{k=1}^m|\langle\varphi_k|\psi_0\rangle|^2>0,$$

and $\sum_{k=1}^m|\langle\varphi_k|\psi\rangle|^2=\sum_{k=1}^m|\langle\varphi_k|\frac{1}{||\psi||}|\psi\rangle|^2||\psi||^2\geq\alpha||\psi||^2.$

■ $\boxed{(|\varphi_k\rangle)_{k=1}^m \text{ frame}} \Rightarrow$ The adjoint of the *synthesis operator* $T:\mathbb{C}^m\longrightarrow\mathcal{H}:(c_k)_{k=1}^m\mapsto\sum_{k=1}^m c_k\,|\varphi_k\rangle$ is the *analysis operator* $T^\dagger:\mathcal{H}\longrightarrow\mathbb{C}^m:|\psi\rangle\mapsto(\langle\varphi_k|\psi\rangle)_{k=1}^m.$

Proof. $\langle\psi|T(c_k)_{k=1}^m\rangle=\sum_{k=1}^m c_k\langle\psi|\varphi_k\rangle=\langle T^\dagger\psi|(c_k)_{k=1}^m\rangle.$

■ $\boxed{(|\varphi_k\rangle)_{k=1}^m \text{ frame}} \Rightarrow$ The *frame operator* $S=TT^\dagger:\mathcal{H}\to\mathcal{H}$

$$\boxed{S=\sum_{k=1}^m|\varphi_k\rangle\langle\varphi_k|}$$

is invertible, $(|\tilde\varphi_k\rangle=S^{-1}|\varphi_k\rangle)_{k=1}^m$ is a frame (called *dual frame*) and $|\psi\rangle=\sum_{k=1}^m|\varphi_k\rangle\langle\tilde\varphi_k|\psi\rangle,\quad\forall|\psi\rangle\in\mathcal{H},$

that is $\boxed{\mathbb{I}=\sum_{k=1}^m|\varphi_k\rangle\langle\tilde\varphi_k|=\sum_{k=1}^m|\tilde\varphi_k\rangle\langle\varphi_k|}$

Proof. The self-adjoint operator S is injective,

$$S|\psi\rangle=0\ \Rightarrow\ 0=\langle\psi|S\psi\rangle=\sum_{k=1}^m|\langle\varphi_k|\psi\rangle|^2\ \Rightarrow\ |\psi\rangle=0,$$

and hence bijective. We have

$$|\psi\rangle=SS^{-1}|\psi\rangle=\sum_{k=1}^m|\varphi_k\rangle\langle\varphi_k|S^{-1}\psi\rangle=\sum_{k=1}^m|\varphi_k\rangle\langle S^{-1}\varphi_k|\psi\rangle.$$

■ $\boxed{\mathrm{span}(\{|\varphi_k\rangle\}_{k=1}^m)=\mathcal{H}} \Leftrightarrow \boxed{(|\varphi_k\rangle)_{k=1}^m \text{is a frame in } \mathcal{H}}$

■ The relation from the definition of a frame can also be written as

$$\langle\alpha\psi|\psi\rangle\leq\langle S\psi|\psi\rangle\leq\langle\beta\psi|\psi\rangle,\ \forall|\psi\rangle\in\mathcal{H}\quad\text{or}\quad\alpha\mathbb{I}\leq S\leq\beta\mathbb{I}.$$

Optimal values: $\alpha=$ smallest eigenvalue of S, $\beta=$ biggest eigenvalue of S.

■ $\boxed{(|\varphi_k\rangle)_{k=1}^m \text{ is a frame}} \Rightarrow \boxed{(S^{-\frac{1}{2}}|\varphi_k\rangle)_{k=1}^m \text{ is a tight frame}}$

Proof. The synthesis operator $V=S^{-\frac{1}{2}}T$ satisfies $VV^\dagger=\mathbb{I}$.

■ *Finite frame quantization*. If $(|\varphi_k\rangle)_{k\in J}$ is a frame, then

$\boxed{\text{to each function } f:J\longrightarrow\mathbb{C}}$ we associate $\boxed{\text{the linear operator } A_f=\sum_{k\in J}f(k)\,|\varphi_k\rangle\langle\tilde\varphi_k|.}$

■ $\boxed{\begin{array}{l}(|\varphi_k\rangle)_{k=1}^m \text{ frame in } \mathcal{H}\\ (|\psi_\ell\rangle)_{\ell=1}^n \text{ frame in } \mathcal{K}\end{array}} \Rightarrow \boxed{\begin{array}{c}(|\varphi_k\rangle\otimes|\psi_\ell\rangle)_{\substack{k\in\{1,2,...,m\}\\ \ell\in\{1,2,...,n\}}}\\ \text{is a frame in } \mathcal{H}\otimes\mathcal{K}\end{array}}$

■ TIGHT FRAMES (TF) IN A HILBERT SPACE $\mathcal{H}$

■ **Definition**. $(|\varphi_k\rangle)_{k=1}^m$ is a **tight frame (TF)** if

$$\boxed{\begin{array}{l}\text{there exists } \alpha\in(0,\infty) \text{ such that}\\ \sum_{k=1}^m|\langle\varphi_k|\psi\rangle|^2=\alpha||\psi||^2,\ \forall\,|\psi\rangle\in\mathcal{H}.\end{array}}$$

Normalized tight frame (NTF)=tight frame with $\alpha=1$.

■ $\boxed{(|\varphi_k\rangle)_{k=1}^m \text{ tight frame}} \Leftrightarrow \boxed{\mathbb{I}=\frac{1}{\alpha}\sum_{k=1}^m|\varphi_k\rangle\langle\varphi_k|} \Updownarrow \boxed{|\psi\rangle=\frac{1}{\alpha}\sum_{k=1}^m|\varphi_k\rangle\langle\varphi_k|\psi\rangle,\ \forall\,|\psi\rangle}$

Proof. Since $U=S-\alpha\mathbb{I}$, where $S=\sum_{k=1}^m|\varphi_k\rangle\langle\varphi_k|$, satisfies

$$4\langle\varphi|U\psi\rangle=\langle\varphi+\psi|U(\varphi+\psi)\rangle-\langle\varphi-\psi|U(\varphi-\psi)\rangle$$
$$+\mathrm{i}\langle\varphi+\mathrm{i}\psi|U(\varphi+\mathrm{i}\psi)\rangle-\mathrm{i}\langle\varphi-\mathrm{i}\psi|U(\varphi-\mathrm{i}\psi)\rangle$$

we get

$$\left.\begin{array}{l}\langle\psi,U\psi\rangle=\langle\psi|S\psi\rangle-\alpha\langle\psi|\psi\rangle\\ =\sum_{k=1}^m|\langle\varphi_k|\psi\rangle|^2-\alpha||\psi||^2=0\end{array}\right|\Rightarrow\begin{array}{c}\langle\varphi|U\psi\rangle=0\\ \text{for any } \varphi,\psi\end{array}\Rightarrow U=0.$$

■ $\boxed{\begin{array}{ll}\{|\Phi_k\rangle\}_{k=1}^m & \text{orthonormal basis in } \mathcal{K}\\ \Pi & \text{orthogonal projector corresponding to } \mathcal{H}\subset\mathcal{K}\end{array}} \Rightarrow \boxed{(\Pi|\Phi_k\rangle)_{k=1}^m \text{ is a NTF in } \mathcal{H}}$

Proof. $\sum_{k=1}^m|\Phi_k\rangle\langle\Phi_k|=\mathbb{I}_\mathcal{K}\Rightarrow\sum_{k=1}^m\Pi|\Phi_k\rangle\langle\Phi_k|\Pi=\mathbb{I}_\mathcal{H}.$

■ $\boxed{\text{Any NTF } (|\varphi_k\rangle)_{k=1}^m \text{ defines an embedding } \mathcal{H}\subset\mathbb{C}^m\text{, and } (|\varphi_k\rangle)_{k=1}^m \text{ is the projection of the canonical basis from } \mathbb{C}^m}$

Proof. Let $(|\varphi_k\rangle)_{k=1}^m$ be a NTF in $\mathcal{H}$, $(|u_n\rangle)_{n=1}^d$ be an orthonormal basis in $\mathcal{H}$, $(|e_k\rangle)_{k=1}^m$ be the canonical basis in $\mathbb{C}^m$.

In $\mathbb{C}^m$: $|\psi_n\rangle=\sum_{k=1}^m|e_k\rangle\langle\varphi_k|u_n\rangle$ satisfy $\langle\psi_n|\psi_\ell\rangle=\delta_{n\ell}$.

By using the embedding of $\mathcal{H}$ into $\mathbb{C}^m$

$$\mathcal{H}\equiv\mathrm{span}(|\psi_n\rangle)_{n=1}^d\subset\mathbb{C}^m\quad|\psi\rangle\equiv\sum_{n=1}^d|\psi_n\rangle\langle u_n|\psi\rangle$$

and the orthogonal projector $\Pi=\sum_{n=1}^d|\psi_n\rangle\langle\psi_n|$ we get

$$|\varphi_k\rangle\equiv\sum_{n=1}^d|\psi_n\rangle\langle u_n|\varphi_k\rangle=\Pi|e_k\rangle.$$

■ $\boxed{\text{Columns of a matrix form a NTF}} \Leftrightarrow \boxed{\text{the rows form an orthonormal system.}}$

■ $\boxed{\mathbb{I}=\sum_{k=1}^m|\varphi_k\rangle\langle\varphi_k|} \Rightarrow \boxed{\begin{array}{l}\text{the Gramian } P=T^\dagger T=(\langle\varphi_k|\varphi_\ell\rangle)\\ \text{is an orthogonal projector:}\\ P^\dagger=P=P^2.\end{array}}$

■ $\boxed{\begin{array}{l}\mathbb{I}=\sum_{k=1}^m|\varphi_k\rangle\langle\varphi_k|\\ |\psi\rangle=\sum_{k=1}^m c_k\,|\varphi_k\rangle\end{array}} \Rightarrow \boxed{\begin{array}{l}|\psi\rangle=\sum_{k=1}^m|\varphi_k\rangle\langle\varphi_k|\psi\rangle \text{ and}\\ \sum_{k=1}^m|c_k|^2\geq\sum_{k=1}^m|\langle\varphi_k|\psi\rangle|^2\end{array}}$

■ $\boxed{\mathbb{I}=\sum_{k=1}^m|\varphi_k\rangle\langle\varphi_k|} \overset{\mathrm{tr}}{\Rightarrow} \boxed{\sum_{k=1}^m\langle\varphi_k|\varphi_k\rangle=d=\sum_{k,\ell=1}^m|\langle\varphi_k|\varphi_\ell\rangle|^2}$

■ $\boxed{\begin{array}{ll}g\mapsto T_g & \text{unitary irreducible representation of a finite group } G \text{ in } \mathcal{H}\\ |\varphi\rangle\neq0 & \text{a fixed element of } \mathcal{H}\end{array}} \Rightarrow \boxed{\{T_g|\varphi\rangle\}_{g\in G} \text{ tight frame in } \mathcal{H}}$

Proof. $S=\sum_{g\in G}T_g|\varphi\rangle\langle\varphi_g|T_g^\dagger$ satisfies $T_hST_h^\dagger=S$, $\forall h\in G$.

Schur's lemma $\Rightarrow$ $S=\alpha\mathbb{I}$.