An overview on $\boxed{\textbf{CONTINUOUS versus DISCRETE WIGNER FUNCTION}}$

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CONTINUOUS WIGNER FUNCTION

■ By using *Fourier transform, position and momentum operators*

$$\mathcal{F}[\psi](p) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}pq}\, \psi(q)\, dq, \qquad (Q\psi)(q) \stackrel{\text{def}}{=} q\,\psi(q), \quad P \stackrel{\text{def}}{=} \mathcal{F}^\dagger Q \mathcal{F} = -\mathrm{i}\frac{d}{dq},$$

one defines the *translation operators*

$$\mathrm{e}^{\mathrm{i}pQ},\ \mathrm{e}^{-\mathrm{i}qP} \quad \text{acting as} \quad \mathrm{e}^{\mathrm{i}pQ}\psi(x) = \mathrm{e}^{\mathrm{i}px}\psi(x), \quad \mathrm{e}^{-\mathrm{i}qP}\psi(x) = \psi(x-q).$$

■ The *displacement operators*

$$D(q,p) \stackrel{\text{def}}{=} \mathrm{e}^{-\mathrm{i}(qP-pQ)} = \mathrm{e}^{-\frac{\mathrm{i}}{2}pq}\, \mathrm{e}^{\mathrm{i}pQ}\, \mathrm{e}^{-\mathrm{i}qP} = \mathrm{e}^{\frac{\mathrm{i}}{2}pq}\, \mathrm{e}^{-\mathrm{i}qP}\, \mathrm{e}^{\mathrm{i}pQ}$$

and the *parity operators*

$$\Pi(q,p) \stackrel{\text{def}}{=} D(q,p)\,\Pi\, D(q,p)^\dagger, \qquad \Pi\psi(x) \stackrel{\text{def}}{=} \psi(-x).$$

act as

$$D(q,p)\psi(x) = \mathrm{e}^{-\frac{\mathrm{i}}{2}pq}\, \mathrm{e}^{\mathrm{i}px}\, \psi(x-q), \qquad \Pi(q,p)\psi(x) = \mathrm{e}^{-2\mathrm{i}p(q-x)}\, \psi(2q-x).$$

■ **Definition**

$$\boxed{\textit{Wigner function of a state } \rho \ \text{ is } \ \mathcal{W}_\rho : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}, \quad \boxed{\mathcal{W}_\rho(q,p) \stackrel{\text{def}}{=} \frac{1}{\pi}\, \mathrm{tr}(\rho\,\Pi(q,p))}}$$

■ $\{\Pi(q,p\}_{q,p\in\mathbb{R}}$ satisfy $\mathrm{tr}(\,\Pi(q,p)\ \Pi(q',p')\,) = \pi\ \delta(q-q')\,\delta(p-p')$ and $\boxed{\rho = \iint_{\mathbb{R}^2} \mathcal{W}_\rho(q,p)\,\Pi(q,p)\,dqdp}$

■ In terms of the Dirac quasibasis $\{|q\rangle \equiv \delta_q\}_{q\in\mathbb{R}}$ $\boxed{\mathcal{W}_\varrho(q,p) = \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-2\mathrm{i}px}\, \langle q+x|\varrho|q-x\rangle dx}$

Proof.
$$\begin{aligned}\mathcal{W}_\rho(q,p) &= \frac{1}{\pi} \iint_{\mathbb{R}^2} \langle a|\varrho|b\rangle\langle b|\Pi(n,k)|a\rangle dadb\\ &= \frac{1}{\pi} \iint_{\mathbb{R}^2} \langle a|\varrho|b\rangle \mathrm{e}^{-2\mathrm{i}p(q-b)}\delta_a(2q-b)dadb\\ &= \frac{1}{\pi} \iint_{\mathbb{R}^2} \langle a|\varrho|q-x\rangle \mathrm{e}^{-2\mathrm{i}px}\delta_a(q+x)dadx\\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-2\mathrm{i}px}\, \langle q+x|\varrho|q-x\rangle dx.\end{aligned}$$

■ For any state ϱ, we have $\boxed{\iint_{\mathbb{R}^2} \mathcal{W}_\rho(q,p)dqdp = 1}$

Proof. $\int_{-\infty}^{\infty} \mathrm{e}^{-2\mathrm{i}px}dp = \pi\,\delta(x) \Rightarrow \iint_{\mathbb{R}^2} \mathcal{W}_\rho(q,p)dqdp = \mathrm{tr}\,\varrho.$

■ Pure state $\varrho = |\psi\rangle\langle\psi|$ $\Rightarrow$ $\boxed{\mathcal{W}_\psi(q,p) = \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-2\mathrm{i}px}\, \overline{\psi(q-x)}\, \psi(q+x)dx}$

Proof.
$$\begin{aligned}\mathcal{W}_\varrho(q,p) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-2\mathrm{i}px}\, \langle q+x|\varrho|q-x\rangle dx\\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-2\mathrm{i}px}\, \langle q+x|\psi\rangle\langle\psi|q-x\rangle dx\\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-2\mathrm{i}px}\, \overline{\psi(q-x)}\, \psi(q+x)dx.\end{aligned}$$

■ Pure state $\varrho = |\psi\rangle\langle\psi|$ $\Rightarrow$ $\boxed{\int_{-\infty}^{\infty} \mathcal{W}_\psi(q,p)dp = |\psi(q)|^2}$

Proof. We have $\int_{-\infty}^{\infty} \mathrm{e}^{-2\mathrm{i}px}dp = \pi\,\delta(x).$

■ Pure state $\varrho = |\psi\rangle\langle\psi|$ $\Rightarrow$ $\boxed{\int_{-\infty}^{\infty} \mathcal{W}_\psi(q,p)dq = |\mathcal{F}[\psi](p)|^2}$

Proof. With the change of variables $(a,b) = (q-x, q+x)$,
$$\begin{aligned}\int_{-\infty}^{\infty} \mathcal{W}_\psi(q,p)dq &= \frac{1}{\pi} \iint_{\mathbb{R}^2} \mathrm{e}^{-2\mathrm{i}px}\, \overline{\psi(q-x)}\, \psi(q+x)dxdq\\ &= \frac{1}{2\pi} \iint_{\mathbb{R}^2} \mathrm{e}^{\mathrm{i}p(a-b)}\, \overline{\psi(a)}\, \psi(b)dadb = |\mathbf{F}[\psi](p)|^2.\end{aligned}$$

DISCRETE WIGNER FUNCTION

($d = 2j+1$ is a positive odd integer)

■ By regarding $\mathbb{C}^d$ as a Hilbert space of functions,

$$\mathbb{C}^d \equiv \{\psi : \{-j, -j+1, ..., j-1, j\} \to \mathbb{C}\} \equiv \{\psi : \mathbb{Z}_d \to \mathbb{C}\},$$

and using *Fourier transform, position and momentum operators*

$$\mathbf{F}[\psi](k) \stackrel{\text{def}}{=} \frac{1}{\sqrt{d}} \sum_{n\in\mathbb{Z}_d} \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}kn}\, \psi(n), \qquad (\mathbf{Q}\psi)(n) \stackrel{\text{def}}{=} \tilde{n}\,\psi(n), \quad \mathbf{P} \stackrel{\text{def}}{=} \mathbf{F}^\dagger\mathbf{Q}\mathbf{F},$$

where $-j \leq \tilde{n} \leq j$ is the representative mod(d) of n,
we define the *translation operators*

$$\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k\mathbf{Q}},\ \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}n\mathbf{P}} \quad \text{acting as} \quad \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k\mathbf{Q}}\psi(n) = \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kn}\psi(n), \quad \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}n\mathbf{P}}\psi(m) = \psi(m-n).$$

■ The *displacement operators*

$$\mathbf{D}(n,k) \stackrel{\text{def}}{=} \mathrm{e}^{-\frac{\pi\mathrm{i}}{d}nk}\, \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k\mathbf{Q}}\, \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}n\mathbf{P}} = \mathrm{e}^{\frac{\pi\mathrm{i}}{d}nk}\, \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}n\mathbf{P}}\, \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k\mathbf{Q}}$$

and the *parity operators*

$$\Pi(n,k) \stackrel{\text{def}}{=} \mathbf{D}(n,k)\,\Pi\,\mathbf{D}(n,k)^\dagger, \qquad \Pi\psi(n) \stackrel{\text{def}}{=} \psi(-n).$$

act as

$$\mathbf{D}(n,k)\psi(m) = \mathrm{e}^{-\frac{\pi\mathrm{i}}{d}nk}\, \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}km}\, \psi(m-n), \qquad \Pi(n,k)\psi(m) = \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}k(n-m)}\, \psi(2n-m).$$

■ **Definition** [JH DeBrota, BC Stacey, arXiv:1912.07554 and references therein]

$$\boxed{\textit{Discrete Wigner function} \text{ of a state } \rho \ \text{ is } \ \mathbf{W}_\rho : \mathbb{Z}_d\times\mathbb{Z}_d \longrightarrow \mathbb{R}, \quad \boxed{\mathbf{W}_\rho(n,k) \stackrel{\text{def}}{=} \frac{1}{d}\ \mathrm{tr}(\rho\,\Pi(n,k))}}$$

■ In the space of Hermitian operators $\{\Pi(n,k\}_{n,k\in\mathbb{Z}_d}$ is orthogonal basis $\mathrm{tr}(\,\Pi(n,k)\ \Pi(m,\ell)\,) = d\ \delta_{nm}\,\delta_{k\ell}$ $\Rightarrow$ $\boxed{\rho = \sum_{n,k\in\mathbb{Z}_d} \mathbf{W}_\rho(n,k)\,\Pi(n,k)}$

■ In terms of the canonical basis $\{|m\rangle \equiv \delta_m\}_{m\in\mathbb{Z}_d}$ $\boxed{\mathbf{W}_\varrho(n,k) = \frac{1}{d} \sum_{m\in\mathbb{Z}_d} \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\, \langle n+m|\varrho|n-m\rangle}$

Proof.
$$\begin{aligned}\mathbf{W}_\rho(n,k) &= \frac{1}{d} \sum_{\ell,s\in\mathbb{Z}_d} \langle\ell|\varrho|s\rangle\langle s|\Pi(n,k)|\ell\rangle\\ &= \frac{1}{d} \sum_{\ell,s\in\mathbb{Z}_d} \langle\ell|\varrho|s\rangle \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}k(n-s)}\delta_\ell(2n-s)\\ &= \frac{1}{d} \sum_{\ell,m\in\mathbb{Z}_d} \langle\ell|\varrho|s\rangle \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\delta_\ell(n+m)\\ &= \frac{1}{d} \sum_{m\in\mathbb{Z}_d} \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\, \langle n+m|\varrho|n-m\rangle.\end{aligned}$$

■ For any state ϱ, we have $\boxed{\sum_{n,k\in\mathbb{Z}_d} \mathbf{W}_\rho(n,k) = 1}$

Proof. $\sum_{k\in\mathbb{Z}_d} \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km} = d\,\delta_{2m,0} = d\,\delta_{m,0} \Rightarrow \sum_{n,k\in\mathbb{Z}_d} \mathbf{W}_\rho(n,k) = \mathrm{tr}\,\varrho.$

■ Pure state $\varrho = |\psi\rangle\langle\psi|$ $\Rightarrow$ $\boxed{\mathbf{W}_\psi(n,k) = \frac{1}{d} \sum_{m\in\mathbb{Z}_d} \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\, \overline{\psi(n-m)}\, \psi(n+m)}$

Proof.
$$\begin{aligned}\mathbf{W}_\psi(n,k) &= \frac{1}{d}\ \mathrm{tr}(\,|\psi\rangle\langle\psi|\ \Pi(n,k)\,) = \frac{1}{d}\ \langle\psi|\Pi(n,k)|\psi\rangle\\ &= \frac{1}{d} \sum_{\ell\in\mathbb{Z}_d} \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}k(n-\ell)}\, \overline{\psi(\ell)}\, \psi(2n-\ell)\\ &= \frac{1}{d} \sum_{m\in\mathbb{Z}_d} \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\, \overline{\psi(n-m)}\, \psi(n+m).\end{aligned}$$

■ Pure state $\varrho = |\psi\rangle\langle\psi|$ $\Rightarrow$ $\boxed{\sum_{k\in\mathbb{Z}_d} \mathbf{W}_\psi(n,k) = |\psi(n)|^2}$

Proof. For odd d, $\sum_{k\in\mathbb{Z}_d} \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km} = d\,\delta_{2m,0} = d\,\delta_{m,0}.$

■ Pure state $\varrho = |\psi\rangle\langle\psi|$ $\Rightarrow$ $\boxed{\sum_{n\in\mathbb{Z}_d} \mathbf{W}_\psi(n,k) = |\mathbf{F}[\psi](k)|^2}$

Proof. $\mathbb{Z}_d\times\mathbb{Z}_d \to \mathbb{Z}_d\times\mathbb{Z}_d : (n,m) \mapsto (n-m, n+m)$ is bijective for odd d. Therefore
$$\begin{aligned}\sum_{n\in\mathbb{Z}_d} \mathbf{W}_\psi(n,k) &= \frac{1}{d} \sum_{n,m\in\mathbb{Z}_d} \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\, \overline{\psi(n-m)}\, \psi(n+m)\\ &= \frac{1}{d} \sum_{\alpha,\beta\in\mathbb{Z}_d} \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k(\alpha-\beta)}\, \overline{\psi(\alpha)}\, \psi(\beta) = |\mathbf{F}[\psi](k)|^2.\end{aligned}$$