PURE AND MIXED DISCRETE VARIABLE GAUSSIAN STATES. ARE THEY SOME NEW SIGNIFICANT QUANTUM STATES ?

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Abstract (The definitions we use and our notations are presented in section D).

- The discrete Wigner function of a *Gaussian function of discrete variable* can be defined in terms of the Wigner function of the corresponding Gaussian function of continuous variable (see Nicolae Cotfas and Daniela Dragoman, J. Phys. A: Math. Theor. 45 (2012) 425305 and $\boxed{B1}$).
- We arrive at a definition for the *Gaussian states of discrete variable systems* (*discrete Gaussian states*) by extending this correspondence to all the Gaussian states of continuous variable quantum systems.
- So, evidently, among the discrete Gaussian states there are the pure Gaussian states described by *discrete Gaussian functions*.
- *Discrete vacuum* $|0\rangle$, and the *discrete coherent states* $|n,k\rangle = D(n,k)|0\rangle$ are discrete Gaussian states.
- Any pure state $|\psi\rangle$ is a linear superposition of discrete Gaussian states: $|\psi\rangle = \frac{1}{d}\sum\limits_{n,k}|n,k\rangle\langle n,k|\psi\rangle$.
- Any Gaussian state of a continuous variable system is the limit of a sequence of discrete Gaussian states.
- If $\det\sigma = 1$, then the discrete Gaussian state with the covariance matrix σ is a pure state.

If the dimension d of the Hilbert space $\mathcal{H}$ of the quantum system is large enough, then our numerical data suggest that:

- Function $\mathcal{W}_\varrho$ from Definition $\boxed{4}$ is the Wigner function of a state: $\sum\limits_{n,k}\mathcal{W}_\varrho(n,k)\Pi(n,k)\geq 0$.
- The purity of a Gaussian state of discrete variable system with the covariance matrix σ is approximately $1/\sqrt{\det\sigma}$.
- The eigenfunctions of the density operator of a *discrete thermal state* with $\sigma = \nu\mathbb{I}$ are almost independent on ν. They can be regarded as *discrete Hermite-Gauss functions*.
- The discrete counterpart $[Q,P] = \mathrm{i}\frac{d}{2\pi}$ of $[\hat{q},\hat{p}] = \mathrm{i}\frac{h}{2\pi}$ is approximately satisfied in a big subspace of $\mathcal{H}$, and this subspace contains the discrete Gaussian states.
- A *discrete Gaussian transform* maps a discrete Gaussian state into an almost discrete Gaussian state.

$\boxed{A}$ SINGLE-MODE GAUSSIAN STATES OF DISCRETE VARIABLE SYSTEMS (odd-dimensional case $d = 2s+1$)

■ Gaussian functions of one continuous/discrete variable ($\kappa\in(0,\infty)$ is a parameter)

$\boxed{1}$ Gaussian function $\mathrm{G}_\kappa:\mathbb{R}\to\mathbb{R}$, $\boxed{\mathrm{G}_\kappa(q) = \mathrm{e}^{-\frac{\kappa}{2}q^2}}$ has the Fourier transform $\boxed{\mathbf{F}[G_\kappa] = \frac{1}{\sqrt{\kappa}}\,G_{\kappa^{-1}}}$

$\boxed{2}$ Discrete Gaussian function $g_\kappa:\mathbb{Z}_d\to\mathbb{R}$, $\boxed{g_\kappa(n) = \sum\limits_{\alpha=-\infty}^{\infty}\mathrm{G}_\kappa\Big((n+\alpha d)\sqrt{\tfrac{2\pi}{d}}\Big)}$ has the Fourier transform $\boxed{F[g_\kappa] = \frac{1}{\sqrt{\kappa}}\,g_{\kappa^{-1}}}$

that is $\boxed{F[g_\kappa](k) = \sum\limits_{\beta=-\infty}^{\infty}\mathbf{F}[\mathrm{G}_\kappa]\Big((k+\beta d)\sqrt{\tfrac{2\pi}{d}}\Big)}$

■ Single-mode Gaussian states

$\boxed{3}$ **Definition.**

A state ρ is a **Gaussian state of a continuous variable quantum system** with the covariance matrix $\sigma = \begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{pmatrix} > 0$ satisfying $\sigma + \mathrm{i}\hbar\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\geq 0$

if its Wigner function has the form

$$\boxed{\mathbf{W}_\rho(q,p) = \frac{1}{\pi\sqrt{\det\sigma}}\,\mathrm{e}^{-(\,q-q_0\;\;p-p_0\,)\begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{pmatrix}^{-1}\begin{pmatrix}q-q_0\\ p-p_0\end{pmatrix}}}$$

where $q_0,\, p_0\in\mathbb{R}$ are two parameters. In the case $q_0 = p_0 = 0$, we write ρ_σ instead of ρ.

$\boxed{4}$ **Definition.**

A state ϱ is a **Gaussian state of a discrete variable quantum system** (*discrete Gaussian state*) with the covariance matrix $\sigma = \begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{pmatrix} > 0$ satisfying $\sigma + \mathrm{i}\hbar\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\geq 0$

if its discrete Wigner function has the form

$$\boxed{\mathcal{W}_\varrho(n,k) = C_d\sum_{\alpha,\beta=-\infty}^{\infty}(-1)^{\alpha\beta}\,\mathbf{W}_\rho\Big((n+\alpha\tfrac{d}{2})\sqrt{\tfrac{2\pi}{d}},\,(k+\beta\tfrac{d}{2})\sqrt{\tfrac{2\pi}{d}}\Big)}$$

that is, the form

$$\boxed{\mathcal{W}_\varrho(n,k) = C_d\sum_{\alpha,\beta=-\infty}^{\infty}(-1)^{\alpha\beta}\,\mathrm{e}^{-\frac{2\pi}{d}(\,n+\alpha\frac{d}{2}-q_0\;\;k+\beta\frac{d}{2}-p_0\,)\begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{pmatrix}^{-1}\begin{pmatrix}n+\alpha\frac{d}{2}-q_0\\ k+\beta\frac{d}{2}-p_0\end{pmatrix}}}$$

where $q_0,\, p_0\in\mathbb{R}$ are two parameters and C_d is a normalizing factor. In the case $q_0 = p_0 = 0$, we write ϱ_σ instead of ϱ.

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B SOME PROPERTIES OF THE SINGLE-MODE DISCRETE GAUSSIAN STATES

1 Theorem.

For any $\kappa\in(0,\infty)$, the pure state
$$\mathbf{g}_\kappa=\frac{1}{\sqrt{\langle g_\kappa,g_\kappa\rangle}}\,g_\kappa$$
is a discrete Gaussian state with the covariance matrix
$$\sigma_\kappa=\begin{pmatrix}\frac{1}{\kappa} & 0\\ 0 & \kappa\end{pmatrix}.$$

Proof.
By using $\boxed{D56}$, the relation ($\boxed{D57}$)
$$F[g_{2\kappa}](2k)=\tfrac{1}{\sqrt{2\kappa}}\,g_{\frac{1}{2\kappa}}(2k)=\tfrac{1}{\sqrt{2\kappa}}\sum_{\alpha=-\infty}^{\infty}\mathrm{e}^{-\frac{\pi}{2\kappa d}(2k+\alpha d)^2}$$
$$=\tfrac{1}{\sqrt{2\kappa}}\sum_{\alpha=-\infty}^{\infty}\mathrm{e}^{-\frac{\pi}{2\kappa d}(2k+2\alpha d)^2}+\tfrac{1}{\sqrt{2\kappa}}\sum_{\alpha=-\infty}^{\infty}\mathrm{e}^{-\frac{\pi}{2\kappa d}(2k+(2\alpha+1)d)^2}$$
$$=\tfrac{1}{\sqrt{2\kappa}}\sum_{\alpha=-\infty}^{\infty}\mathrm{e}^{-\frac{2\pi}{\kappa d}(k+\alpha d)^2}+\tfrac{1}{\sqrt{2\kappa}}\sum_{\alpha=-\infty}^{\infty}\mathrm{e}^{-\frac{2\pi}{\kappa d}(k+\alpha d+\frac{d}{2})^2}$$
$$=\tfrac{1}{\sqrt{2\kappa}}\big(g_{\frac{2}{\kappa}}(k)+g^{+}_{\frac{2}{\kappa}}(k)\big)$$
and the formula
$$\sum_{\alpha,\beta=-\infty}^{\infty}E_{\alpha,\beta}=\sum_{\substack{\alpha,\,\beta\\ \text{both even}\\ \text{or}\\ \text{both odd}}}E_{\alpha,\beta}+\sum_{\substack{\alpha,\,\beta\\ \text{one even}\\ \text{and}\\ \text{other odd}}}E_{\alpha,\beta}$$
$$=\sum_{\mu,\eta=-\infty}^{\infty}E_{\mu+\eta,\mu-\eta}+\sum_{\mu,\eta=-\infty}^{\infty}E_{\mu+\eta+1,\mu-\eta},$$
we get (up to a normalizing multiplicative constant)
$$\mathcal{W}_{g_\kappa}(n,k)=\tfrac{1}{d}\sum_{m=-s}^{s}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\sum_{\alpha,\beta=-\infty}^{\infty}\mathrm{e}^{-\frac{\kappa\pi}{d}(n-m+\alpha d)^2}\mathrm{e}^{-\frac{\kappa\pi}{d}(n+m+\beta d)^2}$$
$$=\tfrac{1}{d}\sum_{m=-s}^{s}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\sum_{\mu,\eta=-\infty}^{\infty}\mathrm{e}^{-\frac{\kappa\pi}{d}(n-m+(\mu+\eta)d)^2}\mathrm{e}^{-\frac{\kappa\pi}{d}(n+m+(\mu-\eta)d)^2}$$
$$+\tfrac{1}{d}\sum_{m=-s}^{s}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\sum_{\mu,\eta=-\infty}^{\infty}\mathrm{e}^{-\frac{\kappa\pi}{d}(n-m+(\mu+\eta+1)d)^2}\mathrm{e}^{-\frac{\kappa\pi}{d}(n+m+(\mu-\eta)d)^2}$$
$$=\tfrac{1}{d}\sum_{\mu=-\infty}^{\infty}\mathrm{e}^{-\frac{2\kappa\pi}{d}(n+\mu d)^2}\sum_{m=-s}^{s}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\sum_{\eta=-\infty}^{\infty}\mathrm{e}^{-\frac{2\kappa\pi}{d}(-m+\eta d)^2}$$
$$+\tfrac{1}{d}\sum_{\mu=-\infty}^{\infty}\mathrm{e}^{-\frac{2\kappa\pi}{d}(n+\frac{d}{2}+\mu d)^2}\sum_{m=-s}^{s}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\sum_{\eta=-\infty}^{\infty}\mathrm{e}^{-\frac{2\kappa\pi}{d}(-m+\frac{d}{2}+\eta d)^2}$$
$$=\tfrac{1}{\sqrt{d}}g_{2\kappa}(n)\,F[g_{2\kappa}](2k)+\tfrac{1}{\sqrt{d}}g^{+}_{2\kappa}(n)\,F[g^{+}_{2\kappa}](2k)$$
$$=\tfrac{1}{\sqrt{2\kappa d}}\,g_{2\kappa}(n)\left(g_{\frac{2}{\kappa}}(k)+g^{+}_{\frac{2}{\kappa}}(k)\right)+\tfrac{1}{\sqrt{2\kappa d}}\,g^{+}_{2\kappa}(n)\left(g_{\frac{2}{\kappa}}(k)-g^{+}_{\frac{2}{\kappa}}(k)\right)$$
$$=\tfrac{1}{\sqrt{2\kappa d}}\sum_{\alpha,\beta=-\infty}^{\infty}\mathrm{e}^{-\frac{2\pi}{d}[\kappa(n+\alpha d)^2+\frac{1}{\kappa}(k+\beta d)^2]}$$
$$+\tfrac{1}{\sqrt{2\kappa d}}\sum_{\alpha,\beta=-\infty}^{\infty}\mathrm{e}^{-\frac{2\pi}{d}[\kappa(n+\alpha d)^2+\frac{1}{\kappa}(k+\frac{d}{2}+\beta d)^2]}$$
$$+\tfrac{1}{\sqrt{2\kappa d}}\sum_{\alpha,\beta=-\infty}^{\infty}\mathrm{e}^{-\frac{2\pi}{d}[\kappa(n+\frac{d}{2}+\alpha d)^2+\frac{1}{\kappa}(k+\beta d)^2]}$$
$$-\tfrac{1}{\sqrt{2\kappa d}}\sum_{\alpha,\beta=-\infty}^{\infty}\mathrm{e}^{-\frac{2\pi}{d}[\kappa(n+\frac{d}{2}+\alpha d)^2+\frac{1}{\kappa}(k+\frac{d}{2}+\beta d)^2]}$$
$$=\tfrac{1}{\sqrt{2\kappa d}}\sum_{\alpha,\beta=-\infty}^{\infty}(-1)^{\alpha\beta}\mathrm{e}^{-\frac{2\pi}{d}[\kappa(n+\alpha\frac{d}{2})^2+\frac{1}{\kappa}(k+\beta\frac{d}{2})^2]}$$
$$=\tfrac{1}{\sqrt{2\kappa d}}\sum_{\alpha,\beta=-\infty}^{\infty}(-1)^{\alpha\beta}\mathrm{e}^{-\frac{2\pi}{d}\left(\,n+\alpha\frac{d}{2}\quad k+\beta\frac{d}{2}\,\right)\begin{pmatrix}\frac{1}{\kappa}&0\\0&\kappa\end{pmatrix}^{-1}\begin{pmatrix}n+\alpha\frac{d}{2}\\k+\beta\frac{d}{2}\end{pmatrix}}$$
$$=\tfrac{\pi}{\sqrt{2\kappa d}}\sum_{\alpha,\beta=-\infty}^{\infty}(-1)^{\alpha\beta}\mathbf{W}_{\rho_{\sigma_\kappa}}\left((n+\alpha\tfrac{d}{2})\sqrt{\tfrac{2\pi}{d}},(k+\beta\tfrac{d}{2})\sqrt{\tfrac{2\pi}{d}}\right).$$

2 Theorem.

For any $\kappa\in(0,\infty)$ and any $n_0,k_0\in\{-s,-s+1,...,s-1,s\}$, the pure state $D(n_0,k_0)\mathbf{g}_\kappa$ *is a discrete Gaussian state with the covariance matrix σ_κ.*

Proof.
The Wigner function of the state
$$\psi(m)=D(n_0,k_0)\mathbf{g}_\kappa(m)=\mathrm{e}^{-\frac{\pi\mathrm{i}}{d}n_0k_0}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k_0m}\,\mathbf{g}_\kappa(m-n_0),$$
is $\mathcal{W}_\psi(n,k)=\frac{1}{d}\sum\limits_{m=-s}^{s}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\,\psi(n+m)\,\overline{\psi(n-m)}$
$$=\tfrac{1}{d}\sum_{m=-s}^{s}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k_0(n-m)}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}k_0(n-m)}$$
$$\times\mathbf{g}_\kappa(n-m-n_0)\,\mathbf{g}_\kappa(n+m-n_0),$$
$$=\tfrac{1}{d}\sum_{m=-s}^{s}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}(k-k_0)m}\,\mathbf{g}_\kappa(n-n_0-m)\,\mathbf{g}_\kappa(n-n_0+m),$$
$$=\mathcal{W}_{\mathbf{g}_\kappa}(n-n_0,k-k_0).$$
$$=\tfrac{\pi}{\sqrt{2\kappa d}}\sum_{\alpha,\beta=-\infty}^{\infty}(-1)^{\alpha\beta}\mathbf{W}_{\rho_{\sigma_\kappa}}\Big((n-n_0+\alpha\tfrac{d}{2})\sqrt{\tfrac{2\pi}{d}},(k-k_0+\beta\tfrac{d}{2})\sqrt{\tfrac{2\pi}{d}}\Big)$$

3 Remark.

Particularly,
- *discrete vacuum* $|0\rangle\equiv|\mathbf{g}_1\rangle$,
- *discrete coherent states* $|n,k\rangle=D(n,k)|0\rangle$

are discrete Gaussian states with the covariance matrix
$$\mathbb{I}=\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

4 Theorem.

Any pure state $\psi:\{-s,-s+1,...,s-1,s\}\to\mathbb{C}$ is a linear superposition of discrete Gaussian states

Proof. See ($\boxed{D55}$)
$$\mathbb{I}=\tfrac{1}{d}\sum_{n,k=-s}^{s}|n,k\rangle\langle n,k|\quad\Rightarrow\quad|\psi\rangle=\tfrac{1}{d}\sum_{n,k=-s}^{s}|n,k\rangle\langle n,k|\psi\rangle.$$

5 Theorem.

Any Gaussian state ρ_σ of a continuous variable system is the limit of a sequence of discrete Gaussian states ϱ_σ
$$\rho_\sigma=\lim_{d\to\infty}\varrho_\sigma.$$

Proof (incomplete ?).
Since $\lim\limits_{q^2+p^2\to\infty}\mathcal{W}_{\rho_\sigma}(q,p)=0$, for d large enough, we have
$$\mathbf{W}_{\rho_\sigma}\Big((n+\alpha\tfrac{d}{2})\sqrt{\tfrac{2\pi}{d}},(k+\beta\tfrac{d}{2})\sqrt{\tfrac{2\pi}{d}}\Big)\approx0\quad\text{for}\quad(\alpha,\beta)\neq(0,0),$$
and consequently
$$\mathcal{W}_{\varrho_\sigma}(n,k)\approx C_d\,\mathbf{W}_{\rho_\sigma}\left(n\sqrt{\tfrac{2\pi}{d}},k\sqrt{\tfrac{2\pi}{d}}\right).$$
But, the set
$$\left\{\left(n\sqrt{\tfrac{2\pi}{d}},k\sqrt{\tfrac{2\pi}{d}}\right)\;\middle|\;\begin{array}{l}d\in\{3,5,7,...\}\\ n,k\in\{-j,-j+1,...,j-1,j\}\end{array}\right\}$$
is dense in the phase space.

6 **Theorem.**

> *For any covariance matrix σ,*
> $\lim\limits_{d\to\infty} \mathrm{tr}\, \varrho_\sigma^2 = \frac{1}{\sqrt{\det\sigma}}$.

Proof (incomplete ?).
For d large enough, we have
$\mathcal{W}_{\varrho_\sigma}(n,k) \approx C_d\, \mathbf{W}_{\rho_\sigma}\left(n\sqrt{\frac{2\pi}{d}}, k\sqrt{\frac{2\pi}{d}}\right)$,
and consequently

$$\mathrm{tr}\, \varrho_\sigma^2 = d \sum_{n,k=-s}^{s} \mathcal{W}^2_{\varrho_\sigma}(n,k) \approx d\, \frac{\sum\limits_{n,k=-s}^{s} \mathbf{W}^2_{\rho_\sigma}\left(n\sqrt{\frac{2\pi}{d}}, k\sqrt{\frac{2\pi}{d}}\right)}{\left(\sum\limits_{n,k=-s}^{s} \mathbf{W}_{\rho_\sigma}\left(n\sqrt{\frac{2\pi}{d}}, k\sqrt{\frac{2\pi}{d}}\right)\right)^2}.$$

By considering a division of the rectangle
$\left[-\frac{d}{2}\sqrt{\frac{2\pi}{d}}, \frac{d}{2}\sqrt{\frac{2\pi}{d}}\right] \times \left[-\frac{d}{2}\sqrt{\frac{2\pi}{d}}, \frac{d}{2}\sqrt{\frac{2\pi}{d}}\right]$
into d^2 squares of area $\frac{2\pi}{d}$ and regarding
the integrals as limits of Riemann sums, we get

$$\mathrm{tr}\, \varrho_\sigma^2 \approx 2\pi\, \frac{\sum\limits_{n,k=-s}^{s} \frac{2\pi}{d}\, \mathbf{W}^2_{\rho_\sigma}\left(n\sqrt{\frac{2\pi}{d}}, k\sqrt{\frac{2\pi}{d}}\right)}{\left(\sum\limits_{n,k=-s}^{s} \frac{2\pi}{d}\, \mathbf{W}_{\rho_\sigma}\left(n\sqrt{\frac{2\pi}{d}}, k\sqrt{\frac{2\pi}{d}}\right)\right)^2}$$

$$\xrightarrow{d\to\infty}\ 2\pi\, \frac{\int_{\mathbb{R}^2} \mathbf{W}^2_{\rho_\sigma}(q,p)\, dq\, dp}{\left(\int_{\mathbb{R}^2} \mathbf{W}_{\rho_\sigma}(q,p)\, dq\, dp\right)^2} = \frac{1}{\sqrt{\det\sigma}}.$$

7 **Theorem.**

> *If* $\det\sigma = 1$, *then the discrete Gaussian state* ϱ_σ *is a pure state:*
> $\det\sigma = 1 \quad\Longrightarrow\quad \mathrm{tr}\, \varrho_\sigma^2 = 1.$

Proof (incomplete ?).

If $\det\sigma = 1$, then there exist $\kappa \in (0,\infty)$
and a canonical transformation (rotation)

$$\begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$$

of the phase space such that

$$\sigma = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{1}{\kappa} & 0 \\ 0 & \kappa \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Consequently

$$\mathbf{W}_{\rho_\sigma}(q,p) = \frac{1}{\pi} \mathrm{e}^{-\left(\kappa q'^2 + \frac{1}{\kappa} p'^2\right)} = \mathbf{W}_{\rho_{\sigma_\kappa}}(q',p')$$

but, we know that the discrete Wigner function $\mathcal{W}_{\varrho_{\sigma_\kappa}}$ corresponding to $\mathbf{W}_{\rho_{\sigma_\kappa}}$ represents a pure state.

8 **Definition.**

We call **discrete thermal state** any discrete Gaussian state with a covariance matrix of the form

$$\sigma = \mathbb{I}_\nu = \begin{pmatrix} \nu & 0 \\ 0 & \nu \end{pmatrix} \qquad \text{with } \nu > 1.$$

The spectrum of $\varrho_{\mathbb{I}_\nu}$ is approximately described by
$N_n = \frac{1}{\sum\limits_{k=0}^{d-1} \left(\frac{\nu-1}{\nu+1}\right)^k} \left(\frac{\nu-1}{\nu+1}\right)^n$, where $n \in \{0,1,2,3,...,d-1\}$.
For example, in the case $\nu=2$, $d=7$, the spectrum is
0.6667, 0.2219, 0.0751, 0.0229, 0.0105, 0.0017, 0.0010,
and the numbers N_0, N_1, ..., N_6 are
0.6669, 0.2223, 0.0741, 0.0247, 0.0082, 0.0027, 0.0009.

Table 1: Variation of the purity in four particular cases.

σ	$\frac{1}{\sqrt{\det\sigma}}$	d	$\mathrm{tr}\, \varrho_\sigma^2$
$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	0.5	3	0.52865
		5	0.50099
		7	0.50003
		9	0.50000
$\begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$	0.7071	3	0.6632
		5	0.7009
		7	0.7064
		9	0.7070
$\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 6 \end{pmatrix}$	0.5773	3	0.7020
		5	0.5948
		7	0.5795
		9	0.5776
$\begin{pmatrix} 7 & -\pi \\ -\pi & 5 \end{pmatrix}$	0.19948	3	0.3389
		5	0.2332
		7	0.2080
		9	0.2017

Table 2: The spectrum of the discrete Gaussian state ϱ_σ in the case $\det\sigma = 1$.

σ	d	Spectrum of the density operator ϱ_σ
$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$	3	$1.,\ -7.26023*10^{-17},\ 4.48467*10^{-17}$
	5	$1.,\ 1.38888*10^{-16},\ 8.37904*10^{-17},\ -5.87708*10^{-17},\ -6.19389*10^{-18}$
	7	$1.,\ 1.58*10^{-16}, -7.84*10^{-17}, -6.73*10^{-17},\ 5.24*10^{-17},\ 3.19*10^{-17},\ -3.10*10^{-17}$
$\begin{pmatrix} 1 & 3 \\ 3 & 10 \end{pmatrix}$	3	$1.,\ 4.47648*10^{-17},\ 1.68388*10^{-17}$
	5	$1.,\ -9.69875*10^{-17},\ 8.67892*10^{-17},\ -3.28385*10^{-17},\ 1.97939*10^{-17}$
	7	$1.,\ 3.21*10^{-16},\ 9.10*10^{-17}, -8.99*10^{-17},\ 4.99*10^{-17}, -3.57*10^{-17},\ 2.98*10^{-17}$
$\begin{pmatrix} 3 & \sqrt{5} \\ \sqrt{5} & 2 \end{pmatrix}$	3	$1.,\ 2.17738*10^{-16},\ 3.20619*10^{-17}$
	5	$1.,\ 2.70799*10^{-16},\ 1.12761*10^{-16},\ -9.79558*10^{-17},\ -2.03061*10^{-17}$
	7	$1.,\ 2.37*10^{-16}, -1.31*10^{-16},\ 1.17*10^{-16},\ -8.50*10^{-17}, -2.36*10^{-17}, -1.01*10^{-18}$
$\begin{pmatrix} 3 & -\sqrt{5} \\ -\sqrt{5} & 2 \end{pmatrix}$	3	$1.,\ 2.28273*10^{-16},\ 7.70378*10^{-17}$
	5	$1.,\ 2.85697*10^{-16},\ -8.77195*10^{-17},\ 8.2357*10^{-17},\ -4.52847*10^{-17}$
	7	$1.,\ 2.63*10^{-16}, -1.46*10^{-16},\ 1.30*10^{-16},\ -4.82*10^{-17}, -1.63*10^{-17}, -4.67*10^{-19}$

$\boxed{9}$ In the case $d\!=\!61$, the eigenvalues of $[Q,P]-\mathrm{i}\frac{d}{2\pi}$ are

$$\begin{array}{rrrrr}
-4.05333*10^{-15}\mathrm{i}, & -3.36139*10^{-14}\mathrm{i}, & -6.82471*10^{-14}\mathrm{i}, & -2.30564*10^{-11}\mathrm{i}, & -0.00513798\,\mathrm{i}\\
9.31866*10^{-15}\mathrm{i}, & 3.14251*10^{-14}\mathrm{i}, & 6.89526*10^{-14}\mathrm{i}, & 1.41605*10^{-10}\mathrm{i}, & 0.0198162\,\mathrm{i}\\
7.48845*10^{-15}\mathrm{i}, & 3.68406*10^{-14}\mathrm{i}, & 7.0783*10^{-14}\mathrm{i}, & -8.34352*10^{-10}\mathrm{i}, & -0.0733599\,\mathrm{i}\\
-1.17774*10^{-15}\mathrm{i}, & -3.41122*10^{-14}\mathrm{i}, & -7.25619*10^{-14}\mathrm{i}, & 4.73159*10^{-9}\mathrm{i}, & 0.260404\,\mathrm{i}\\
-3.78568*10^{-15}\mathrm{i}, & -3.71574*10^{-14}\mathrm{i}, & 8.03083*10^{-14}\mathrm{i}, & -2.58237*10^{-8}\mathrm{i}, & -0.88457\,\mathrm{i}\\
1.14028*10^{-14}\mathrm{i}, & 4.1543*10^{-14}\mathrm{i}, & -8.1415*10^{-14}\mathrm{i}, & 1.35662*10^{-7}\mathrm{i}, & 2.87316\,\mathrm{i}\\
-1.69386*10^{-14}\mathrm{i}, & -4.48723*10^{-14}\mathrm{i}, & 8.70762*10^{-14}\mathrm{i}, & -6.86077*10^{-7}\mathrm{i}, & -8.86191\,\mathrm{i}\\
-2.31158*10^{-14}\mathrm{i}, & 4.08535*10^{-14}\mathrm{i}, & -8.9346*10^{-14}\mathrm{i}, & 3.34023*10^{-6}\mathrm{i}, & 26.1438\,\mathrm{i}\\
2.23703*10^{-14}\mathrm{i}, & 4.78884*10^{-14}\mathrm{i}, & -9.94767*10^{-14}\mathrm{i}, & -0.0000156547\,\mathrm{i}, & -70.9696\,\mathrm{i}\\
2.76156*10^{-14}\mathrm{i}, & -4.97772*10^{-14}\mathrm{i}, & 1.13415*10^{-13}\mathrm{i}, & 0.0000706183\,\mathrm{i}, & 189.607\,\mathrm{i}\\
2.73091*10^{-14}\mathrm{i}, & -4.63059*10^{-14}\mathrm{i}, & -5.72182*10^{-13}\mathrm{i}, & -0.000306537\,\mathrm{i}, & -404.509\,\mathrm{i}\\
-3.32093*10^{-14}\mathrm{i}, & 6.80431*10^{-14}\mathrm{i}, & 3.61784*10^{-12}\mathrm{i}, & 0.00127992\,\mathrm{i}, & 944.717\,\mathrm{i}\\
 & & & & -1270.53\,\mathrm{i}
\end{array}$$

There exists a subspace, for example, of dimension 44 where $[Q,P]\approx\mathrm{i}\frac{d}{2\pi}$.

$\boxed{10}$ In the case $d\!=\!11$, the eigenvalues of the operator $[Q,P]-\mathrm{i}\frac{d}{2\pi}$, considered in the increasing order of their modulus are

$$\begin{array}{l}
\lambda_1=-1.4*10^{-8}\mathrm{i},\\
\lambda_2=7.9*10^{-7}\mathrm{i},\\
\lambda_3=2*10^{-5}\mathrm{i},\\
\lambda_4=3.3*10^{-4}\mathrm{i},\\
\lambda_5=-3.9*10^{-3}\mathrm{i},\\
\lambda_6=3.4*10^{-2}\mathrm{i},\\
\lambda_7=-0.24\,\mathrm{i},\\
\lambda_8=1.33\,\mathrm{i},\\
\lambda_9=-5.45\,\mathrm{i},\\
\lambda_{10}=19.99\,\mathrm{i},\\
\lambda_{11}=-34.92\,\mathrm{i}.
\end{array}$$

In the eigenbasis $\{e_n\}$ formed by the corresponding eigenvectors,
the matrices $(|\langle e_n|\mathbf{g}_1\rangle|)$ and $(|\langle e_n\,|\varrho_\sigma|\,e_m\rangle|)$ for $\sigma=\begin{pmatrix}2&1\\1&1\end{pmatrix}$ are

$$\begin{pmatrix}0.9999\\1*10^{-10}\\2*10^{-11}\\2*10^{-13}\\0.0079\\4*10^{-15}\\1*10^{-15}\\2*10^{-16}\\0.0004\\6*10^{-18}\\2*10^{-17}\end{pmatrix},
\begin{pmatrix}
0.8954 & 6*10^{-11} & 0.2837 & 1*10^{-12} & 0.1059 & 1*10^{-14} & 0.0393 & 3*10^{-16} & 0.0164 & 3*10^{-17} & 0.0084\\
1*10^{-10} & 5*10^{-18} & 5*10^{-11} & 1*10^{-17} & 1*10^{-11} & 8*10^{-18} & 7*10^{-12} & 1*10^{-17} & 3*10^{-12} & 1*10^{-17} & 1*10^{-12}\\
0.2837 & 2*10^{-11} & 0.0899 & 3*10^{-13} & 0.0335 & 4*10^{-15} & 0.0124 & 1*10^{-16} & 0.0052 & 1*10^{-}17 & 0.0026\\
2*10^{-13} & 1*10^{-17} & 7*10^{-14} & 3*10^{-18} & 2*10^{-14} & 3*10^{-17} & 1*10^{-14} & 4*10^{-17} & 4*10^{-15} & 1*10^{-17} & 2*10^{-15}\\
0.1059 & 7*10^{-12} & 0.0335 & 1*10^{-13} & 0.0125 & 1*10^{-15} & 0.0046 & 4*10^{-17} & 0.0019 & 3*10^{-18} & 0.0010\\
1*10^{-15} & 1*10^{-17} & 5*10^{-16} & 2*10^{-17} & 2*10^{-16} & 1*10^{-17} & 8*10^{-17} & 1*10^{-17} & 3*10^{-17} & 1*10^{-17} & 7*10^{-18}\\
0.0393 & 2*10^{-12} & 0.0124 & 5*10^{-14} & 0.0046 & 5*10^{-16} & 0.0017 & 2*10^{-17} & 0.0007 & 6*10^{-18} & 0.0003\\
2*10^{-16} & 1*10^{-17} & 8*10^{-17} & 4*10^{-17} & 3*10^{-17} & 7*10^{-18} & 2*10^{-17} & 4*10^{-18} & 8*10^{-18} & 3*10^{-17} & 5*10^{-18}\\
0.0164 & 1*10^{-12} & 0.0052 & 2*10^{-14} & 0.0019 & 2*10^{-16} & 0.0007 & 1*10^{-17} & 0.0003 & 8*10^{-19} & 0.0001\\
2*10^{-17} & 1*10^{-17} & 8*10^{-18} & 1*10^{-17} & 2*10^{-18} & 1*10^{-17} & 5*10^{-18} & 3*10^{-17} & 1*10^{-18} & 4*10^{-17} & 5*10^{-18}\\
0.0084 & 6*10^{-13} & 0.0026 & 1*10^{-14} & 0.0010 & 1*10^{-16} & 0.00037 & 2*10^{-18} & 0.0001 & 5*10^{-18} & 0.0000
\end{pmatrix}.$$

The most significant elements of $(\langle e_n|\mathbf{g}_1\rangle)$ and $(\langle e_n\,|\varrho_\sigma|\,e_m\rangle)$ are those corresponding to small $|\lambda_n|$ and $|\lambda_m|$.

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In the general case, let:

- $\lambda_1,\ \lambda_2,\ \dots,\lambda_d$ be the eigenvalues of $[Q,P]-\mathrm{i}\frac{d}{2\pi}$, considered in the increasing order of their modulus;
- $|e_1\rangle,\ |e_2\rangle,\ \dots|e_d\rangle$ be the corresponding eigenvectors;
- $d'<d$ be such that

$$\boxed{[Q,P]\approx\mathrm{i}\tfrac{d}{2\pi}}\ \text{ in the subspace }\ \boxed{\mathcal{H}_{d'}=\mathrm{span}(\{|e_1\rangle,\ |e_2\rangle,\ \dots|e_{d'}\rangle\})=\{\ \psi\in\mathcal{H}\ \mid\ \langle e_n|\psi\rangle=0\ \text{ for } n>d'\ \}};$$

We approximate any linear operator $A:\mathcal{H}\to\mathcal{H}$ by the linear operator $A':\mathcal{H}\to\mathcal{H}$ with the matrix elements

$$\langle e_n|A'|e_m\rangle=\begin{cases}\langle e_n|A|e_m\rangle & \text{for } 1\le n,m\le d'\\ 0 & \text{in other cases.}\end{cases}$$

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For $H:\mathcal{H}\to\mathcal{H}$ of the form

$$\begin{aligned} H &= \alpha\,\mathfrak{a}^\dagger\mathfrak{a}+\beta\,(\mathfrak{a}^\dagger)^2+\bar\beta\,\mathfrak{a}^2+\alpha\,\mathfrak{a}\,\mathfrak{a}^\dagger \\ &= (\mathfrak{a}^\dagger \quad \mathfrak{a})\begin{pmatrix}\alpha & \beta\\ \bar\beta & \alpha\end{pmatrix}\begin{pmatrix}\mathfrak{a}\\ \mathfrak{a}^\dagger\end{pmatrix} \quad \text{with} \quad \begin{array}{l}\alpha\in\mathbb{R}\\ \beta\in\mathbb{C}\end{array}\end{aligned}$$

the analogy with the case of continuous variable and the relation $\lim\limits_{d\to\infty}\varrho_\sigma=\rho_\sigma$ suggest that

$$\boxed{\mathrm{e}^{-\frac{\mathrm{i}}{2}H}\varrho_\sigma\mathrm{e}^{\frac{\mathrm{i}}{2}H}\approx\varrho_{S\sigma S^T}} \qquad \text{for } d \text{ large enough.}$$

In certain cases (see Table 3), a good approximation is obtained for relatively small values of d.

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Table 3: Norm of $\mathrm{e}^{-\frac{\mathrm{i}}{2}H}\varrho_\sigma\mathrm{e}^{\frac{\mathrm{i}}{2}H}-\varrho_{S\sigma S^T}$ in the case of a phase shift and the case of a single-mode squeezing transformation.

α	β	σ	d	$\left\Vert\mathrm{e}^{-\frac{\mathrm{i}}{2}H}\varrho_\sigma\mathrm{e}^{\frac{\mathrm{i}}{2}H}-\varrho_{S\sigma S^T}\right\Vert$
$\frac{\pi}{4}$	0	$\begin{pmatrix}2&1\\1&1\end{pmatrix}$	5	0.1701
			7	0.0932
			9	0.0489
			11	0.0256
			13	0.0135
			15	0.0071
0	$\frac{1}{2}\,\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$	$\begin{pmatrix}3&2\\2&2\end{pmatrix}$	5	0.1605
			7	0.1217
			9	0.1057
			11	0.0887
			13	0.0729
			15	0.0593

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In the continuous case, the relation $[\hat q,\hat p]=\mathrm{i}\hbar$ is satisfied in a subspace dense in $L^2(\mathbb{R})$, but the corresponding relation $[Q,P]=\mathrm{i}\frac{d}{2\pi}$ is only approximately satisfied in a subspace $\mathcal{H}_{d'}$ of $\mathcal{H}$, and only for d large enough.

The discrete annihilation and creation operators satisfy

$$\boxed{[\mathfrak{a},\mathfrak{a}^\dagger]\approx 1} \qquad \text{in} \quad \mathcal{H}_{d'}.$$

In the usual way one can obtain a discrete counterpart of the Bogolubov transformations approximately satisfied in $\mathcal{H}_{d'}$.

For $H:\mathcal{H}\to\mathcal{H}$ of the form

$$H=(\mathfrak{a}^\dagger \quad \mathfrak{a})\begin{pmatrix}\alpha & \beta\\ \bar\beta & \alpha\end{pmatrix}\begin{pmatrix}\mathfrak{a}\\ \mathfrak{a}^\dagger\end{pmatrix} \quad \text{with} \quad \begin{array}{l}\alpha\in\mathbb{R}\\ \beta\in\mathbb{C}\end{array}$$

one gets the relations

$$\begin{aligned}\left[-\tfrac{\mathrm{i}}{2}H,\mathfrak{a}\right]&\approx\mathrm{i}\left(\alpha\,\mathfrak{a}+\beta\,\mathfrak{a}^\dagger\right)\\ \left[-\tfrac{\mathrm{i}}{2}H,\mathfrak{a}^\dagger\right]&\approx\mathrm{i}\left(-\bar\beta\,\mathfrak{a}-\alpha\,\mathfrak{a}^\dagger\right)\end{aligned}$$

which can be written together as

$$\left[-\frac{\mathrm{i}}{2}H,\begin{pmatrix}\mathfrak{a}\\ \mathfrak{a}^\dagger\end{pmatrix}\right]\approx\mathrm{i}\begin{pmatrix}\alpha & \beta\\ -\bar\beta & -\alpha\end{pmatrix}\begin{pmatrix}\mathfrak{a}\\ \mathfrak{a}^\dagger\end{pmatrix} \qquad \text{in} \quad \mathcal{H}_{d'}.$$

In $\mathcal{H}_{d'}$, by iteration, we obtain

$$\left[-\frac{\mathrm{i}}{2}H,\left[-\frac{\mathrm{i}}{2}H,\begin{pmatrix}\mathfrak{a}\\ \mathfrak{a}^\dagger\end{pmatrix}\right]\right]\approx\mathrm{i}^2\begin{pmatrix}\alpha & \beta\\ -\bar\beta & -\alpha\end{pmatrix}^2\begin{pmatrix}\mathfrak{a}\\ \mathfrak{a}^\dagger\end{pmatrix},$$

$$\left[-\frac{\mathrm{i}}{2}H,\left[-\frac{\mathrm{i}}{2}H,\left[-\frac{\mathrm{i}}{2}H,\begin{pmatrix}\mathfrak{a}\\ \mathfrak{a}^\dagger\end{pmatrix}\right]\right]\right]\approx\mathrm{i}^3\begin{pmatrix}\alpha & \beta\\ -\bar\beta & -\alpha\end{pmatrix}^3\begin{pmatrix}\mathfrak{a}\\ \mathfrak{a}^\dagger\end{pmatrix}, \text{ etc..}$$

From the relation

$$\mathrm{e}^XY\mathrm{e}^{-X}=Y+\frac{1}{1!}[X,Y]+\frac{1}{2!}[X,[X,Y]]+\frac{1}{3!}[X,[X,[X,Y]]]+\cdots$$

it follows that

$$\mathrm{e}^{-\frac{\mathrm{i}}{2}H}\begin{pmatrix}\mathfrak{a}\\ \mathfrak{a}^\dagger\end{pmatrix}\mathrm{e}^{\frac{\mathrm{i}}{2}H}\approx\mathrm{e}^{\mathrm{i}\begin{pmatrix}\alpha & \beta\\ -\bar\beta & -\alpha\end{pmatrix}}\begin{pmatrix}\mathfrak{a}\\ \mathfrak{a}^\dagger\end{pmatrix} \qquad \text{in} \quad \mathcal{H}_{d'}.$$

Since

$$\begin{pmatrix}\mathfrak{a}\\ \mathfrak{a}^\dagger\end{pmatrix}=\sqrt{\frac{\pi}{d}}\begin{pmatrix}1 & \mathrm{i}\\ 1 & -\mathrm{i}\end{pmatrix}\begin{pmatrix}Q\\ P\end{pmatrix},$$

the previous relation can be written as

$$\mathrm{e}^{-\frac{\mathrm{i}}{2}H}\begin{pmatrix}Q\\ P\end{pmatrix}\mathrm{e}^{\frac{\mathrm{i}}{2}H}\approx S^{-1}\begin{pmatrix}Q\\ P\end{pmatrix} \qquad \text{in} \quad \mathcal{H}_{d'},$$

where

$$\begin{aligned}S&=\tfrac{1}{2}\begin{pmatrix}1 & 1\\ -\mathrm{i} & \mathrm{i}\end{pmatrix}\mathrm{e}^{-\mathrm{i}\begin{pmatrix}\alpha & \beta\\ -\bar\beta & -\alpha\end{pmatrix}}\begin{pmatrix}1 & \mathrm{i}\\ 1 & -\mathrm{i}\end{pmatrix}\\ &=\mathrm{e}^{-\frac{1}{2}\begin{pmatrix}(\beta-\bar\beta)\mathrm{i} & -2\alpha+\beta+\bar\beta\\ 2\alpha+\beta+\bar\beta & -(\beta-\bar\beta)\mathrm{i}\end{pmatrix}}\end{aligned}$$

This matrix with real elements and $\det S=1$ belongs to the symplectic group $\mathrm{Sp}(2,\mathbb{R})$.

For example, the phase shift $U=\mathrm{e}^{\mathrm{i}\varphi\,\hat a^\dagger\hat a}$ corresponds to

$$S_\varphi=\begin{pmatrix}\cos\frac{\varphi}{2} & \sin\frac{\varphi}{2}\\ -\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2}\end{pmatrix}$$

and the single-mode squeezing $U=\mathrm{e}^{\frac{s}{2}\left(\mathrm{e}^{\mathrm{i}\theta}(\hat a^\dagger)^2-\mathrm{e}^{-\mathrm{i}\theta}\,\hat a^2\right)}$ to

$$S_{s,\theta}=\begin{pmatrix}\cosh s+\cos\theta\,\sinh s & \sin\theta\,\sinh s\\ \sin\theta\,\sinh s & \cosh s-\cos\theta\,\sinh s\end{pmatrix}.$$

This may explain why the discrete counterpart of the Gaussian transform maps approximately discrete Gaussian states into discrete Gaussian states.

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Theorem (Discrete thermal states).

If $\nu>1$ and $\mu>1$, then
- $\varrho_{\mathbb{I}_\nu}\ \varrho_{\mathbb{I}_\mu}=\varrho_{\mathbb{I}_\mu}\ \varrho_{\mathbb{I}_\nu}$ *for* $d\!=\!3$, *and*
- $\varrho_{\mathbb{I}_\nu}\ \varrho_{\mathbb{I}_\mu}\approx\varrho_{\mathbb{I}_\mu}\ \varrho_{\mathbb{I}_\nu}$ *for d large enough.*

Remark.
For d large enough, the density operators of all the discrete thermal states have almost the same eigenfunctions. They can be regarded as a discrete version of the Hermite-Gauss functions.

Proof.
In the case of the parity operators, we get

$$\Pi(n,k)\,\Pi(n',k')=\mathrm{e}^{\frac{4\pi\mathrm{i}}{d}(nk'-kn')}\ \Pi(n\!-\!n',k\!-\!k')\,\Pi(0,0).$$

We have

$$\begin{aligned}\varrho_{\mathbb{I}_\nu}\ \varrho_{\mathbb{I}_\mu}&=\sum_{n,k=-s}^{s}\mathcal{W}_{\varrho_{\mathbb{I}_\nu}}(n,k)\Pi(n,k)\sum_{n',k'=-s}^{s}\mathcal{W}_{\varrho_{\mathbb{I}_\mu}}(n',k')\Pi(n',k')\\&=\sum_{n,k=-s}^{s}\sum_{n',k'=-s}^{s}\mathrm{e}^{\frac{4\pi\mathrm{i}}{d}(nk'-kn')}\ \mathcal{W}_{\varrho_{\mathbb{I}_\nu}}(n,k)\ \mathcal{W}_{\varrho_{\mathbb{I}_\mu}}(n',k')\,\Pi(n\!-\!n',k\!-\!k')\,\Pi(0,0)\\&=\sum_{m,\ell=-s}^{s}\left(\sum_{n,k=-s}^{s}\mathrm{e}^{\frac{4\pi\mathrm{i}}{d}(km-n\ell)}\ \mathcal{W}_{\varrho_{\mathbb{I}_\nu}}(n,k)\ \mathcal{W}_{\varrho_{\mathbb{I}_\mu}}(n\!-\!m,k\!-\!\ell)\right)\Pi(m,\ell)\,\Pi(0,0)\end{aligned}$$

and

$$\begin{aligned}\varrho_{\mathbb{I}_\mu}\ \varrho_{\mathbb{I}_\nu}&=\sum_{n,k=-s}^{s}\mathcal{W}_{\varrho_{\mathbb{I}_\mu}}(n,k)\,\Pi(n,k)\sum_{n',k'=-s}^{s}\mathcal{W}_{\varrho_{\mathbb{I}_\nu}}(n',k')\,\Pi(n',k')\\&=\sum_{n,k=-s}^{s}\sum_{n',k'=-s}^{s}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}(nk'-kn')}\ \mathcal{W}_{\varrho_{\mathbb{I}_\nu}}(n,k)\ \mathcal{W}_{\varrho_{\mathbb{I}_\mu}}(n',k')\ \Pi(n'\!-\!n,k'\!-\!k)\,\Pi(0,0)\\&=\sum_{m,\ell=-s}^{s}\left(\sum_{n,k=-s}^{s}\mathrm{e}^{\frac{4\pi\mathrm{i}}{d}(km-n\ell)}\ \mathcal{W}_{\varrho_{\mathbb{I}_\nu}}(n,k)\ \mathcal{W}_{\varrho_{\mathbb{I}_\mu}}(n\!+\!m,k\!+\!\ell)\right)\Pi(m,\ell)\,\Pi(0,0).\end{aligned}$$

By using $n\mapsto -n$, $\ k\mapsto -k$, the identity $\mathcal{W}_\varrho(-n,-k)=\mathcal{W}_\varrho(n,k)$ and the notation

$$f(m,\ell)=\sum_{n,k=-s}^{s}\mathrm{e}^{\frac{4\pi\mathrm{i}}{d}(km-n\ell)}\ \mathcal{W}_{\varrho_{\mathbb{I}_\nu}}(n,k)\ \mathcal{W}_{\varrho_{\mathbb{I}_\mu}}(n\!+\!m,k\!+\!\ell)$$

these two relations can be written as

$$\begin{aligned}\varrho_{\mathbb{I}_\nu}\ \varrho_{\mathbb{I}_\mu}&=\sum_{m,\ell=-s}^{s}f(\ell,m)\ \Pi(m,\ell)\,\Pi(0,0),\\\varrho_{\mathbb{I}_\mu}\ \varrho_{\mathbb{I}_\nu}&=\sum_{m,\ell=-s}^{s}f(m,\ell)\ \Pi(m,\ell)\,\Pi(0,0).\end{aligned}$$

The functions $f(m,\ell)$ satisfy the relations

$$\begin{aligned}&f(m,\ell)=f(m\!+\!d,\ell)=f(m,\ell\!+\!d)\\&f(m,\ell)=f(-\ell,m)=f(\ell,-m)=f(-m,-\ell)\end{aligned}\qquad\text{for any } m,\ell\in\mathbb{Z},$$

In the case $d=3$, we have $f(m,\ell)=f(\ell,m)$ for any $m,\ell\in\{-1,0,1\}$, and consequently $\varrho_{\mathbb{I}_\nu}\ \varrho_{\mathbb{I}_\mu}=\varrho_{\mathbb{I}_\mu}\ \varrho_{\mathbb{I}_\nu}$.
In the case $d>3$, we can have $f(m,\ell)\neq f(\ell,m)$, but the numerical data suggest that

$$\lim_{d\to\infty}\big(\ f(m,\ell)-f(\ell,m)\ \big)=0.$$

In the investigated cases, the convergence is very fast.

Table 4. Variation of the difference $f(m,\ell)-f(\ell,m)$ in some particular cases.

ν	μ	m	ℓ	d	$f(m,\ell)-f(\ell,m)$
2	3	1	2	5	$-0.00041567\,\mathrm{i}$
				7	$-0.0000111636\,\mathrm{i}$
				9	$6.67928*10^{-7}\,\mathrm{i}$
				11	$1.71064*10^{-7}\,\mathrm{i}$
2	5	1	2	5	$-0.000846551\,\mathrm{i}$
				7	$-3.34924*10^{-7}\,\mathrm{i}$
				9	$1.3713*10^{-6}\,\mathrm{i}$
				11	$6.06216*10^{-7}\,\mathrm{i}$
2	3	1	3	7	$-0.0000383665\,\mathrm{i}$
				9	$-2.29785*10^{-6}\,\mathrm{i}$
				11	$-6.79155*10^{-8}\,\mathrm{i}$
2	5	1	3	7	$-0.000173011\,\mathrm{i}$
				9	$-9.87889*10^{-7}\,\mathrm{i}$
				11	$-1.34018*10^{-7}\,\mathrm{i}$

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For d large enough, the discrete vacuum

$$|0\rangle = \frac{1}{\sqrt{\langle g_1|g_1\rangle}}|g_1\rangle \qquad (\textit{see}\ \boxed{D15})$$

approximately satisfies the relation

$$\mathfrak{a}|0\rangle = 0,$$

where $\mathfrak{a}$ is the discrete annihilation operator

$$\mathfrak{a} = \sqrt{\tfrac{\pi}{d}}(Q + \mathrm{i}P).$$

- In the case $d=3$, we have

$$\mathfrak{a}|0\rangle = \begin{pmatrix} 5.55112 * 10^{-17} \\ 0 \\ -5.55112 * 10^{-17} \end{pmatrix}$$

- In the case $d=5$, we have

$$\mathfrak{a}|0\rangle = \begin{pmatrix} -0.0203313 \\ 0.0057757 \\ -2.77556 * 10^{-17} \\ -0.0057757 \\ 0.0203313 \end{pmatrix}$$

- In the case $d=11$, we have

$$\mathfrak{a}|0\rangle = \begin{pmatrix} -0.0000547532 - 1.85212 * 10^{-19}\mathrm{i} \\ -0.0000747858 - 1.96854 * 10^{-18}\mathrm{i} \\ 0.0000609102 - 4.42783 * 10^{-18}\mathrm{i} \\ -0.0000447974 - 4.16755 * 10^{-18}\mathrm{i} \\ 0.0000238314 - 1.90795 * 10^{-18}\mathrm{i} \\ -8.99888 * 10^{-18} + 1.20371 * 10^{-35}\mathrm{i} \\ -0.0000238314 + 1.90795 * 10^{-18}\mathrm{i} \\ 0.0000447974 + 4.16755 * 10^{-18}\mathrm{i} \\ -0.0000609102 + 4.42783 * 10^{-18}\mathrm{i} \\ 0.0000747858 + 1.96854 * 10^{-18}\mathrm{i} \\ 0.0000547532 + 1.85212 * 10^{-19}\mathrm{i} \end{pmatrix}$$

- In the case $d=21$, we have

$$\mathfrak{a}|0\rangle = \begin{pmatrix} -8.82766 * 10^{-8} - 5.81819 * 10^{-22}\mathrm{i} \\ 2.37575 * 10^{-8} - 5.95229 * 10^{-21}\mathrm{i} \\ -2.48875 * 10^{-8} - 4.16377 * 10^{-20}\mathrm{i} \\ 2.34472 * 10^{-8} - 2.1582 * 10^{-19}\mathrm{i} \\ -2.15409 * 10^{-8} - 8.29074 * 10^{-19}\mathrm{i} \\ 1.90966 * 10^{-8} - 2.35842 * 10^{-18}\mathrm{i} \\ -1.61127 * 10^{-8} - 4.95373 * 10^{-18}\mathrm{i} \\ 1.26189 * 10^{-8} - 7.60903 * 10^{-18}\mathrm{i} \\ -8.68622 * 10^{-9} - 8.25789 * 10^{-18}\mathrm{i} \\ 4.42942 * 10^{-9} - 5.46052 * 10^{-18}\mathrm{i} \\ -3.37182 * 10^{-17} + 0\,\mathrm{i} \\ -4.42942 * 10^{-9} + 5.46052 * 10^{-18}\mathrm{i} \\ 8.68622 * 10^{-9} + 8.25789 * 10^{-18}\mathrm{i} \\ -1.26189 * 10^{-8} + 7.60903 * 10^{-18}\mathrm{i} \\ 1.61127 * 10^{-8} + 4.95373 * 10^{-18}\mathrm{i} \\ -1.90966 * 10^{-8} + 2.35842 * 10^{-18}\mathrm{i} \\ 2.15409 * 10^{-8} + 8.29074 * 10^{-19}\mathrm{i} \\ -2.34472 * 10^{-8} + 2.1582 * 10^{-19}\mathrm{i} \\ 2.48875 * 10^{-8} + 4.16377 * 10^{-20}\mathrm{i} \\ -2.37575 * 10^{-8} + 5.95229 * 10^{-21}\mathrm{i} \\ 8.82766 * 10^{-8} + 5.81819 * 10^{-22}\mathrm{i} \end{pmatrix}$$

- In the case $d=31$, we have

$$\mathfrak{a}|0\rangle = \begin{pmatrix} -1.75698 * 10^{-11} - 3.13441 * 10^{-18}\mathrm{i} \\ -1.01971 * 10^{-11} - 7.66892 * 10^{-18}\mathrm{i} \\ 9.0788 * 10^{-12} - 8.52369 * 10^{-18}\mathrm{i} \\ -8.89096 * 10^{-12} - 6.87123 * 10^{-18}\mathrm{i} \\ 8.59664 * 10^{-12} - 5.71414 * 10^{-18}\mathrm{i} \\ -8.22226 * 10^{-12} - 7.17248 * 10^{-18}\mathrm{i} \\ 7.76246 * 10^{-12} - 1.09477 * 10^{-17}\mathrm{i} \\ -7.21462 * 10^{-12} - 1.48722 * 10^{-17}\mathrm{i} \\ 6.57647 * 10^{-12} - 1.64631 * 10^{-17}\mathrm{i} \\ -5.84835 * 10^{-12} - 1.43307 * 10^{-17}\mathrm{i} \\ 5.03331 * 10^{-12} - 8.98847 * 10^{-18}\mathrm{i} \\ -4.13745 * 10^{-12} - 2.6566 * 10^{-18}\mathrm{i} \\ 3.17109 * 10^{-12} + 2.07228 * 10^{-18}\mathrm{i} \\ -2.14752 * 10^{-12} + 3.77019 * 10^{-18}\mathrm{i} \\ 1.08399 * 10^{-12} + 2.66099 * 10^{-18}\mathrm{i} \\ -6.64054 * 10^{-17} + 1.89107 * 10^{-34}\mathrm{i} \\ -1.08405 * 10^{-12} - 2.66099 * 10^{-18}\mathrm{i} \\ 2.14749 * 10^{-12} - 3.77019 * 10^{-18}\mathrm{i} \\ -3.17113 * 10^{-12} - 2.07228 * 10^{-18}\mathrm{i} \\ 4.13744 * 10^{-12} + 2.6566 * 10^{-18}\mathrm{i} \\ -5.03331 * 10^{-12} + 8.98847 * 10^{-18}\mathrm{i} \\ 5.84835 * 10^{-12} + 1.43307 * 10^{-17}\mathrm{i} \\ -6.57648 * 10^{-12} + 1.64631 * 10^{-17}\mathrm{i} \\ 7.21463 * 10^{-12} + 1.48722 * 10^{-17}\mathrm{i} \\ -7.76246 * 10^{-12} + 1.09477 * 10^{-17}\mathrm{i} \\ 8.22226 * 10^{-12} + 7.17248 * 10^{-18}\mathrm{i} \\ -8.59664 * 10^{-12} + 5.71414 * 10^{-18}\mathrm{i} \\ 8.89095 * 10^{-12} + 6.87123 * 10^{-18}\mathrm{i} \\ -9.07881 * 10^{-12} + 8.52369 * 10^{-18}\mathrm{i} \\ 1.01971 * 10^{-11} + 7.66892 * 10^{-18}\mathrm{i} \\ 1.75698 * 10^{-11} + 3.13441 * 10^{-18}\mathrm{i} \end{pmatrix}$$

$\boxed{\textbf{C}}$ TWO-MODE GAUSSIAN STATES OF DISCRETE VARIABLE QUANTUM SYSTEMS

- in preparation -

D SUPPLEMENTARY INFORMATION: Some definitions, notations and results

■ **Definition**. Alternative representations of the Hilbert space $\mathcal{H}$ of dimension d:

- $\mathcal{H}\equiv\mathbb{C}^d=\{\ x=(x_0,x_1,...,x_{d-1})\ |\ x_k\in\mathbb{C}\ \}$
- $\mathcal{H}\equiv\{\psi:\{0,1,...,d-1\}\to\mathbb{C}\ |\ \psi \text{ is a function}\}$
- $\mathcal{H}\equiv\{\ \psi:\mathbb{Z}_d\to\mathbb{C}\ |\ \psi \text{ is a function}\ \}$
- $\mathcal{H}\equiv\{\psi:\{-s,-s+1,...,s\}\to\mathbb{C}\ |\ \psi \text{ is a function}\}$

1 $$\langle\varphi|\psi\rangle=\sum_{n=-s}^{s}\overline{\varphi(n)}\,\psi(n)\equiv\sum_{n\in\mathbb{Z}_d}\overline{\varphi(n)}\,\psi(n).$$

2 The *canonical orthonormal basis* is formed by $\delta_{-s},\delta_{-s+1},...,\delta_s:\{-s,-s+1,...,s-1,s\}\to\mathbb{C}$,
$$\delta_m(n)=\delta_{nm}=\begin{cases}1 & \text{for} \quad n=m,\\ 0 & \text{for} \quad n\neq m.\end{cases}$$
By using the notation $|m\rangle=|\delta_m\rangle$, we have
$$\boxed{\langle j|m\rangle=\delta_{jm}\quad,\quad\sum_{m=-s}^{s}|m\rangle\langle m|=\mathbb{I}}.$$

Alternatively, we can regard it as the set $\{\ \delta_m:\mathbb{Z}_d\to\mathbb{C}\ |\ m\in\mathbb{Z}_d\ \}$, where
$$\delta_m(n)=\delta_{nm}^{[d]}=\begin{cases}1 & \text{for} \quad n=m \bmod(d),\\ 0 & \text{for} \quad n\neq m \bmod(d).\end{cases}$$

3 *Finite Fourier transform* $F:\mathcal{H}\to\mathcal{H}$ is
$$\begin{array}{c}\psi:\mathbb{Z}_d\longrightarrow\mathbb{C}\\ \downarrow\\ F[\psi]:\mathbb{Z}_d\longrightarrow\mathbb{C}\end{array}\qquad \boxed{F[\psi](k)=\frac{1}{\sqrt{d}}\sum_{n\in\mathbb{Z}_d}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}kn}\,\psi(n)}$$

4 A *density operator* is a Hermitian operator $\varrho:\mathcal{H}\to\mathcal{H}$ satisfying $\boxed{\varrho\geq0,\ \mathrm{tr}\,\varrho=1.}$

5 *Mean value of an observable* A in a state $\varrho:\mathcal{H}\to\mathcal{H}$ $\boxed{\langle A\rangle_\varrho=\mathrm{tr}(\varrho A)}$

6 *Position* operator $Q:\mathcal{H}\to\mathcal{H}$, $\boxed{Q\psi(n)=\tilde{n}\psi(n)}$

7 *Momentum* operator $P:\mathcal{H}\to\mathcal{H}$, $\boxed{P=F^\dagger QF}$

where $\tilde{n}$ is the representative modulo d of n satisfying $-s\leq\tilde{n}\leq s$.

8 *Discrete annihilation operator* $\mathfrak{a}:\mathcal{H}\to\mathcal{H}$, $\boxed{\mathfrak{a}=\sqrt{\tfrac{\pi}{d}}(Q+\mathrm{i}P)}$

9 *Discrete creation operator* $\mathfrak{a}^\dagger:\mathcal{H}\to\mathcal{H}$, $\boxed{\mathfrak{a}^\dagger=\sqrt{\tfrac{\pi}{d}}(Q-\mathrm{i}P)}$

10 *Translation* operators $(n,k\in\mathbb{Z}_d)$ $\mathcal{H}\to\mathcal{H}$, $$\boxed{\begin{array}{l}\psi\mapsto\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}\psi\\ \psi\mapsto\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}\psi\end{array}}$$

11 *Displacement* operators $(-s\leq n,k\leq s)$ $\boxed{D(n,k)=\mathrm{e}^{-\frac{\pi\mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}}$

12 *Parity operators* $(n,k\in\mathbb{Z}_d)$ $\boxed{\Pi(n,k)=D(n,k)\,\Pi\,D(n,k)^\dagger}$ where $\Pi\psi(n)=\psi(-n)$.

13 *Weyl transform* of a linear operator $A:\mathcal{H}\to\mathcal{H}$ is $A^{\mathrm{w}}:\mathbb{Z}_d\times\mathbb{Z}_d\longrightarrow\mathbb{C}$, $\boxed{A^{\mathrm{w}}(n,k)\overset{\mathrm{def}}{=}\mathrm{tr}(A\,\Pi(n,k))}$

14 *Wigner function* of a linear operator $A:\mathcal{H}\to\mathcal{H}$ is $\mathcal{W}_A:\mathbb{Z}_d\times\mathbb{Z}_d\longrightarrow\mathbb{C}$, $\mathcal{W}_A=\frac{1}{d}A^{\mathrm{w}}$, $\boxed{\mathcal{W}_A(n,k)\overset{\mathrm{def}}{=}\frac{1}{d}\,\mathrm{tr}(A\,\Pi(n,k))}$

15 Gaussian function $g_\kappa:\mathbb{Z}_d\to\mathbb{R}$, $\boxed{g_\kappa(n)=\sum_{m=-\infty}^{\infty}\mathrm{e}^{-\frac{\kappa\pi}{d}(md+n)^2}}$

16 Complementary Gaussian function $g_\kappa^+:\mathbb{Z}_d\to\mathbb{R}$, $\boxed{g_\kappa^+(n)=\sum_{m=-\infty}^{\infty}\mathrm{e}^{-\frac{\kappa\pi}{d}((m+\frac{1}{2})d+n)^2}}$

17 Discrete vacuum $\boxed{|0\rangle=\frac{1}{\sqrt{\langle g_1|g_1\rangle}}|g_1\rangle}$

18 Discrete coherent states $\boxed{|n,k\rangle=D(n,k)|0\rangle}$

■ **Remark.**

19 The definition of $D(n,k)$ is not modulo d invariant:
$$D(n+d,k)=(-1)^k\,D(n,k),$$
$$D(n,k+d)=(-1)^n\,D(n,k),$$
but, the definition of $\Pi(n,k)$ is modulo d invariant.

■ **Lemma.** For odd $d=2s+1$, we have:

20 $$\boxed{\begin{array}{c}2n=0\ (\mathrm{mod}\,d)\\ \Downarrow\\ n=0\ (\mathrm{mod}\,d)\end{array}}\qquad\boxed{\begin{array}{c}\mathbb{Z}_d\times\mathbb{Z}_d\to\mathbb{Z}_d\times\mathbb{Z}_d\\ (n,m)\mapsto(n+m,n-m)\\ \text{is a one-to-one map}\end{array}}$$

21 $$\boxed{\frac{1}{d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{\pm\frac{2\pi\mathrm{i}}{d}km}=\delta_0(k)}\qquad\boxed{\frac{1}{d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{\pm\frac{4\pi\mathrm{i}}{d}km}=\delta_0(k)}$$

■ **Theorem.**

22 Finite Fourier transform is a unitary transform:
$$\boxed{F^{-1}=F^\dagger}\qquad\boxed{F^\dagger[\psi](k)=\frac{1}{\sqrt{d}}\sum_{n\in\mathbb{Z}_d}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kn}\,\psi(n)}$$

23 $$\boxed{P\psi(n)=\frac{1}{d}\sum_{k,m}\tilde{k}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k(n-m)}\psi(m)}$$

24 $$\boxed{\begin{array}{l}Q^\dagger=Q\\ P^\dagger=P\end{array}}\qquad\boxed{\begin{array}{l}\langle m|Q|j\rangle=\delta_{mj}\\ \langle m|P|j\rangle=\frac{1}{d}\sum_k\tilde{k}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k(m-j)}\end{array}}$$

25 $$\boxed{\begin{array}{l}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}\psi(n)=\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kn}\psi(n),\\ \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}\psi(m)=\psi(m-n)\end{array}}\qquad\boxed{\begin{array}{l}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}|m\rangle=\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}km}|m\rangle,\\ \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}|m\rangle=|m+n\rangle\end{array}}$$

26 $$\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}=\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}nk}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ},$$

27 $$\begin{aligned}D(n,k)&\equiv\mathrm{e}^{-\frac{\pi\mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}\\ &=\mathrm{e}^{\frac{\pi\mathrm{i}}{d}nk}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}\end{aligned}$$

28 $$\boxed{\begin{array}{l}D(n,k)\psi(m)=\mathrm{e}^{-\frac{\pi\mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}km}\,\psi(m-n)\\ \Pi(n,k)\psi(m)=\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}2k(n-m)}\,\psi(2n-m)\end{array}}$$

29 $$\boxed{\begin{array}{l}\langle m|D(n,k)|j\rangle=\mathrm{e}^{\frac{\pi\mathrm{i}}{d}k(m+j)}\,\delta_m(j+n)\\ \langle m|\Pi(n,k)|j\rangle=\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k(m-j)}\,\delta_{2n}(m+j)\end{array}}$$

30 $$D(n,k)\,D(m,j)=\mathrm{e}^{-\frac{\pi\mathrm{i}}{d}(nj-km)}D(\widetilde{n+m},\widetilde{k+j}).$$

31 $$D(n,k)^\dagger=D(-n,-k)=D^{-1}(n,k),$$

32 $\mathcal{L}(\mathcal{H})\equiv\left\{A:\mathcal{H}\to\mathcal{H}\ \middle|\ \begin{array}{c}\text{linear}\\ \text{operator}\end{array}\right\}$ with $\langle A,B\rangle=\mathrm{tr}(A^\dagger B)$ is a d^2-dimensional complex Hilbert space.

33 $\left\{\frac{1}{\sqrt{d}}D(n,k)\right\}$, $\left\{\frac{1}{\sqrt{d}}\Pi(n,k)\right\}$ are orthonormal bases in $\mathcal{L}(\mathcal{H})$.

34 $A\in\mathcal{L}(\mathcal{H}) \;\Rightarrow\; \boxed{\begin{array}{l} A^{\rm w}(n,k)=\langle\Pi(n,k),A\rangle \\ \mathcal{W}_A(n,k)=\frac{1}{d}\langle\Pi(n,k),A\rangle\end{array}}$

35 $\boxed{\begin{array}{l}\langle A^\dagger,B\rangle=\overline{\langle A,B^\dagger\rangle}\\ \Pi(n,k)^\dagger=\Pi(n,k)\end{array}}$ $\boxed{\begin{array}{l}\widetilde{A^\dagger}(n,k)=\overline{A^{\rm w}(n,k)}\\ \mathcal{W}_{A^\dagger}(n,k)=\overline{\mathcal{W}_A(n,k)}\end{array}}$

36 $A\in\mathcal{L}(\mathcal{H}) \;\Rightarrow\; \boxed{A=\sum_{n,k}\mathcal{W}_A(n,k)\,\Pi(n,k)}$

37 $A\in\mathcal{L}(\mathcal{H}) \Rightarrow \boxed{\mathcal{W}_A(n,k)=\frac{1}{d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\,\langle n+m|A|n-m\rangle}$

38 Pure state $\psi\in\mathcal{H}$ $\Rightarrow \boxed{\mathcal{W}_\psi(n,k)=\frac{1}{d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\,\psi(n+m)\,\overline{\psi(n-m)}}$

39 $A:\mathcal{H}\to\mathcal{H}$ linear operator $\Rightarrow \boxed{\sum_{n,k\in\mathbb{Z}_d}\mathcal{W}_A(n,k)=\mathrm{tr}\,A}$

40 $A,B\in\mathcal{L}(\mathcal{H}) \Rightarrow$
$$\boxed{\begin{aligned}\mathrm{tr}(AB)&=\frac{1}{d}\sum_{n,k}A^{\rm w}(n,k)\,B^{\rm w}(n,k)\\ &=\sum_{n,k}A^{\rm w}(n,k)\,\mathcal{W}_B(n,k)\\ &=d\sum_{n,k}\mathcal{W}_A(n,k)\,\mathcal{W}_B(n,k)\end{aligned}}$$

41 $\begin{array}{l}\varrho_\psi=|\psi\rangle\langle\psi|\\ \varrho_\varphi=|\varphi\rangle\langle\varphi|\end{array} \Rightarrow \boxed{\begin{aligned}\mathrm{tr}(\varrho_\psi\varrho_\varphi)&=d\sum_{n,k}\mathcal{W}_\psi(n,k)\,\mathcal{W}_\varphi(n,k)\\ &=|\langle\psi|\varphi\rangle|^2\end{aligned}}$

42 $\begin{array}{l}A:\mathcal{H}\to\mathcal{H}\\ \varrho=|\psi\rangle\langle\psi|\end{array} \Rightarrow \boxed{\langle A\rangle_\varrho\equiv\mathrm{tr}(\,\varrho\,A)=\langle\psi|\,A|\psi\rangle.}$

43 $\boxed{\mathbb{I}^{\rm w}(n,k)=1}$ $\boxed{\begin{array}{l}Q^{\rm w}(n,k)=\tilde{n}\\ P^{\rm w}(n,k)=\tilde{k}\end{array}}$ $\boxed{\begin{array}{l}\langle Q\rangle_\varrho=\sum\tilde{n}\,\mathcal{W}_\varrho(n,k)\\ \langle P\rangle_\varrho=\sum\tilde{k}\,\mathcal{W}_\varrho(n,k)\end{array}}$

44 $\mathcal{A}(\mathcal{H})\equiv\left\{A\in\mathcal{L}(\mathcal{H})\;|\;A^\dagger=A\right\}$ with $\langle A,B\rangle=\mathrm{tr}(AB)$ is a d^2-dimensional real Hilbert space.

45 $\left\{\frac{1}{\sqrt{d}}\Pi(n,k)\right\}$ is an orthonormal basis in $\mathcal{A}(\mathcal{H})$.

46 $A\in\mathcal{A}(\mathcal{H}) \;\Rightarrow\;$ $A^{\rm w}:\mathbb{Z}_d\times\mathbb{Z}_d\longrightarrow\mathbb{R}$ is a real function,
$\mathcal{W}_A:\mathbb{Z}_d\times\mathbb{Z}_d\longrightarrow\mathbb{R}$ is a real function.

47 Pure state $\psi\in\mathcal{H}$ $\Rightarrow \boxed{\sum_{k\in\mathbb{Z}_d}\mathcal{W}_\psi(n,k)=|\psi(n)|^2}$

48 Pure state $\psi\in\mathcal{H}$ $\Rightarrow \boxed{\sum_{n\in\mathbb{Z}_d}\mathcal{W}_\psi(n,k)=|F[\psi](k)|^2}$

49 $\varphi(n)=\psi(n-a) \;\Rightarrow\; \boxed{\mathcal{W}_\varphi(n,k)=\mathcal{W}_\psi(n-a,k)}$

50 $\varphi(n)=\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}bn}\,\psi(n) \;\Rightarrow\; \boxed{\mathcal{W}_\varphi(n,k)=\mathcal{W}_\psi(n,k-b)}$

51 Pure state $\psi\in\mathcal{H}$ $\Rightarrow \boxed{\sum_{n,k\in\mathbb{Z}_d}\mathcal{W}_\psi(n,k)^2=\frac{1}{d}}$

52 Pure state $\psi\in\mathcal{H}$ $\Rightarrow \boxed{|\mathcal{W}_\psi(n,k)|\le\frac{1}{d},\quad\text{for any } n,k}$

53 Even state $\psi(-n)=\psi(n)$ $\Rightarrow\; \mathcal{W}_\psi(0,0)|=\frac{1}{d}.$

54 Odd state $\psi(-n)=-\psi(n)$ $\Rightarrow\; \mathcal{W}_\psi(0,0)|=-\frac{1}{d}.$

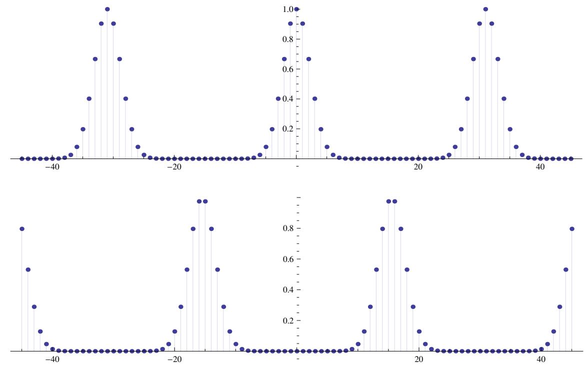

Figure 1: The functions g_1 and g_1^+ in the case $d=31$.

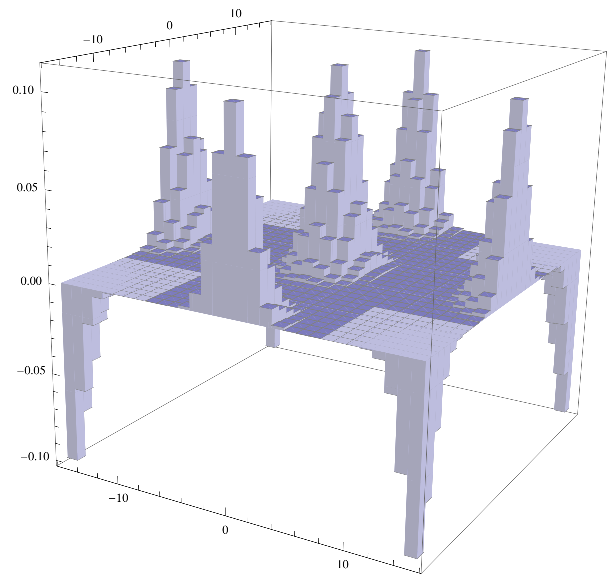

Figure 2: The Wigner function of g_1 in the case $d=31$.

■ **Remark.** The corresponding <u>*discrete phase space*</u> is

$$\mathfrak{S}=\{-s,-s+1,...,s-1,s\}\times\{-s,-s+1,...,s-1,s\}.$$

It is not $\mathbb{Z}_d\times\mathbb{Z}_d$ (see, for example, the relations 43).

■ **Theorem.**

55 Resolution of the identity $\boxed{\mathbb{I}=\frac{1}{d}\sum_{(n,k)\in\mathfrak{S}}|n,k\rangle\langle n,k|}$

56 $\boxed{g_\kappa(n)=\sum_{\alpha=-\infty}^{\infty}\mathrm{G}_\kappa\Big((n+\alpha d)\sqrt{\tfrac{2\pi}{d}}\Big)}$ where $\boxed{\mathrm{G}_\kappa(q)=\mathrm{e}^{-\frac{\kappa}{2}q^2}}$

$\boxed{F[g_\kappa]=\frac{1}{\sqrt{\kappa}}\,g_{\frac{1}{\kappa}}}$

57 Fourier transform $\boxed{F[g_\kappa](k)=\sum_{\beta=-\infty}^{\infty}\mathbf{F}[\mathrm{G}_\kappa]\Big((k+\beta d)\sqrt{\tfrac{2\pi}{d}}\Big)}$

$\boxed{F[g_{2\kappa}^+](2k)=\frac{1}{\sqrt{2\kappa}}(g_{\frac{2}{\kappa}}(k)-g_{\frac{2}{\kappa}}^+(k))}$

58 Wigner function of g_κ
$$\boxed{\begin{aligned}&\text{Up to a normalizing constant,}\\ &\mathcal{W}_{g_\kappa}(n,k)=g_{2\kappa}(n)\,\Big(g_{\frac{2}{\kappa}}(k)+g_{\frac{2}{\kappa}}^+(k)\Big)\\ &\qquad\qquad+g_{2\kappa}^+(n)\,\Big(g_{\frac{2}{\kappa}}(k)-g_{\frac{2}{\kappa}}^+(k)\Big)\\ &=C_\kappa\sum_{\alpha,\beta=-\infty}^{\infty}(-1)^{\alpha\beta}\mathbf{W}_{\mathrm{G}_\kappa}\Big((n+\alpha\tfrac{d}{2})\sqrt{\tfrac{2\pi}{d}},(k+\beta\sqrt{\tfrac{2\pi}{d}}\Big)\end{aligned}}$$

where $\mathbf{W}_{\mathrm{G}_\kappa}$ = Wigner function of G_κ
C_κ = normalizing multiplicative constant.

■ **Definition** (in a particular representaton).

59 The *tensor product* of the Hilbert spaces
$\mathcal{H}_1 = \{\ \psi : \mathbb{Z}_{d_1} \to \mathbb{C}\ |\ \psi \text{ is a function }\}$ of dim $d_1 = 2s_1+1$
$\mathcal{H}_2 = \{\ \psi : \mathbb{Z}_{d_2} \to \mathbb{C}\ |\ \psi \text{ is a function }\}$ of dim $d_2 = 2s_2+1$
is the Hilbert space

$$\boxed{\mathcal{H}_1 \otimes \mathcal{H}_2 = \{\ \Psi : \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \to \mathbb{C}\ |\ \Psi \text{ is a function }\}}$$

where

60 $$\langle \Psi, \Phi \rangle = \sum_{n\in\mathbb{Z}_{d_1}} \sum_{m\in\mathbb{Z}_{d_2}} \overline{\Psi(n,m)}\ \Phi(n,m) \quad \text{and}$$

61 $$(\psi\otimes\varphi)(n,m) = \psi(n)\,\varphi(m) \qquad \text{for any} \quad \begin{matrix}\psi\in\mathcal{H}_1\\ \varphi\in\mathcal{H}_2\end{matrix}$$

62 The *canonical basis* of $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is
$$\{\ |nm\rangle \equiv |n\rangle_1 \otimes |m\rangle_2 \ \ |\ \ n\in\mathbb{Z}_{d_1},\ m\in\mathbb{Z}_{d_2}\}$$
where
$$\{\ |n\rangle_1\ |\ \ n\in\mathbb{Z}_{d_1}\} = \text{canonical basis of } \mathcal{H}_1$$
$$\{\ |m\rangle_2\ |\ \ m\in\mathbb{Z}_{d_2}\} = \text{canonical basis of } \mathcal{H}_2.$$

63 *Parity operators* $(n_1, k_1 \in \mathbb{Z}_{d_1},\ n_2, k_2 \in \mathbb{Z}_{d_2})$
$$\boxed{\Pi(n_1,n_2,k_1,k_2) = \Pi(n_1,k_1)\otimes\Pi(n_2,k_2)}$$

64 *Weyl transform* of a linear operator $A : \mathcal{H} \to \mathcal{H}$ is
$$\tilde{A} : \mathbb{Z}_{d_1}\times\mathbb{Z}_{d_2}\times\mathbb{Z}_{d_1}\times\mathbb{Z}_{d_2} \longrightarrow \mathbb{C},$$
$$\boxed{\tilde{A}(n_1,n_2,k_1,k_2) = \mathrm{tr}(A\,\Pi(n_1,n_2,k_1,k_2))}$$

65 *Wigner function* of a linear operator $A : \mathcal{H} \to \mathcal{H}$ is
$$\mathcal{W}_A : \mathbb{Z}_{d_1}\times\mathbb{Z}_{d_2}\times\mathbb{Z}_{d_1}\times\mathbb{Z}_{d_2} \longrightarrow \mathbb{C},$$
$$\boxed{\mathcal{W}_A(n_1,n_2,k_1,k_2) = \tfrac{1}{d_1 d_2}\mathrm{tr}(A\,\Pi(n_1,n_2,k_1,k_2))}$$

■ **Lemma.**

66 $$\mathrm{tr}(A\otimes B) = \mathrm{tr}A\ \mathrm{tr}B, \qquad (A\otimes B)^\dagger = A^\dagger \otimes B^\dagger.$$

■ **Theorem.**

67 $$\left\{\tfrac{1}{\sqrt{d_1 d_2}}\,\Pi(n_1,n_2,k_1,k_2)\right\} \text{ is an } \begin{matrix}\text{orthonormal basis}\\ \text{in } \ \mathcal{A}(\mathcal{H}_1\otimes\mathcal{H}_2)\\ \text{and } \mathcal{L}(\mathcal{H}_1\otimes\mathcal{H}_2).\end{matrix}$$

68 $$A\in\mathcal{L}(\mathcal{H}) \Rightarrow \boxed{\mathcal{W}_A(n_1,n_2,k_1,k_2) = \tfrac{1}{d_1 d_2}\langle\Pi(n_1,n_2,k_1,k_2), A\rangle}$$
$$\Downarrow$$
$$\boxed{A = \sum \mathcal{W}_A(n_1,n_2,k_1,k_2)\,\Pi(n_1,n_2,k_1,k_2)}$$

69 *Wigner function* of $A : \mathcal{H}_1\otimes\mathcal{H}_2 \to \mathcal{H}_1\otimes\mathcal{H}_2$ is
$$\boxed{\begin{aligned}\mathcal{W}_A(n_1,n_2,k_1,k_2) = \tfrac{1}{d_1 d_2} \sum_{m_1\in\mathbb{Z}_{d_1}} \sum_{m_2\in\mathbb{Z}_{d_2}} \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_1}k_1 m_1}\, \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_2}k_2 m_2}\\ \times\langle n_1+m_1, n_2+m_2|A|n_1-m_1, n_2-m_2\rangle\end{aligned}}$$

70 *Wigner function* of a pure state $\Psi\in\mathcal{H}_1\otimes\mathcal{H}_2$ is
$$\boxed{\begin{aligned}\mathcal{W}_\Psi(n_1,n_2,k_1,k_2) = \tfrac{1}{d_1 d_2} \sum_{m_1\in\mathbb{Z}_{d_1}} \sum_{m_2\in\mathbb{Z}_{d_2}} \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_1}k_1 m_1}\, \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_2}k_2 m_2}\\ \times\,\Psi(n_1+m_1, n_2+m_2)\,\overline{\Psi(n_1-m_1, n_2-m_2)}\end{aligned}}$$

71 $$\begin{matrix}\psi\in\mathcal{H}_1\\ \varphi\in\mathcal{H}_2\end{matrix} \Rightarrow \boxed{\mathcal{W}_{\psi\otimes\varphi}(n_1,n_2,k_1,k_2) = \mathcal{W}_\psi(n_1,k_1)\,\mathcal{W}_\varphi(n_2,k_2)}$$

72 Partial trace of a pure state (in terms of Schmidt form)
$$\Psi\in\mathcal{H}_1\otimes\mathcal{H}_2,\ \ \varrho = |\Psi\rangle\langle\Psi|$$
$$|\Psi\rangle = \sum_a \sqrt{\lambda_a}\,|\psi_a\rangle\otimes|\varphi_a\rangle$$
$$\Downarrow$$
$$\boxed{\varrho_1 = \mathrm{tr}_2\,\varrho = \sum_a \lambda_a\,|\psi_a\rangle\langle\psi_a|}$$

73 Wigner function of a *partial trace*
$$\mathcal{H}_1\otimes\mathcal{H}_2 \xrightarrow{\varrho} \mathcal{H}_1\otimes\mathcal{H}_2$$
linear operator
$$\Downarrow$$
$$\boxed{\boxed{\mathcal{W}_{\varrho_1}(n_1,k_1) = \sum_{n_2,k_2\in\mathbb{Z}_{d_2}} \mathcal{W}_\varrho(n_1,n_2,k_1,k_2)}}$$

74 $$\boxed{\widetilde{A\otimes B}(n_1,n_2,k_1,k_2) = \tilde{A}(n_1,k_1)\,\tilde{B}(n_2,k_2)}$$

75 $$\boxed{\mathcal{W}_{A\otimes B}(n_1,n_2,k_1,k_2) = \mathcal{W}_A(n_1,k_1)\,\mathcal{W}_B(n_2,k_2)}$$

76 $$\boxed{\mathrm{tr}(\varrho(A\otimes\mathbb{I})) = \mathrm{tr}(\varrho_1\,A)}$$

77 *Purification* Ψ of a state ϱ and Wigner function of ϱ
$$\boxed{\begin{matrix}\rho : \mathcal{H}_1 \to \mathcal{H}_1 \text{ density operator}\\ \varrho = \sum\lambda_j|\varphi_j\rangle\langle\varphi_j| \quad \text{(spectral decomposition)}\end{matrix}}$$
$$\Downarrow$$
$$\boxed{\boxed{\begin{aligned}&\mathcal{W}_\varrho(n_1,k_1) = \sum_{n_2,k_2}\mathcal{W}_\Psi(n_1,n_2,k_1,k_2)\\ &\text{where}\\ &\qquad |\Psi\rangle = \sum\sqrt{\lambda_j}\,|\varphi_j\rangle\otimes|\varphi_j\rangle\end{aligned}}}$$

■ **Definition.**

78 For $\sigma = \begin{pmatrix} a & b\\ b & c\end{pmatrix} > 0$ with $a, b, c\in\mathbb{R}$, we define
$$\boxed{g_\sigma(n_1,n_2) = \sum_{\alpha_1,\alpha_2=-\infty}^{\infty} \mathrm{e}^{-\frac{\pi}{d}(n_1+\alpha_1 d \quad n_2+\alpha_2 d)\begin{pmatrix} a & b\\ b & c\end{pmatrix}\begin{pmatrix} n_1+\alpha_1 d\\ n_2+\alpha_2 d\end{pmatrix}}}$$
$$g_\sigma^{+0}(n_1,n_2) = \sum_{\alpha_1,\alpha_2=-\infty}^{\infty} \mathrm{e}^{-\frac{\pi}{d}(n_1+(\alpha_1+\frac{1}{2})d \quad n_2+\alpha_2 d)\begin{pmatrix} a & b\\ b & c\end{pmatrix}\begin{pmatrix} n_1+(\alpha_1+\frac{1}{2})d\\ n_2+\alpha_2 d\end{pmatrix}},$$
$$g_\sigma^{0+}(n_1,n_2) = \sum_{\alpha_1,\alpha_2=-\infty}^{\infty} \mathrm{e}^{-\frac{\pi}{d}(n_1+\alpha_1 d \quad n_2+(\alpha_2+\frac{1}{2})d)\begin{pmatrix} a & b\\ b & c\end{pmatrix}\begin{pmatrix} n_1+\alpha_1 d\\ n_2+(\alpha_2+\frac{1}{2})d\end{pmatrix}},$$
$$g_\sigma^{++}(n_1,n_2) = \sum_{\alpha_1,\alpha_2=-\infty}^{\infty} \mathrm{e}^{-\frac{\pi}{d}(n_1+(\alpha_1+\frac{1}{2})d \quad n_2+(\alpha_2+\frac{1}{2})d)\begin{pmatrix} a & b\\ b & c\end{pmatrix}\begin{pmatrix} n_1+(\alpha_1+\frac{1}{2})d\\ n_2+(\alpha_2+\frac{1}{2})d\end{pmatrix}}.$$

■ **Theorem** (Fourier transform)

79 $$\boxed{g_\sigma(n_1,n_2) = \sum_{\alpha_1,\alpha_2=-\infty}^{\infty} \mathrm{G}_\sigma\left((n_1+\alpha_1 d)\sqrt{\tfrac{2\pi}{d}}, (n_2+\alpha_2 d)\sqrt{\tfrac{2\pi}{d}}\right)}$$

80 $$\boxed{F[g_\sigma](k_1,k_2) = \sum_{\beta_1,\beta_2=-\infty}^{\infty} \mathbf{F}[\mathrm{G}_\sigma]\left((k_1+\beta_1 d)\sqrt{\tfrac{2\pi}{d}}, (k_2+\beta_2 d)\sqrt{\tfrac{2\pi}{d}}\right)}$$

81 $$\boxed{F[g_\sigma] = \tfrac{1}{\sqrt{\det\sigma}}\,g_{\sigma^{-1}}} \text{ where } \mathrm{G}_\sigma(q_1,q_2) = \mathrm{e}^{-\frac{1}{2}(q_1\ \ q_2)\begin{pmatrix} a & b\\ b & c\end{pmatrix}\begin{pmatrix} q_1\\ q_2\end{pmatrix}}$$

■ **Theorem** (Wigner function).

82 $$\boxed{\begin{aligned}\mathcal{W}_{g_\sigma}(n_1,n_2,k_1,k_2) = &\tfrac{1}{2d\sqrt{\det\sigma}}\ g_{2\sigma}(n_1,n_2)\left[g_{2\sigma^{-1}}(k_1,k_2) + g^{+0}_{2\sigma^{-1}}(k_1,k_2) + g^{0+}_{2\sigma^{-1}}(k_1,k_2) + g^{++}_{2\sigma^{-1}}(k_1,k_2)\right]\\ &+\tfrac{1}{2d\sqrt{\det\sigma}}\ g^{+0}_{2\sigma}(n_1,n_2)\left[g_{2\sigma^{-1}}(k_1,k_2) - g^{+0}_{2\sigma^{-1}}(k_1,k_2) + g^{0+}_{2\sigma^{-1}}(k_1,k_2) - g^{++}_{2\sigma^{-1}}(k_1,k_2)\right]\\ &+\tfrac{1}{2d\sqrt{\det\sigma}}\ g^{0+}_{2\sigma}(n_1,n_2)\left[g_{2\sigma^{-1}}(k_1,k_2) + g^{+0}_{2\sigma^{-1}}(k_1,k_2) - g^{0+}_{2\sigma^{-1}}(k_1,k_2) - g^{++}_{2\sigma^{-1}}(k_1,k_2)\right]\\ &+\tfrac{1}{2d\sqrt{\det\sigma}}\ g^{++}_{2\sigma}(n_1,n_2)\left[g_{2\sigma^{-1}}(k_1,k_2) - g^{+0}_{2\sigma^{-1}}(k_1,k_2) - g^{0+}_{2\sigma^{-1}}(k_1,k_2) + g^{++}_{2\sigma^{-1}}(k_1,k_2)\right].\\ = C_\sigma \sum_{\alpha_1,\alpha_2=-\infty}^{\infty}\ \sum_{\beta_1,\beta_2=-\infty}^{\infty} &(-1)^{\alpha_1\beta_1+\alpha_2\beta_2}\mathbf{W}_{\mathrm{G}_\sigma}\Big((n_1+\alpha_1\tfrac{d}{2})\sqrt{\tfrac{2\pi}{d}}, (n_2+\alpha_2\tfrac{d}{2})\sqrt{\tfrac{2\pi}{d}}, (k_1+\beta_1\tfrac{d}{2})\sqrt{\tfrac{2\pi}{d}}, (k_2+\beta_2\tfrac{d}{2})\sqrt{\tfrac{2\pi}{d}}\Big)\end{aligned}}$$

$\boxed{\text{E}}$ PROOFS

$\boxed{D20}$

$$\begin{array}{l} 2n=0 \pmod d\\ \Downarrow\\ 2n=kd\\ \Downarrow\\ k=2m \quad\Rightarrow\quad 2n=2md\\ \qquad\qquad\qquad\Downarrow\\ \qquad\qquad\qquad n=md\\ \qquad\qquad\qquad\Downarrow\\ \qquad\qquad\qquad n=0 \pmod d. \end{array}$$

Surjectivity

$$\begin{cases} n+m=k\\ n-m=\ell \end{cases} \Rightarrow \begin{cases} n=(k+\ell)(-s)\\ m=(k-\ell)(-s). \end{cases}$$

Injectivity

$$\begin{cases} n+m=n'+m'\\ n-m=n'-m' \end{cases} \Rightarrow 2n=2n' \Updownarrow 2(n-n')=0 \Updownarrow 2(n-n')\in\mathbb{Z}d \Updownarrow 2(n-n')=2kd \Updownarrow n=n'+kd \Updownarrow n=n' \bmod(d).$$

$\boxed{D21}$

$$k\neq 0 \;\Rightarrow\; \frac{1}{d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{\pm\frac{2\pi\mathrm{i}}{d}km}=\frac{1}{d}\sum_{m=0}^{d-1}\mathrm{e}^{\pm\frac{2\pi\mathrm{i}}{d}km}=\frac{1}{d}\,\frac{1-\left(\mathrm{e}^{\pm\frac{2\pi\mathrm{i}}{d}k}\right)^d}{1-\mathrm{e}^{\pm\frac{2\pi\mathrm{i}}{d}k}}=0$$

$$\Rightarrow\; \frac{1}{d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{\pm\frac{2\pi\mathrm{i}}{d}km}=\delta_0(k) \;\Rightarrow\; \frac{1}{d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{\pm\frac{4\pi\mathrm{i}}{d}km}=\delta_0(2k)=\delta_0(k).$$

$\boxed{D22}$

$$\begin{aligned} F[F^\dagger[\psi]](n) &= \frac{1}{\sqrt{d}}\sum_{k\in\mathbb{Z}_d}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}kn}\,F^\dagger[\psi](k)\\ &=\frac{1}{d}\sum_{k,m\in\mathbb{Z}_d}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}kn}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}km}\,\psi(m)\\ &=\sum_{m\in\mathbb{Z}_d}\frac{1}{d}\sum_{k\in\mathbb{Z}_d}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k(m-n)}\,\psi(m)\\ &\Downarrow \boxed{D21}\\ &=\sum_{m\in\mathbb{Z}_d}\delta_{mn}\,\psi(m)\\ &=\psi(n). \end{aligned}$$

$$\begin{aligned} F^\dagger[F[\psi]](n) &= \frac{1}{\sqrt{d}}\sum_{k\in\mathbb{Z}_d}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kn}\,F[\psi](k)\\ &=\frac{1}{d}\sum_{k,m\in\mathbb{Z}_d}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kn}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}km}\,\psi(m)\\ &=\sum_{m\in\mathbb{Z}_d}\frac{1}{d}\sum_{k\in\mathbb{Z}_d}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k(n-m)}\,\psi(m)\\ &\Downarrow \boxed{D21}\\ &=\sum_{m\in\mathbb{Z}_d}\delta_{mn}\,\psi(m)\\ &=\psi(n). \end{aligned}$$

$\boxed{D23}$

$$\begin{aligned} P\psi(n) &= (F^\dagger QF)\psi(n)\\ &=F^\dagger(QF\psi)(n)\\ &=\frac{1}{\sqrt{d}}\sum_{k\in\mathbb{Z}_d}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kn}(QF\psi)(k)\\ &=\frac{1}{\sqrt{d}}\sum_{k\in\mathbb{Z}_d}\tilde{k}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kn}(F\psi)(k)\\ &=\frac{1}{d}\sum_{k,m\in\mathbb{Z}_d}\tilde{k}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kn}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}km}\psi(m)\\ &=\frac{1}{d}\sum_{k,m\in\mathbb{Z}_d}\tilde{k}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k(n-m)}\psi(m). \end{aligned}$$

$\boxed{D24}$

$$\begin{aligned} \langle Q\psi|\varphi\rangle &= \sum_n \overline{Q\psi(n)}\,\varphi(n)\\ &=\sum_n \tilde{n}\,\overline{\psi(n)}\,\varphi(n) \qquad\Rightarrow\qquad Q^\dagger=Q.\\ &=\langle\psi|Q\varphi\rangle \end{aligned}$$

$$P^\dagger=(F^\dagger QF)^\dagger=F^\dagger QF=P.$$

$$\begin{aligned} \langle m|Q|j\rangle &= Q\delta_j(m)\\ &=\tilde{m}\,\delta_j(m)\\ &=\tilde{m}\,\delta_{mj}. \end{aligned}$$

$$\begin{aligned} \langle m|P|j\rangle &= P\delta_j(m)\\ &\Downarrow \boxed{D23}\\ &=\frac{1}{d}\sum_{k,n}\tilde{k}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k(m-n)}\delta_j(n)\\ &=\frac{1}{d}\sum_{k}\tilde{k}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k(m-j)} \end{aligned}$$

$\boxed{D25}$

$$\begin{aligned} \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}\psi(m) &\equiv (\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}\psi)(m)\\ &=(\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nF^\dagger QF}\psi)(m)\\ &=(F^\dagger\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nQ}F\psi)(m)\\ &=F^\dagger(\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nQ}F\psi)(m)\\ &=\frac{1}{\sqrt{d}}\sum_k\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}km}(\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nQ}F\psi)(k)\\ &=\frac{1}{\sqrt{d}}\sum_k\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}km}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nQ}(F\psi)(k)\\ &=\frac{1}{\sqrt{d}}\sum_k\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}km}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nk}(F\psi)(k)\\ &=\frac{1}{d}\sum_{k,\ell}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k(m-n)}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}k\ell}\psi(\ell)\\ &=\frac{1}{d}\sum_{k,\ell}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k(m-n-\ell)}\psi(\ell)\\ &\Downarrow \boxed{D21}\\ &=\sum_\ell\delta_{m-n}(\ell)\,\psi(\ell)\\ &=\psi(m-n), \end{aligned}$$

$$\begin{aligned} \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}\delta_m(\ell) &= \delta_m(\ell-n)\\ &=\delta_{m+n}(\ell)\\ &\Downarrow\\ \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}|m\rangle &= |m+n\rangle \end{aligned}$$

and

$$\begin{aligned} \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}\delta_m(\ell) &= \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k\ell}\delta_m(\ell)\\ &\Downarrow\\ \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}|m\rangle &= \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}km}|m\rangle \end{aligned}$$

$\boxed{D26}$

$$\begin{aligned} \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}\psi(m) &\equiv ((\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP})\psi)(m)\\ &=\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}\,(\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}\psi)(m)\\ &=\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}km}\,(\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}\psi)(m)\\ &=\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}km}\,\psi(m-n) \end{aligned}$$

and

$$\begin{aligned} \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}\psi(m) &\equiv ((\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ})\psi)(m)\\ &=\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}\,(\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}\psi)(m)\\ &=(\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}\psi)(m-n)\\ &=\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k(m-n)}\,\psi(m-n)\\ &=\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}kn}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}km}\,\psi(m-n)\\ &\Downarrow \end{aligned}$$

$$\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}=\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}nk}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ},$$

$\boxed{D27}$

$$\begin{aligned} D(n,k) &= \mathrm{e}^{-\frac{\pi\mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}\\ &\Downarrow \boxed{D26}\\ D(n,k) &= \mathrm{e}^{\frac{\pi\mathrm{i}}{d}nk}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}. \end{aligned}$$

$\boxed{D28}$

$$\begin{aligned}
D(n,k)\psi(m) &= \mathrm{e}^{-\frac{\pi \mathrm{i}}{d}nk}\,(\mathrm{e}^{\frac{2\pi \mathrm{i}}{d}kQ}\,\mathrm{e}^{-\frac{2\pi \mathrm{i}}{d}nP}\psi)(m)\\
&= \mathrm{e}^{-\frac{\pi \mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi \mathrm{i}}{d}km}\,\mathrm{e}^{-\frac{2\pi \mathrm{i}}{d}nP}\psi(m)\\
&= \mathrm{e}^{-\frac{\pi \mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi \mathrm{i}}{d}km}\,\psi(m-n).\\
\Pi(n,k)\psi(m) &\equiv (\Pi(n,k)\psi)(m)\\
&= (D(n,k)\,\Pi\,D(n,k)^{\dagger}\psi)(m)\\
&= \mathrm{e}^{-\frac{\pi \mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi \mathrm{i}}{d}km}(\Pi\,D(n,k)^{\dagger}\psi)(m-n)\\
&= \mathrm{e}^{-\frac{\pi \mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi \mathrm{i}}{d}km}(D(n,k)^{\dagger}\psi)(n-m)\\
&= \mathrm{e}^{-\frac{\pi \mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi \mathrm{i}}{d}km}(D(-n,-k)\psi)(n-m)\\
&= \mathrm{e}^{-\frac{2\pi \mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi \mathrm{i}}{d}km}\,\mathrm{e}^{-\frac{2\pi \mathrm{i}}{d}k(n-m)}\,\psi(2n-m)\\
&= \mathrm{e}^{-\frac{2\pi \mathrm{i}}{d}2k(n-m)}\,\psi(2n-m).
\end{aligned}$$

$\boxed{D29}$

$$\begin{aligned}
\langle m|D(n,k)|j\rangle &= \mathrm{e}^{-\frac{\pi \mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi \mathrm{i}}{d}km}\,\delta_j(m-n)\\
&= \mathrm{e}^{-\frac{\pi \mathrm{i}}{d}(m-j)k}\,\mathrm{e}^{\frac{2\pi \mathrm{i}}{d}km}\,\delta_m(j+n)\\
&= \mathrm{e}^{\frac{\pi \mathrm{i}}{d}k(m+j)}\,\delta_m(j+n)\\
\langle m|\Pi(n,k)|j\rangle &= \mathrm{e}^{-\frac{2\pi \mathrm{i}}{d}2k(n-m)}\,\delta_j(2n-m)\\
&= \mathrm{e}^{\frac{2\pi \mathrm{i}}{d}k(m-j)}\,\delta_{2n}(m+j).
\end{aligned}$$

$\boxed{D30}$

$$\begin{aligned}
D(n,k)\,D(m,j)\psi(\ell) &\equiv (D(n,k)\,D(m,j)\psi)(\ell)\\
&= D(n,k)(D(m,j)\psi)(\ell)\\
&= \mathrm{e}^{-\frac{\pi \mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi \mathrm{i}}{d}km}\,(D(m,j)\psi)(\ell-n)\\
&= \mathrm{e}^{-\frac{\pi \mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi \mathrm{i}}{d}km}\,\mathrm{e}^{-\frac{\pi \mathrm{i}}{d}mj}\,\mathrm{e}^{\frac{2\pi \mathrm{i}}{d}j(\ell-n)}\,\psi(\ell-n-m)\\
&= \mathrm{e}^{-\frac{\pi \mathrm{i}}{d}(nj-km)}D(\widetilde{n+m},\widetilde{k+j}).
\end{aligned}$$

$\boxed{D31}$

$$\begin{aligned}
D(n,k) &= \mathrm{e}^{-\frac{\pi \mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi \mathrm{i}}{d}kQ}\,\mathrm{e}^{-\frac{2\pi \mathrm{i}}{d}nP}\\
&\Downarrow\\
D(n,k)^{\dagger} &= \mathrm{e}^{\frac{\pi \mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi \mathrm{i}}{d}nP}\,\mathrm{e}^{-\frac{2\pi \mathrm{i}}{d}kQ}\\
&\Downarrow\ \boxed{D27}\\
D(n,k)^{\dagger} &= \mathrm{e}^{-\frac{\pi \mathrm{i}}{d}nk}\,\mathrm{e}^{-\frac{2\pi \mathrm{i}}{d}kQ}\,\mathrm{e}^{\frac{2\pi \mathrm{i}}{d}nP}\\
&\Downarrow\\
D(n,k)^{\dagger} &= D(-n,-k).\\
&\boxed{D30}\\
&\Downarrow\\
D(n,k)D(-n,-k) &= D(0,0) = \mathbb{I},\\
D(-n,-k)D(n,k) &= D(0,0) = \mathbb{I}
\end{aligned}$$

$\boxed{D32}$

$$\begin{aligned}
\langle A,B\rangle &= \sum_{n,m=-s}^{s}\overline{\langle n|A|m\rangle}\,\langle n|B|m\rangle\\
&= \sum_{n,m=-s}^{s}\langle m|A^{\dagger}|n\rangle\,\langle n|B|m\rangle\\
&= \sum_{m=-s}^{s}\langle m|A^{\dagger}B|m\rangle\\
&= \mathrm{tr}\,(A^{\dagger}B).
\end{aligned}$$

$\boxed{D33}$

$$\begin{aligned}
\langle D(n,k),D(m,\ell)\rangle &= \mathrm{tr}(D(n,k)^{\dagger}D(m,\ell))\\
&= \mathrm{tr}(D(-n,-k)D(m,\ell))\\
&\Downarrow\ \boxed{D30}\\
&= \mathrm{tr}(\mathrm{e}^{\frac{\pi \mathrm{i}}{d}(n\ell-km)}D(\widetilde{m-n},\widetilde{\ell-k}))\\
&= \mathrm{e}^{\frac{\pi \mathrm{i}}{d}(n\ell-km)}\mathrm{tr}\,D(\widetilde{m-n},\widetilde{\ell-k})\\
&= \mathrm{e}^{\frac{\pi \mathrm{i}}{d}(n\ell-km)}\,\sum_a\langle a|D(\widetilde{m-n},\widetilde{\ell-k})|a\rangle\\
&\Downarrow\ \boxed{D29}\\
&= \mathrm{e}^{\frac{\pi \mathrm{i}}{d}(n\ell-km)}\,\sum_a\mathrm{e}^{\frac{2\pi \mathrm{i}}{d}a(\ell-k)}\,\delta_a(a+m-n)\\
&= \mathrm{e}^{\frac{\pi \mathrm{i}}{d}(n\ell-km)}\,\sum_a\mathrm{e}^{\frac{2\pi \mathrm{i}}{d}a(\ell-k)}\,\delta_m(n)\\
&\Downarrow\ \boxed{D21}\\
&= d\,\mathrm{e}^{\frac{\pi \mathrm{i}}{d}(n\ell-km)}\,\delta_{mn}\,\delta_{k\ell}.\\
&= d\,\delta_{mn}\,\delta_{k\ell}.
\end{aligned}$$

$$\begin{aligned}
\langle \Pi(n,k)\,\Pi(m,\ell)\rangle &= \mathrm{tr}(\Pi(n,k)\,\Pi(m,\ell))\\
&= \sum_{a,b}\langle a|\Pi(n,k)|b\rangle\langle b|\Pi(m,\ell)|a\rangle\\
&= \sum_{a,b}\mathrm{e}^{\frac{2\pi \mathrm{i}}{d}k(a-b)}\,\delta_{2n}(a+b)\mathrm{e}^{-\frac{2\pi \mathrm{i}}{d}\ell(a-b)}\,\delta_{2m}(a+b)\\
&= \sum_{a,b}\mathrm{e}^{\frac{2\pi \mathrm{i}}{d}(k-\ell)(a-b)}\,\delta_{2n}(a+b)\,\delta_{nm}\\
&= \sum_{a}\mathrm{e}^{\frac{2\pi \mathrm{i}}{d}(k-\ell)(2a-2n)}\,\delta_{nm}\\
&= \mathrm{e}^{-\frac{4\pi \mathrm{i}}{d}n(k-\ell)}\,\sum_{a}\mathrm{e}^{\frac{4\pi \mathrm{i}}{d}a(k-\ell)}\,\delta_{nm}\\
&\Downarrow\ \boxed{D21}\\
&= \mathrm{e}^{-\frac{4\pi \mathrm{i}}{d}n(k-\ell)}\,d\ \delta_{k\ell}\,\delta_{nm}\\
&= d\,\delta_{nm}\,\delta_{k\ell}.
\end{aligned}$$

$\boxed{D34}$

$$\begin{aligned}
A^{\mathrm{w}}(n,k) &= \mathrm{tr}(A\Pi(n,k))\\
&= \mathrm{tr}(\Pi(n,k)A)\\
&= \mathrm{tr}(\Pi(n,k)^{\dagger}A)\\
&= \langle\Pi(n,k),A\rangle.
\end{aligned}$$

$$\begin{aligned}
\mathcal{W}_A(n,k) &= \tfrac{1}{d}\mathrm{tr}(A\Pi(n,k))\\
&= \tfrac{1}{d}A^{\mathrm{w}}(n,k)\\
&= \tfrac{1}{d}\langle\Pi(n,k),A\rangle.
\end{aligned}$$

$\boxed{D35}$

$$\begin{aligned}
\langle A^{\dagger},B\rangle &= \mathrm{tr}(AB)\\
&= \mathrm{tr}(BA)\\
&= \overline{\langle B^{\dagger},A\rangle}\\
&= \overline{\langle A,B^{\dagger}\rangle}.
\end{aligned}$$

$$\begin{aligned}
(\Pi\psi)(n) &= \psi(-n)\\
&\Downarrow\\
\langle\Pi\psi,\varphi\rangle &= \sum_n\overline{(\Pi\psi)(n)}\,\varphi(n)\\
&= \sum_n\overline{\psi(-n)}\,\varphi(n)\\
&\Downarrow\ n\mapsto -n\\
&= \sum_n\overline{\psi(n)}\,\varphi(-n)\\
&= \sum_n\overline{\psi(n)}\,(\Pi\varphi)(n)\\
&= \langle\psi,\Pi\varphi\rangle\\
&\Downarrow\\
\Pi^{\dagger} &= \Pi
\end{aligned}$$

and

$$\begin{aligned}
\Pi(n,k)^{\dagger} &= (D(n,k)\,\Pi\,D(n,k)^{\dagger})^{\dagger}\\
&= D(n,k)\,\Pi^{\dagger}\,D(n,k)^{\dagger}\\
&= D(n,k)\,\Pi\,D(n,k)^{\dagger}\\
&= \Pi(n,k).
\end{aligned}$$

$$\begin{aligned}
\widetilde{A^{\dagger}}(n,k) &= \langle\Pi(n,k),A^{\dagger}\rangle\\
&= \overline{\langle\Pi(n,k)^{\dagger},A\rangle}\\
&= \overline{\langle\Pi(n,k),A\rangle}\\
&= \overline{A^{\mathrm{w}}(n,k)}.
\end{aligned}$$

$$\begin{aligned}
\mathcal{W}_{A^{\dagger}}(n,k) &= \tfrac{1}{d}\widetilde{A^{\dagger}}(n,k)\\
&= \tfrac{1}{d}\overline{A^{\mathrm{w}}(n,k)}\\
&= \overline{\mathcal{W}_A(n,k)}.
\end{aligned}$$

$\boxed{D36}$

$$\left\{\tfrac{1}{\sqrt{d}}\Pi(n,k)\right\} = \text{orthonormal basis in } \mathcal{L}(\mathcal{H})$$
$$\Downarrow$$
$$\begin{aligned} A &= \sum_{n,k}\left\langle \tfrac{1}{\sqrt{d}}\Pi(n,k), A\right\rangle \tfrac{1}{\sqrt{d}}\Pi(n,k) \\ &= \sum_{n,k}\tfrac{1}{d}\ \langle \Pi(n,k), A\rangle\ \Pi(n,k) \\ &\Downarrow \boxed{D34} \\ &= \sum_{n,k}\mathcal{W}_A(n,k)\,\Pi(n,k). \end{aligned}$$

$\boxed{D37}$

$$\begin{aligned} \mathcal{W}_A(n,k) &= \tfrac{1}{d}\ \mathrm{tr}(A\,\Pi(n,k)) \\ &= \tfrac{1}{d}\sum_{a\in\mathbb{Z}_d}\langle a|A\Pi(n,k)|a\rangle \\ &= \tfrac{1}{d}\sum_{a,b\in\mathbb{Z}_d}\langle a|A|b\rangle\langle b|\Pi(n,k)|a\rangle \\ &\Downarrow \boxed{D29} \\ &= \tfrac{1}{d}\sum_{a,b\in\mathbb{Z}_d}\langle a|A|b\rangle e^{-\frac{2\pi i}{d}k(a-b)}\ \delta_{2n}(a+b) \\ &\Downarrow \text{b=2n-a} \\ &= \tfrac{1}{d}\sum_{a\in\mathbb{Z}_d}\langle a|A|2n-a\rangle e^{-\frac{4\pi i}{d}k(a-n)} \\ &\Downarrow \text{m=a-n} \\ &= \tfrac{1}{d}\sum_{a\in\mathbb{Z}_d}\langle n+m|A|n-m\rangle e^{-\frac{4\pi i}{d}km} \\ &= \tfrac{1}{d}\sum_{m\in\mathbb{Z}_d} e^{-\frac{4\pi i}{d}km}\ \langle n+m|A|n-m\rangle. \end{aligned}$$

$\boxed{D38}$

$$\begin{aligned} &\boxed{D37} \\ &\Downarrow \\ \mathcal{W}_\varrho(n,k) &= \tfrac{1}{d}\sum_{m\in\mathbb{Z}_d} e^{-\frac{4\pi i}{d}km}\ \langle n+m|\varrho|n-m\rangle \\ &\Downarrow \varrho=|\psi\rangle\langle\psi| \\ \mathcal{W}_\varrho(n,k) &= \tfrac{1}{d}\sum_{m\in\mathbb{Z}_d} e^{-\frac{4\pi i}{d}km}\ \langle n+m|\psi\rangle\langle\psi|n-m\rangle \\ &= \tfrac{1}{d}\sum_{m\in\mathbb{Z}_d} e^{-\frac{4\pi i}{d}km}\ \psi(n+m)\ \overline{\psi(n-m)}. \end{aligned}$$

$\boxed{D39}$

$$\begin{aligned} \sum_{n,k\in\mathbb{Z}_d}\mathcal{W}_A(n,k) &= \tfrac{1}{d}\sum_{n,k\in\mathbb{Z}_d}\ \sum_{m\in\mathbb{Z}_d} e^{-\frac{4\pi i}{d}km}\ \langle n+m|A|n-m\rangle \\ &= \sum_{n,m\in\mathbb{Z}_d}\tfrac{1}{d}\sum_{k\in\mathbb{Z}_d} e^{-\frac{4\pi i}{d}km}\ \langle n+m|A|n-m\rangle \\ &\Downarrow \boxed{D21} \\ &= \sum_{n,m\in\mathbb{Z}_d}\delta_{m0}\ \langle n+m|A|n-m\rangle \\ &= \sum_{n\in\mathbb{Z}_d}\langle n|A|n\rangle \\ &= \mathrm{tr}\,A. \end{aligned}$$

$\boxed{D40}$

$$\begin{aligned} \mathrm{tr}(AB) &= \langle A^\dagger, B\rangle \\ &\Downarrow \boxed{D33} \\ &= \tfrac{1}{d}\sum\overline{\langle\Pi(n,k),A^\dagger\rangle}\langle\Pi(n,k),B\rangle \\ &\Downarrow \boxed{D35} \\ &= \tfrac{1}{d}\sum\langle\Pi(n,k)^\dagger,A\rangle\langle\Pi(n,k),B\rangle \\ &= \tfrac{1}{d}\sum\langle\Pi(n,k),A\rangle\langle\Pi(n,k),B\rangle \\ &= \tfrac{1}{d}\sum A^{\mathrm{w}}(n,k)\,B^{\mathrm{w}}(n,k) \\ &= \sum A^{\mathrm{w}}(n,k)\,\mathcal{W}_B(n,k) \\ &= d\sum\mathcal{W}_A(n,k)\,\mathcal{W}_B(n,k). \end{aligned}$$

$\boxed{D41}$

$$\begin{aligned} \mathrm{tr}(\varrho_\psi\varrho_\varphi) &= \mathrm{tr}(|\psi\rangle\langle\psi|\varphi\rangle\langle\varphi|) \\ &= \sum_n\langle n|\psi\rangle\langle\psi|\varphi\rangle\langle\varphi|n\rangle \\ &= \sum_n\langle\varphi|n\rangle\langle n|\psi\rangle\langle\psi|\varphi\rangle \\ &= \langle\varphi|\psi\rangle\langle\psi|\varphi\rangle \\ &= |\langle\psi|\varphi\rangle|^2. \end{aligned}$$

$\boxed{D42}$

$$\begin{aligned} \mathrm{tr}(\,\varrho\,A) &= \mathrm{tr}(\,|\psi\rangle\langle\psi|\,A\,) \\ &= \sum_{m=-s}^{s}\langle m|\psi\rangle\langle\psi|\,A|m\rangle \\ &= \sum_{m=-s}^{s}\langle\psi|\,A|m\rangle\langle m|\psi\rangle \\ &= \langle\psi|\,A|\psi\rangle. \end{aligned}$$

$\boxed{D43}$

$$\begin{aligned} \mathbb{I}^{\mathrm{w}}(n,k) &= \mathrm{tr}(\mathbb{I}\Pi(n,k)) \\ &= \mathrm{tr}(\Pi(n,k)) \\ &= \sum_m\langle m|\Pi(n,k)|m\rangle \\ &\Downarrow \boxed{D29} \\ &= \sum_m\delta_{2n}(2m) \\ &= 1. \end{aligned}$$

$$\begin{aligned} Q^{\mathrm{w}}(n,k) &= \mathrm{tr}(Q\,\Pi(n,k)) \\ &= \mathrm{tr}(\Pi(n,k)Q) \\ &= \sum_m\langle m|\Pi(n,k)Q|m\rangle \\ &= \sum_m\tilde{m}\ \langle m|\Pi(n,k)|m\rangle \\ &\Downarrow \boxed{D29} \\ &= \sum_m\tilde{m}\,\delta_{nm} \\ &= \tilde{n}. \end{aligned}$$

$$\begin{aligned} P^{\mathrm{w}}(n,k) &= \mathrm{tr}(P\,\Pi(n,k)) \\ &= \sum_m\langle m|P\Pi(n,k)|m\rangle \\ &= \sum_{m,j}\langle m|P|j\rangle\langle \jmath|\Pi(n,k)|m\rangle \\ &= \tfrac{1}{d}\sum_{m,j,\ell}\tilde{\ell}\,e^{\frac{2\pi i}{d}\ell(m-j)}\ e^{\frac{2\pi i}{d}k(j-m)}\ \delta_{2n}(j+m) \\ &= \tfrac{1}{d}\sum_{m,\ell}\tilde{\ell}\,e^{\frac{2\pi i}{d}\ell(2m-2n)}\ e^{\frac{2\pi i}{d}k(2n-2m)} \\ &= \tfrac{1}{d}\sum_{m,\ell}\tilde{\ell}\,e^{\frac{2\pi i}{d}n(2k-2\ell)}\ e^{\frac{2\pi i}{d}m(2\ell-2k)} \\ &= \sum_\ell\tilde{\ell}\,e^{\frac{2\pi i}{d}n(2k-2\ell)}\,\tfrac{1}{d}\sum_m e^{\frac{2\pi i}{d}m(2\ell-2k)} \\ &\Downarrow \boxed{D21} \\ &= \sum_\ell\tilde{\ell}\,e^{\frac{2\pi i}{d}n(2k-2\ell)}\delta_{\ell k} \\ &= \tilde{k}. \end{aligned}$$

$\boxed{D44}$

$$\begin{aligned} \langle A,B\rangle &= \sum_{n,m=-s}^{s}\overline{\langle n|A|m\rangle}\ \langle n|B|m\rangle \\ &= \sum_{n,m=-s}^{s}\langle m|A^\dagger|n\rangle\ \langle n|B|m\rangle \\ &= \sum_{m=-s}^{s}\langle m|AB|m\rangle \\ &= \mathrm{tr}\,(AB). \end{aligned}$$

$\boxed{D45}$

See $\boxed{D33}$ at page 13.

$\boxed{D46}$

$$\boxed{D35}\ \Downarrow$$

$\mathcal{W}_A(n,k)$ are real numbers.
$A^{\mathrm{w}}(n,k)$ are real numbers.

$\boxed{D47}$

$$\begin{aligned}\sum_{k\in\mathbb{Z}_d}\mathcal{W}_\psi(n,k) &= \tfrac{1}{d}\sum_{k\in\mathbb{Z}_d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\ \psi(n+m)\,\overline{\psi(n-m)}\\ &= \tfrac{1}{d}\sum_{m\in\mathbb{Z}_d}\sum_{k\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\ \psi(n+m)\,\overline{\psi(n-m)}\\ &\Downarrow\ \boxed{D21}\\ &= \sum_{m\in\mathbb{Z}_d}\delta_0(m)\,\psi(n+m)\,\overline{\psi(n-m)}\\ &= \psi(n)\,\overline{\psi(n)}\\ &= |\psi(n)|^2.\end{aligned}$$

$\boxed{D48}$

$$\begin{aligned}\sum_{n\in\mathbb{Z}_d}\mathcal{W}_\psi(n,k) &= \tfrac{1}{d}\sum_{n\in\mathbb{Z}_d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\ \psi(n+m)\,\overline{\psi(n-m)}\\ &\Downarrow\ \boxed{D20}\ (n,m)\mapsto(a,b)=(n+m,n-m)\ \text{bijective}\\ &= \tfrac{1}{d}\sum_{a\in\mathbb{Z}_d}\sum_{b\in\mathbb{Z}_d}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}k(a-b)}\ \psi(a)\,\overline{\psi(b)}\\ &= \tfrac{1}{\sqrt{d}}\sum_{a\in\mathbb{Z}_d}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}ka}\psi(a)\ \ \tfrac{1}{\sqrt{d}}\sum_{b\in\mathbb{Z}_d}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kb}\,\overline{\psi(b)}\\ &= F[\psi](k)\ \overline{F[\psi](k)}\\ &= |F[\psi](k)|^2.\end{aligned}$$

$\boxed{D49}$

$$\begin{aligned}\mathcal{W}_\varphi(n,k) &= \tfrac{1}{d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\,\varphi(n+m)\,\overline{\varphi(n-m)}\\ &= \tfrac{1}{d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\,\psi(n+m-a)\,\overline{\psi(n-m-a)}\\ &= \mathcal{W}_\psi(n-a,k).\end{aligned}$$

$\boxed{D50}$

$$\begin{aligned}\mathcal{W}_\varphi(n,k) &= \tfrac{1}{d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\,\varphi(n+m)\,\overline{\varphi(n-m)}\\ &= \tfrac{1}{d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}b(n+m)}\,\psi(n+m)\\ &\qquad\qquad \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}b(n-m)}\,\overline{\psi(n-m)}\\ &= \tfrac{1}{d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}(k-b)m}\,\psi(n+m)\,\overline{\psi(n-m)}\\ &= \mathcal{W}_\psi(n,k-b).\end{aligned}$$

$\boxed{D52}$

$$\begin{aligned}\mathcal{W}_\psi(n,k) &= \tfrac{1}{d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\,\psi(n+m)\,\overline{\psi(n-m)}\\ &= \tfrac{1}{d}\langle\varphi_1,\varphi_2\rangle,\end{aligned}$$

where $\varphi_1(m)=\psi(n-m)$
$\varphi_2(m)=\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\,\psi(n+m)$

and $||\varphi_1||^2=\sum_m|\varphi_1(m)|^2=\sum_m|\psi(n-m)|^2=||\psi||^2=1,$
$||\varphi_2||^2=\sum_m|\varphi_2(m)|^2=\sum_m|\psi(n+m)|^2=||\psi||^2=1.$

But, $\mathcal{W}_\psi(n,k)=\tfrac{1}{d}\langle\varphi_1,\varphi_2\rangle$

$$\Downarrow$$

$$|\mathcal{W}_\psi(n,k)|=\tfrac{1}{d}|\langle\varphi_1,\varphi_2\rangle|\le\tfrac{1}{d}||\varphi_1||\ ||\varphi_2||=\tfrac{1}{d}.$$

$\boxed{D55}$

$$\mathfrak{g}\equiv\frac{1}{\sqrt{\langle g_1|g_1\rangle}}\,g_1$$
$$\Downarrow\ \boxed{D18}$$
$$\langle a|n,k\rangle=\mathrm{e}^{-\frac{\pi\mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}ka}\,\mathfrak{g}(a-n)$$
$$\Downarrow$$

$$\begin{aligned}\left\langle a\left|\tfrac{1}{d}\sum_{(n,k)\in\mathfrak{S}}|n,k\rangle\langle n,k|\right|b\right\rangle &= \tfrac{1}{d}\sum_{(n,k)\in\mathfrak{S}}\langle a|n,k\rangle\langle n,k|b\rangle\\ &= \tfrac{1}{d}\sum_{(n,k)\in\mathfrak{S}}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}ka}\,\mathfrak{g}(a-n)\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}kb}\,\mathfrak{g}(b-n)\\ &= \tfrac{1}{d}\sum_{(n,k)\in\mathfrak{S}}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k(a-b)}\,\mathfrak{g}(a-n)\,\mathfrak{g}(b-n)\\ &= \sum_{n=-s}^{s}\tfrac{1}{d}\sum_{k=-s}^{s}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k(a-b)}\,\mathfrak{g}(a-n)\,\mathfrak{g}(b-n)\\ &\Downarrow\ \boxed{D21}\\ &= \sum_{n=-s}^{s}\delta_0(a-b)\,\mathfrak{g}(a-n)\,\mathfrak{g}(b-n)\\ &= \sum_{n=-s}^{s}\delta_{ab}\,\mathfrak{g}(a-n)\,\mathfrak{g}(b-n)\\ &= \delta_{ab}\sum_{n=-s}^{s}\mathfrak{g}(a-n)^2\\ &= \delta_{ab}\sum_{n=-s}^{s}\mathfrak{g}(n-a)^2\\ &\Downarrow\ n-a\mapsto m\\ &= \delta_{ab}\sum_{n=-s}^{s}\mathfrak{g}(m)^2\\ &= \delta_{ab}\,||\mathfrak{g}||^2\\ &= \delta_{ab}\\ &= \langle a|\mathbb{I}|b\rangle.\end{aligned}$$

$\boxed{D57a}$

The periodic function

$$\mathcal{G}_\kappa:\mathbb{R}\to\mathbb{R},\qquad \mathcal{G}_\kappa(x)=\sum_{\alpha=-\infty}^{\infty}\mathrm{e}^{-\frac{\kappa\pi}{d}(x+\alpha d)^2}$$

admits the Fourier expansion

$$\mathcal{G}_\kappa(x)=\sum_{\ell=-\infty}^{\infty}a_\ell\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}\ell x}$$

with

$$\begin{aligned}a_\ell &= \tfrac{1}{d}\int_0^d\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}\ell x}\sum_{\alpha=-\infty}^{\infty}\mathrm{e}^{-\frac{\kappa}{2}\left(\sqrt{\frac{2\pi}{d}}\,(x+\alpha d)\right)^2}dx\\ &= \tfrac{1}{d}\sum_{\alpha=-\infty}^{\infty}\int_0^d\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}\ell x}\mathrm{e}^{-\frac{\kappa}{2}\left(\sqrt{\frac{2\pi}{d}}\,(x+\alpha d)\right)^2}dx.\end{aligned}$$

By denoting $t=\sqrt{\frac{2\pi}{d}}\,(x+\alpha d)$ and using the relation

$$\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\xi t}\,\mathrm{e}^{-at^2}\,dt=\sqrt{\tfrac{\pi}{a}}\ \mathrm{e}^{-\frac{\xi^2}{4a}}$$

we get

$$\begin{aligned}a_\ell &= \frac{1}{\sqrt{2\pi d}}\sum_{\alpha=-\infty}^{\infty}\int_{\alpha\sqrt{2\pi d}}^{(\alpha+1)\sqrt{2\pi d}}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}\ell\left(t\sqrt{\frac{d}{2\pi}}\right)}\mathrm{e}^{-\frac{\kappa}{2}t^2}dt\\ &= \frac{1}{\sqrt{2\pi d}}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\ell t\sqrt{\frac{2\pi}{d}}}\mathrm{e}^{-\frac{\kappa}{2}t^2}dt\\ &= \frac{1}{\sqrt{\kappa d}}\,\mathrm{e}^{-\frac{\pi}{\kappa d}\ell^2}\end{aligned}$$

whence, we have the relation

$$\mathcal{G}_\kappa(x)=\frac{1}{\sqrt{\kappa d}}\sum_{\ell=-\infty}^{\infty}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}\ell x}\,\mathrm{e}^{-\frac{\pi}{\kappa d}\ell^2}$$

leading to

$$\begin{aligned}g_\kappa(k) &= \tfrac{1}{\sqrt{\kappa d}} \sum_{\ell=-\infty}^{\infty} \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}\ell k}\,\mathrm{e}^{-\frac{\pi}{\kappa d}\ell^2}\\ &= \tfrac{1}{\sqrt{\kappa d}} \sum_{n=-s}^{s} \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kn} \sum_{\alpha=-\infty}^{\infty} \mathrm{e}^{-\frac{\pi}{\kappa d}(n+\alpha d)^2}\\ &= \tfrac{1}{\sqrt{\kappa d}} \sum_{n=-s}^{s} \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kn}\, g_{\frac{1}{\kappa}}(n)\\ &= \tfrac{1}{\sqrt{\kappa}}\, F^{-1}[g_{\frac{1}{\kappa}}](k).\end{aligned}$$

$\boxed{D57c}$

The periodic function

$$\mathcal{G}^+_{2\kappa}:\mathbb{R}\to\mathbb{R},\qquad \mathcal{G}^+_{2\kappa}(x) = \sum_{\alpha=-\infty}^{\infty} \mathrm{e}^{-\frac{2\kappa\pi}{d}\left(x+\left(\alpha+\frac{1}{2}\right)d\right)^2}$$

admits the Fourier expansion

$$\mathcal{G}^+_{2\kappa}(x) = \sum_{\ell=-\infty}^{\infty} c_\ell\, \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}\ell x}$$

with

$$\begin{aligned}c_\ell &= \tfrac{1}{d}\int_0^d \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}\ell x} \sum_{\alpha=-\infty}^{\infty} \mathrm{e}^{-\kappa\left(\sqrt{\frac{2\pi}{d}}\left(x+\left(\alpha+\frac{1}{2}\right)d\right)\right)^2} dx\\ &= \tfrac{1}{d}\sum_{\alpha=-\infty}^{\infty}\int_0^d \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}\ell x}\mathrm{e}^{-\kappa\left(\sqrt{\frac{2\pi}{d}}\left(x+\left(\alpha+\frac{1}{2}\right)d\right)\right)^2} dx.\end{aligned}$$

By denoting $t=\sqrt{\frac{2\pi}{d}}\,\left(x+\left(\alpha+\frac{1}{2}\right)d\right)$ and using

$$\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\xi t}\,\mathrm{e}^{-at^2}\,dt = \sqrt{\tfrac{\pi}{a}}\ \mathrm{e}^{-\frac{\xi^2}{4a}}$$

we get

$$\begin{aligned}c_\ell &= \tfrac{1}{\sqrt{2\pi d}} \sum_{\alpha=-\infty}^{\infty} \int_{(\alpha-1/2)\sqrt{2\pi d}}^{(\alpha+1/2)\sqrt{2\pi d}} \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}\ell\left(t\sqrt{\frac{d}{2\pi}}-\left(\alpha+\frac{1}{2}\right)d\right)}\mathrm{e}^{-\kappa t^2}\,dt\\ &= \tfrac{(-1)^\ell}{\sqrt{2\pi d}} \sum_{\alpha=-\infty}^{\infty} \int_{(\alpha-1/2)\sqrt{2\pi d}}^{(\alpha+1/2)\sqrt{2\pi d}} \mathrm{e}^{-\mathrm{i}\ell t\sqrt{\frac{2\pi}{d}}}\mathrm{e}^{-\kappa t^2}\,dt\\ &= \tfrac{(-1)^\ell}{\sqrt{2\pi d}} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\ell t\sqrt{\frac{2\pi}{d}}}\mathrm{e}^{-\kappa t^2}\,dt\\ &= \tfrac{(-1)^\ell}{\sqrt{2\kappa d}}\,\mathrm{e}^{-\frac{\pi}{2\kappa d}\ell^2}\end{aligned}$$

whence

$$G^+_{2\kappa}(x) = \tfrac{1}{\sqrt{2\kappa d}} \sum_{\ell=-\infty}^{\infty} \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}\ell x}\,(-1)^\ell\,\mathrm{e}^{-\frac{\pi}{2\kappa d}\ell^2}.$$

Particularly, we have

$$\begin{aligned}g^+_{2\kappa}(m) &= \mathcal{G}^+_{2\kappa}(m) = \tfrac{1}{\sqrt{2\kappa d}} \sum_{\ell=-\infty}^{\infty} \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}m\ell}\,(-1)^\ell\,\mathrm{e}^{-\frac{\pi}{2\kappa d}\ell^2}\\ &= \tfrac{1}{\sqrt{2\kappa d}} \sum_{n=-s}^{s}\sum_{\alpha=-\infty}^{\infty} \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}m(n+\alpha d)}\,(-1)^{(n+\alpha d)}\mathrm{e}^{-\frac{\pi}{2\kappa d}(n+\alpha d)^2}\\ &= \tfrac{1}{\sqrt{d}} \sum_{n=-s}^{s} \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}mn}\tfrac{(-1)^n}{\sqrt{2\kappa}} \sum_{\alpha=-\infty}^{\infty} (-1)^\alpha\,\mathrm{e}^{-\frac{\pi}{2\kappa d}(n+\alpha d)^2}\end{aligned}$$

whence

$$F[g^+_{2\kappa}](n) = \tfrac{(-1)^n}{\sqrt{2\kappa}} \sum_{\alpha=-\infty}^{\infty} (-1)^\alpha\,\mathrm{e}^{-\frac{\pi}{2\kappa d}(n+\alpha d)^2}$$

and we get

$$\begin{aligned}F[g^+_{2\kappa}](2k) &= \tfrac{1}{\sqrt{2\kappa}} \sum_{\alpha=-\infty}^{\infty} (-1)^\alpha\,\mathrm{e}^{-\frac{\pi}{2\kappa d}(2k+\alpha d)^2}\\ &= \tfrac{1}{\sqrt{2\kappa}} \sum_{\alpha=-\infty}^{\infty} \mathrm{e}^{-\frac{\pi}{2\kappa d}(2k+2\alpha d)^2} - \tfrac{1}{\sqrt{2\kappa}} \sum_{\alpha=-\infty}^{\infty} \mathrm{e}^{-\frac{\pi}{2\kappa d}(2k+(2\alpha+1)d)^2}\\ &= \tfrac{1}{\sqrt{2\kappa}} \sum_{\alpha=-\infty}^{\infty} \mathrm{e}^{-\frac{2\pi}{\kappa d}(k+\alpha d)^2} - \tfrac{1}{\sqrt{2\kappa}} \sum_{\alpha=-\infty}^{\infty} \mathrm{e}^{-\frac{2\pi}{\kappa d}(k+\alpha d+\frac{d}{2})^2}\\ &= \tfrac{1}{\sqrt{2\kappa}}\left(g_{\frac{2}{\kappa}}(k) - g^+_{\frac{2}{\kappa}}(k)\right).\end{aligned}$$

$\boxed{D58}$

By using $\boxed{D56}$, $\boxed{D57}$, the relation

$$\begin{aligned}F[g_{2\kappa}](2k) &= \tfrac{1}{\sqrt{2\kappa}}\, g_{\frac{1}{2\kappa}}(2k)\\ &= \tfrac{1}{\sqrt{2\kappa}} \sum_{\alpha=-\infty}^{\infty} \mathrm{e}^{-\frac{\pi}{2\kappa d}(2k+\alpha d)^2}\\ &= \tfrac{1}{\sqrt{2\kappa}} \sum_{\alpha=-\infty}^{\infty} \mathrm{e}^{-\frac{\pi}{2\kappa d}(2k+2\alpha d)^2} + \tfrac{1}{\sqrt{2\kappa}} \sum_{\alpha=-\infty}^{\infty} \mathrm{e}^{-\frac{\pi}{2\kappa d}(2k+(2\alpha+1)d)^2}\\ &= \tfrac{1}{\sqrt{2\kappa}} \sum_{\alpha=-\infty}^{\infty} \mathrm{e}^{-\frac{2\pi}{\kappa d}(k+\alpha d)^2} + \tfrac{1}{\sqrt{2\kappa}} \sum_{\alpha=-\infty}^{\infty} \mathrm{e}^{-\frac{2\pi}{\kappa d}(k+\alpha d+\frac{d}{2})^2}\\ &= \tfrac{1}{\sqrt{2\kappa}}(g_{\frac{2}{\kappa}}(k) + g^+_{\frac{2}{\kappa}}(k))\end{aligned}$$

and the formula

$$\begin{aligned}\sum_{\alpha,\beta=-\infty}^{\infty} E_{\alpha,\beta} &= \sum_{\substack{\alpha,\,\beta\\ \text{both even}\\ \text{or}\\ \text{both odd}}} E_{\alpha,\beta} + \sum_{\substack{\alpha,\,\beta\\ \text{one even}\\ \text{and}\\ \text{other odd}}} E_{\alpha,\beta}\\ &= \sum_{\mu,\eta=-\infty}^{\infty} E_{\mu+\eta,\mu-\eta} + \sum_{\mu,\eta=-\infty}^{\infty} E_{\mu+\eta+1,\mu-\eta},\end{aligned}$$

we get (up to a normalizing multiplicative constant)

$$\begin{aligned}\mathcal{W}_{g_\kappa}(n,k) &= \tfrac{1}{d}\sum_{m=-s}^{s} \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km} \sum_{\alpha,\beta=-\infty}^{\infty} \mathrm{e}^{-\frac{\kappa\pi}{d}(n-m+\alpha d)^2}\mathrm{e}^{-\frac{\kappa\pi}{d}(n+m+\beta d)^2}\\ &= \tfrac{1}{d}\sum_{m=-s}^{s} \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km} \sum_{\mu,\eta=-\infty}^{\infty} \mathrm{e}^{-\frac{\kappa\pi}{d}(n-m+(\mu+\eta)d)^2}\mathrm{e}^{-\frac{\kappa\pi}{d}(n+m+(\mu-\eta)d)^2}\\ &\quad+\tfrac{1}{d}\sum_{m=-s}^{s} \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km} \sum_{\mu,\eta=-\infty}^{\infty} \mathrm{e}^{-\frac{\kappa\pi}{d}(n-m+(\mu+\eta+1)d)^2}\mathrm{e}^{-\frac{\kappa\pi}{d}(n+m+(\mu-\eta)d)^2}\\ &= \tfrac{1}{d}\sum_{\mu=-\infty}^{\infty} \mathrm{e}^{-\frac{2\kappa\pi}{d}(n+\mu d)^2} \sum_{m=-s}^{s} \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km} \sum_{\eta=-\infty}^{\infty} \mathrm{e}^{-\frac{2\kappa\pi}{d}(-m+\eta d)^2}\\ &\quad+\tfrac{1}{d}\sum_{\mu=-\infty}^{\infty} \mathrm{e}^{-\frac{2\kappa\pi}{d}(n+\frac{d}{2}+\mu d)^2} \sum_{m=-s}^{s} \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km} \sum_{\eta=-\infty}^{\infty} \mathrm{e}^{-\frac{2\kappa\pi}{d}(-m+\frac{d}{2}+\eta d)^2}\\ &= \tfrac{1}{\sqrt{d}} g_{2\kappa}(n)\, F[g_{2\kappa}](2k) + \tfrac{1}{\sqrt{d}} g^+_{2\kappa}(n)\, F[g^+_{2\kappa}](2k)\\ &= \tfrac{1}{\sqrt{2\kappa d}}\, g_{2\kappa}(n)\,\left(g_{\frac{2}{\kappa}}(k) + g^+_{\frac{2}{\kappa}}(k)\right) + \tfrac{1}{\sqrt{2\kappa d}}\, g^+_{2\kappa}(n)\,\left(g_{\frac{2}{\kappa}}(k) - g^+_{\frac{2}{\kappa}}(k)\right).\end{aligned}$$

$\boxed{D66a}$

$$\begin{aligned}\mathrm{tr}(A\otimes B) &= \sum_{a,b}\langle ab|A\otimes B|ab\rangle\\ &= \sum_a\langle a|A|a\rangle\ \sum_b\langle b|B|b\rangle\\ &= \mathrm{tr}\,A\ \mathrm{tr}\,B\end{aligned}$$

$\boxed{D66b}$

$$\begin{aligned}\langle\varphi_1\otimes\psi_1,(A\otimes B)^\dagger(\varphi_2\otimes\psi_2)\rangle &= \langle(A\otimes B)(\varphi_1\otimes\psi_1),\varphi_2\otimes\psi_2\rangle\\ &= \langle(A\varphi_1)\otimes(B\psi_1),\varphi_2\otimes\psi_2\rangle\\ &= \langle A\varphi_1,\varphi_2\rangle\langle B\psi_1,\psi_2\rangle\\ &= \langle\varphi_1,A^\dagger\varphi_2\rangle\langle\psi_1,B^\dagger\psi_2\rangle\\ &= \langle\varphi_1\otimes\psi_1,(A^\dagger\varphi_2)\otimes(B^\dagger\psi_2)\rangle\\ &= \langle\varphi_1\otimes\psi_1,(A^\dagger\otimes B^\dagger)(\varphi_2\otimes\psi_2)\rangle\end{aligned}$$

for any $\varphi_1,\varphi_2\in\mathcal{H}_1,\ \psi_1,\psi_2\in\mathcal{H}_2$

$$\Downarrow$$

$$\langle nk|(A\otimes B)^\dagger|m\ell\rangle = \langle nk|A^\dagger\otimes B^\dagger|m\ell\rangle$$

(Matrices of the two operators coincide).

$\boxed{D67}$

$$\Big\langle \tfrac{1}{\sqrt{d_1d_2}}\,\Pi(n_1,n_2,k_1,k_2),\ \tfrac{1}{\sqrt{d_1d_2}}\,\Pi(n_1',n_2',k_1',k_2')\Big\rangle$$
$$=\tfrac{1}{d_1d_2}\ \langle \Pi(n_1,k_1)\otimes\Pi(n_2,k_2),\Pi(n_1',k_1')\otimes\Pi(n_2',k_2')\rangle$$
$$=\tfrac{1}{d_1d_2}\ \mathrm{tr}((\Pi(n_1,k_1)\otimes\Pi(n_2,k_2))(\Pi(n_1',k_1')\otimes\Pi(n_2',k_2')))$$
$$=\tfrac{1}{d_1d_2}\ \mathrm{tr}((\Pi(n_1,k_1)\Pi(n_1',k_1')\otimes\Pi(n_2,k_2)\Pi(n_2',k_2'))$$
$$=\tfrac{1}{d_1d_2}\ \mathrm{tr}((\Pi(n_1,k_1)\Pi(n_1',k_1'))\ \mathrm{tr}(\Pi(n_2,k_2)\Pi(n_2',k_2'))$$
$$=\delta_{n_1n_1'}\ \delta_{k_1k_1'}\ \delta_{n_2n_2'}\ \delta_{k_2k_2'}.$$

$\boxed{D68}$

$$\mathcal{W}_A(n_1,n_2,k_1,k_2)=\tfrac{1}{d_1d_2}\mathrm{tr}(A\,\Pi(n_1,n_2,k_1,k_2)).$$
$$=\tfrac{1}{d_1d_2}\langle\Pi(n_1,n_2,k_1,k_2),A\rangle$$

$\Big\{\tfrac{1}{\sqrt{d_1d_2}}\,\Pi(n_1,n_2,k_1,k_2)\Big\}$ is an orthonormal basis in $\mathcal{A}(\mathcal{H}_1\otimes\mathcal{H}_2)$.

$$\Downarrow$$
$$A=\sum_{n_1,n_2,k_1,k_2}\tfrac{1}{d_1d_2}\langle\Pi(n_1,n_2,k_1,k_2),A\rangle\ \Pi(n_1,n_2,k_1,k_2)$$
$$=\sum_{n_1,n_2,k_1,k_2}\mathcal{W}_A(n_1,n_2,k_1,k_2)\,\Pi(n_1,n_2,k_1,k_2)$$

$\boxed{D69}$

$$\mathcal{W}_A(n_1,n_2,k_1,k_2)=\tfrac{1}{d_1d_2}\mathrm{tr}(A\,\Pi(n_1,n_2,k_1,k_2))$$
$$=\tfrac{1}{d_1d_2}\sum_{a_1,a_2,b_1,b_2}\langle a_1a_2|A|b_1b_2\rangle\ \langle b_1b_2|\Pi(n_1,k_1)\otimes\Pi(n_2,k_2)|a_1a_2\rangle$$
$$=\tfrac{1}{d_1d_2}\sum_{a_1,a_2,b_1,b_2}\langle a_1a_2|A|b_1b_2\rangle \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_1}k_1(a_1-b_1)}\ \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_2}k_2(a_2-b_2)}\delta_{2n_1}(a_1+b_1)\delta_{2n_2}(a_2+b_2)$$
$$\Downarrow\ b_1=2n_1-a_1,\ \ b_2=2n_2-a_2$$
$$=\tfrac{1}{d_1d_2}\sum_{a_1,a_2,b_1,b_2}\langle a_1a_2|A|2n_1-a_1,2n_2-a_2\rangle\ \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_1}k_1(a_1-b_1)}\ \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_2}k_2(a_2-b_2)}$$
$$\Downarrow\ \ m_1=a_1-n_1,\ \ m_2=a_2-n_2$$
$$=\tfrac{1}{d_1d_2}\sum_{m_1\in\mathbb{Z}_{d_1}}\ \sum_{m_2\in\mathbb{Z}_{d_2}}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_1}k_1m_1}\ \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_2}k_2m_2}\ \times\langle n_1+m_1,n_2+m_2|A|n_1-m_1,n_2-m_2\rangle$$

$\boxed{D70}$

$$\mathcal{W}_\varrho(n_1,n_2,k_1,k_2)=\tfrac{1}{d_1d_2}\sum_{m_1\in\mathbb{Z}_{d_1}}\ \sum_{m_2\in\mathbb{Z}_{d_2}}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_1}k_1m_1}\ \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_2}k_2m_2}\ \times\langle n_1+m_1,n_2+m_2|\varrho|n_1-m_1,n_2-m_2\rangle$$
$$\Downarrow\ \ \varrho=|\Psi\rangle\langle\Psi|$$
$$\mathcal{W}_\Psi(n_1,n_2,k_1,k_2)=\tfrac{1}{d_1d_2}\sum_{m_1\in\mathbb{Z}_{d_1}}\ \sum_{m_2\in\mathbb{Z}_{d_2}}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_1}k_1m_1}\ \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_2}k_2m_2}\ \times\ \Psi(n_1+m_1,n_2+m_2)\ \overline{\Psi(n_1-m_1,n_2-m_2)}$$

$\boxed{D71}$

$$\mathcal{W}_{\psi\otimes\varphi}(n_1,n_2,k_1,k_2)=$$
$$=\tfrac{1}{d_1d_2}\sum_{m_1\in\mathbb{Z}_{d_1}}\ \sum_{m_2\in\mathbb{Z}_{d_2}}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_1}k_1m_1}\ \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_2}k_2m_2}\ \times\ \psi(n_1+m_1)\,\varphi(n_2+m_2)\ \overline{\psi(n_1-m_1)}\ \overline{\varphi(n_2-m_2)}$$
$$=\mathcal{W}_\psi(n_1,k_1)\,\mathcal{W}_\varphi(n_2,k_2)$$

$\boxed{D72}$

$$|\Psi\rangle=\sum_a\sqrt{\lambda_a}|\psi_a\rangle\otimes|\varphi_a\rangle\equiv\sum_a\sqrt{\lambda_a}|\psi_a\varphi_a\rangle\ \text{(Schmidt form)}$$
$$\Downarrow$$
$$|\Psi\rangle\langle\Psi|=\sum_{a,b}\sqrt{\lambda_a\lambda_b}|\psi_a\varphi_a\rangle\langle\psi_b\varphi_b|$$
$$=\sum_{a,b}\sqrt{\lambda_a\lambda_b}|\psi_a\rangle\langle\psi_b|\otimes|\varphi_a\rangle\langle\varphi_b|$$
$$\Downarrow$$
$$\mathrm{tr}_2|\Psi\rangle\langle\Psi|=\sum_{a,b}\sqrt{\lambda_a\lambda_b}\ \langle\varphi_b|\varphi_a\rangle\ |\psi_a\rangle\langle\psi_b|$$
$$=\sum_{a,b}\sqrt{\lambda_a\lambda_b}\ \delta_{ab}\ |\psi_a\rangle\langle\psi_b|$$
$$=\sum_a\ \lambda_a\ |\psi_a\rangle\langle\psi_a|.$$

$\boxed{D73}$

$\mathcal{H}_1\otimes\mathcal{H}_2\xrightarrow{\varrho}\mathcal{H}_1\otimes\mathcal{H}_2$
linear operator

$$\Downarrow\ \text{(spectral decomposition)}$$
$$\varrho=\sum_j p_j\,|\Psi_j\rangle\langle\Psi_j|$$
$$\Downarrow\ \text{(Schmidt form)}$$
$$\varrho=\sum_j p_j\sum_{a_j,b_j}\sqrt{\lambda_{a_j}\lambda_{b_j}}|\psi_{a_j}\varphi_{a_j}\rangle\langle\psi_{b_j}\varphi_{b_j}|$$
$$\Downarrow\ \boxed{D72}$$
$$\varrho_1=\mathrm{tr}_2\,\varrho=\sum_j p_j\sum_{a_j}\lambda_{a_j}\,|\psi_{a_j}\rangle\langle\psi_{a_j}|$$

and

$$\varrho=\sum_j p_j\sum_{a_j,b_j}\sqrt{\lambda_{a_j}\lambda_{b_j}}|\psi_{a_j}\varphi_{a_j}\rangle\langle\psi_{b_j}\varphi_{b_j}|$$
$$\Downarrow\ \boxed{D65}$$
$$\mathcal{W}_\varrho=\sum_j p_j\sum_{a_j,b_j}\sqrt{\lambda_{a_j}\lambda_{b_j}}\ \mathcal{W}_{|\psi_{a_j}\varphi_{a_j}\rangle\langle\psi_{b_j}\varphi_{b_j}|}$$
$$\Downarrow\ \boxed{D70}$$
$$\sum_{n_2,k_2=-s_2}^{s_2}\mathcal{W}_\varrho(n_1,n_2,k_1,k_2)=\tfrac{1}{d_1d_2}$$
$$\times\sum_j p_j\sum_{a_j,b_j}\sqrt{\lambda_{a_j}\lambda_{b_j}}\sum_{m_1\in\mathbb{Z}_{d_1}}\ \sum_{m_2\in\mathbb{Z}_{d_2}}\ \sum_{n_2,k_2=-s_2}^{s_2}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_1}k_1m_1}\ \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_2}k_2m_2}$$
$$\times\psi_{a_j}(n_1+m_1)\,\varphi_{a_j}(n_2+m_2)\,\overline{\psi_{b_j}(n_1-m_1)}\,\overline{\varphi_{b_j}(n_2-m_2)}$$
$$\Downarrow\ \boxed{D21}$$
$$=\tfrac{1}{d_1}\sum_j p_j\sum_{a_j,b_j}\sqrt{\lambda_{a_j}\lambda_{b_j}}\sum_{m_1\in\mathbb{Z}_{d_1}}\ \sum_{n_2=-s_2}^{s_2}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_1}k_1m_1}$$
$$\times\psi_{a_j}(n_1+m_1)\,\varphi_{a_j}(n_2)\,\overline{\psi_{b_j}(n_1-m_1)}\,\overline{\varphi_{b_j}(n_2)}$$

$$=\tfrac{1}{d_1}\sum_j p_j\sum_{a_j,b_j}\sqrt{\lambda_{a_j}\lambda_{b_j}}\sum_{m_1\in\mathbb{Z}_{d_1}}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_1}k_1m_1}\langle\varphi_{b_j}|\varphi_{a_j}\rangle$$
$$\times\psi_{a_j}(n_1+m_1)\,\overline{\psi_{b_j}(n_1-m_1)}$$

$$=\tfrac{1}{d_1}\sum_j p_j\sum_{a_j}\lambda_{a_j}\sum_{m_1\in\mathbb{Z}_{d_1}}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_1}k_1m_1}\psi_{a_j}(n_1+m_1)\,\overline{\psi_{a_j}(n_1-m_1)}$$
$$=\tfrac{1}{d_1}\sum_j p_j\sum_{a_j}\lambda_{a_j}\sum_{m_1\in\mathbb{Z}_{d_1}}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_1}k_1m_1}\langle n_1+m_1|\psi_{a_j}\rangle\langle\psi_{a_j}|n_1-m_1)\rangle$$
$$\Downarrow\ \boxed{D72}$$
$$=\tfrac{1}{d_1}\sum_{m_1\in\mathbb{Z}_{d_1}}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_1}k_1m_1}\langle n_1+m_1|\varrho_1|n_1-m_1\rangle$$
$$=\mathcal{W}_{\varrho_1}(n_1,k_1).$$

$\boxed{D74}$

$$\begin{aligned}\widetilde{A\otimes B}(n_1,n_2,k_1,k_2) &= \mathrm{tr}(A\otimes B\ \Pi(n_1,n_2,k_1,k_2))\\ &= \mathrm{tr}(A\otimes B\ \Pi(n_1,k_1)\otimes\Pi(n_2,k_2))\\ &= \mathrm{tr}((A\,\Pi(n_1,k_1))\otimes(B\,\Pi(n_2,k_2)))\\ &= \mathrm{tr}(A\,\Pi(n_1,k_1))\ \mathrm{tr}(B\,\Pi(n_2,k_2))\\ &= \tilde{A}(n_1,k_1)\,\tilde{B}(n_2,k_2).\end{aligned}$$

$\boxed{D75}$

$$\begin{aligned}\mathcal{W}_{A\otimes B}(n_1,n_2,k_1,k_2) &= \tfrac{1}{d_1 d_2}\,\widetilde{A\otimes B}(n_1,n_2,k_1,k_2)\\ &= \tfrac{1}{d_1 d_2}\,\tilde{A}(n_1,k_1)\,\tilde{B}(n_2,k_2)\\ &= \mathcal{W}_A(n_1,k_1)\,\mathcal{W}_B(n_2,k_2).\end{aligned}$$

$\boxed{D76}$ *Proof 1.*

$$\begin{aligned}\mathrm{tr}(\varrho(A\otimes\mathbb{I})) &= \sum_{a,b}\langle ab|\varrho(A\otimes\mathbb{I})|ab\rangle\\ &= \sum_{a,b}\sum_{n,k}\langle ab|\varrho|nk\rangle\langle nk|A\otimes\mathbb{I}|ab\rangle\\ &= \sum_{a,b}\sum_{n,k}\langle ab|\varrho|nk\rangle\langle n|A|a\rangle\langle k|b\rangle\\ &= \sum_{a,n}\sum_{b}\langle ab|\varrho|nb\rangle\langle n|A|a\rangle\\ &= \sum_{a,n}\langle a|\varrho_1|n\rangle\langle n|A|a\rangle\\ &= \sum_{a}\langle a|\varrho_1 A|a\rangle\\ &= \mathrm{tr}(\varrho_1\,A).\end{aligned}$$

Proof 2.

$$\begin{aligned}\mathrm{tr}(\varrho(A\otimes\mathbb{I})) &= \sum_{n_1,n_2,k_1,k_2}\mathcal{W}_\varrho(n_1,n_2,k_1,k_2)\,\widetilde{A\otimes\mathbb{I}}(n_1,n_2,k_1,k_2)\\ &= \sum_{n_1,n_2,k_1,k_2}\mathcal{W}_\varrho(n_1,n_2,k_1,k_2)\,\widetilde{A\otimes\mathbb{I}}(n_1,n_2,k_1,k_2)\\ &\Downarrow\ \boxed{D74}\\ &= \sum_{n_1,n_2,k_1,k_2}\mathcal{W}_\varrho(n_1,n_2,k_1,k_2)\,\tilde{A}(n_1,k_1)\ \tilde{\mathbb{I}}(n_2,k_2)\\ &\Downarrow\ \boxed{D42}\\ &= \sum_{n_1,n_2,k_1,k_2}\mathcal{W}_\varrho(n_1,n_2,k_1,k_2)\,\tilde{A}(n_1,k_1)\\ &= \sum_{n_1,k_1}\sum_{n_2,k_2}\mathcal{W}_\varrho(n_1,n_2,k_1,k_2)\,\tilde{A}(n_1,k_1)\\ &\Downarrow\ \boxed{D73}\\ &= \sum_{n_1,k_1}\mathcal{W}_{\varrho_1}(n_1,k_1)\,\tilde{A}(n_1,k_1)\\ &\Downarrow\ \boxed{D40}\\ &= \mathrm{tr}(\varrho_1\,A).\end{aligned}$$

$\boxed{D77}$

$$\begin{aligned}|\Psi\rangle &= \sum\sqrt{\lambda_j}|\varphi_j\rangle\otimes|\varphi_j\rangle\\ &\Downarrow\\ |\Psi\rangle\langle\Psi| &= \sum\sqrt{\lambda_j\lambda_k}|\varphi_j\varphi_j\rangle\langle\varphi_k\varphi_k|\\ &= \sum\sqrt{\lambda_j\lambda_k}|\varphi_j\rangle\langle\varphi_k|\otimes|\varphi_j\rangle\langle\varphi_k|\\ &\Downarrow\\ \mathcal{W}_\Psi(n_1,n_2,k_1,k_2) &= \sum_{j,k}\sqrt{\lambda_j\lambda_k}\,\mathcal{W}_{|\varphi_j\rangle\langle\varphi_k|}(n_1,k_1)\\ &\qquad\times\mathcal{W}_{|\varphi_j\rangle\langle\varphi_k|}(n_2,k_2)\\ &\Downarrow\ \boxed{D39}\\ \sum_{n_2,k_2}\mathcal{W}_\Psi(n_1,n_2,k_1,k_2) &= \sum_{j,k}\sqrt{\lambda_j\lambda_k}\,\mathcal{W}_{|\varphi_j\rangle\langle\varphi_k|}(n_1,k_1)\\ &\qquad\times\mathrm{tr}(|\varphi_j\rangle\langle\varphi_k|)\\ &= \sum_{j,k}\sqrt{\lambda_j\lambda_k}\,\mathcal{W}_{|\varphi_j\rangle\langle\varphi_k|}(n_1,k_1)\\ &\qquad\times\langle\varphi_k|\varphi_j\rangle\\ &= \sum_{j,k}\sqrt{\lambda_j\lambda_k}\,\mathcal{W}_{|\varphi_j\rangle\langle\varphi_k|}(n_1,k_1)\\ &\qquad\times\delta_{kj}\\ &= \sum_j\lambda_j\,\mathcal{W}_{|\varphi_j\rangle\langle\varphi_j|}(n_1,k_1)\\ &\Downarrow\ \varrho=\textstyle\sum\lambda_j|\varphi_j\rangle\langle\varphi_j|\\ &= \mathcal{W}_\varrho(n_1,k_1).\end{aligned}$$

$\boxed{D81}$

See [14].

$\boxed{D82}$

See [14].

$\boxed{\text{F}}$ SOURCE CODES

$\boxed{1}$

The discrete Wigner function for given d and σ, can be computed by using the following program in Mathematica:

```
d=3;     sigma={{2,1},{1,1}};       j=(d-1)/2
sigma11=sigma[[1]][[1]]; sigma22=sigma[[2]][[2]]; sigma12=sigma[[1]][[2]];
Eigenvalues[{{sigma11,sigma12+I},{sigma12-I,sigma22}}]
w[q_,p_]:= Exp[-(2 Pi/d) (sigma22 q^2-2 sigma12 q p+sigma11 p^2)/Det[sigma]]
Cw=N[Sum[(-1)^(alpha beta) w[alpha/2,beta/2],{alpha,-30,30},{beta,-30,30} ]]
DiscreteWigner[n_,k_]:=N[Sum[(-1)^(alpha beta) w[n+alpha d/2,k+beta d/2],
                                                     {alpha,-30,30},{beta,-30,30}]/Cw]
MatrixForm[Table[DiscreteWigner[a,b],{a,-j,j},{b,-j,j}]]
DiscretePlot3D[ DiscreteWigner[n, k], {n,-j,j}, {k,-j,j}, ExtentSize -> 0.5, AxesLabel -> Automatic,
                                       ColorFunction -> Function[{x,y,z}, If[z >= 0, White, Black]],
                                             ColorFunctionScaling -> False, BoxRatios -> {1.2,1,2}]
```

$\boxed{2}$

For given d and σ, the matrix of the density operator ϱ can be obtained by using the following program

```
d=3;     sigma={{2,1},{1,1}};       j=(d-1)/2
sigma11=sigma[[1]][[1]]; sigma22=sigma[[2]][[2]]; sigma12=sigma[[1]][[2]];
Eigenvalues[{{sigma11,sigma12+I},{sigma12-I,sigma22}}]
w[q_,p_]:= Exp[-(2 Pi/d) (sigma22 q^2-2 sigma12 q p+sigma11 p^2)/Det[sigma]]
Cw=N[Sum[(-1)^(alpha beta) w[alpha/2,beta/2],{alpha,-30,30},{beta,-30,30} ]]
DiscreteWigner[n_,k_]:=N[Sum[(-1)^(alpha beta) w[n+alpha d/2,k+beta d/2],
                                                     {alpha,-30,30},{beta,-30,30}]/Cw]
rho:=Table[Sum[Exp[2 Pi I k (a-b)/d] DiscreteDelta[Mod[2 n-a-b,d]] DiscreteWigner[n,k],
                                                 {n,-j,j},{k,-j,j}],{a,-j,j},{b,-j,j}]
MatrixForm[rho]
```

$\boxed{3}$

By using the program

```
d=3;     sigma11=3;    sigma22=2;    sigma12=Sqrt[sigma11 sigma22-1];   j=(d-1)/2
w[q_,p_]:=Exp[-(2 Pi/d) (sigma22 q^2-2 sigma12 q p+sigma11 p^2)]
Cw=N[Sum[(-1)^(alpha beta) w[alpha/2,beta/2],{alpha,-30,30},{beta,-30,30} ]]
DiscreteWigner[n_,k_]:=N[Sum[(-1)^(alpha beta) w[n+alpha d/2,k+beta d/2],
                                                     {alpha,-30,30},{beta,-30,30}]/Cw]
rho:=Table[Sum[Exp[2 Pi I k (a-b)/d] DiscreteDelta[Mod[2 n-a-b,d]] DiscreteWigner[n,k],
                                                 {n,-j,j},{k,-j,j}],{a,-j,j},{b,-j,j}]
Eigenvalues[rho]
MatrixForm[rho.rho-rho]
```

one can compute the eigenvalues of ϱ_σ and see that $\boxed{\det\sigma=1 \;\Rightarrow\; \varrho_\sigma^2=\varrho_\sigma}$ (ϱ_σ is a pure state).

$\boxed{4}$

By using the program

```
d=3;        nu1=2;        nu2=3;     j=(d-1)/2
w[q_, p_,nu_]:= Exp[-(2 Pi/d) (q^2+p^2)/nu]
Cw[nu_]:=N[Sum[(-1)^(alpha beta) w[alpha/2,beta/2,nu],{alpha,-30,30},{beta,-30,30} ]]
DiscreteWigner[n_,k_,nu_]:=N[Sum[(-1)^(alpha beta) w[n+alpha d/2,k+beta d/2,nu],
                                                    {alpha,-30,30},{beta,-30,30}]/Cw[nu]]
rho[nu_]:=Table[Sum[Exp[2 Pi I k (a-b)/d] DiscreteDelta[Mod[2 n-a-b,d]] DiscreteWigner[n,k,nu],
                                                      {n,-j,j},{k,-j,j}],{a,-j,j},{b,-j,j}]
A=rho[nu1]
B=rho[nu2]
MatrixForm[A.B-B.A]
MatrixForm[Eigenvectors[A]]
MatrixForm[Eigenvectors[B]]
```

one can see that the density operators of two thermal states commute, and compute the corresponding common eigenvectors.

$\boxed{5}$

The values from Table 3 have been obtained by using the program

```
d := 5;   A := N[Pi/4];    B := 0;   sigmai := {{1, 1}, {1, 2}};    j := (d - 1)/2
S := (1/2) {{1, 1}, {-I, I}} . MatrixExp[-I {{A, B}, {-Conjugate[B], -A}}] . {{1, I}, {1, -I}}
sigmaf := S.sigmai.Transpose[S]
Q := Table[N[n DiscreteDelta[n - m]], {n, -j, j}, {m, -j, j}]
P := Table[  N[(1/d) Sum[k Exp[2 Pi I (n - m) k/d], {k, -j, j}]], {n, -j,    j}, {m, -j, j}]
ap := Sqrt[Pi/d] (Q + I P)
am := Sqrt[Pi/d] (Q - I P)
H := A am . ap + B am . am + Conjugate[B] ap . ap + A  ap.am
U := MatrixExp[(-I/2) H]
............................................
rhoi[n, k] := ......... (computed in Fortran)
...........................................
rhoint := Table[rhoi[n, k], {n, -j, j}, {k, -j, j}]
............................................
rhof[n, k] := ......... (computed in Fortran)
...........................................
rhofin := Table[rhof[n, k], {n, -j, j}, {k, -j, j}]
rhorho := U.rhoint.ConjugateTranspose[U] - rhofin
Print["S=", MatrixForm[S]]
Print["U=", MatrixForm[U]]
Print["sigma initial=", MatrixForm[sigmai]]
Print["sigma final=", MatrixForm[sigmaf]]
Print["rho initial=", MatrixForm[rhoint]]
Print["rho final=", MatrixForm[rhofin]]
MatrixForm[Abs[rhorho]]
Norm[rhorho]
```

with the density matrices computed by using the program in Fortran

```
PROGRAM DENSITY
INTEGER :: D=13
REAL :: sig11=2.0
REAL :: sig12=1.0
REAL :: sig22=1.0
INTEGER :: N,K,L,M, J, DELTA
REAL :: Rd, Cd, WIGNER
COMPLEX :: I=(0,1)
COMPLEX :: RHO
J=(D-1)/2
Rd=2*3.141592654/(D*sig11*sig22-D*sig12**2)
Cd=0
DO 50 Lb=-100,100
DO 60 Mb=-100,100
Cd=Cd+(-1)**(Lb*Mb)*EXP(-Rd*(sig22*Lb**2-2*sig12*Lb*Mb+sig11*Mb**2)/4)
60  CONTINUE
50  CONTINUE
DO 10 N=-J,J
DO 20 K=-J,J
RHO=0
DO 30 L=-J,J
DO 40 M=-J,J
RHO=RHO+WIGNER(L,M,sig11,sig22,sig12,Rd,D)*EXP(2*3.141592654*I*M*(N-K)/D)*DELTA(N+K,2*L,D)
40  CONTINUE
30  CONTINUE
RHO=RHO/Cd
IF(AIMAG(RHO).LT.0) THEN
PRINT 5, ' rhoi[', N,',',K,']:=',REAL(RHO), ' - I ', -AIMAG(RHO)
ELSE
PRINT 5, ' rhoi[', N,',',K,']:=',REAL(RHO), ' + I ', AIMAG(RHO)
END IF
5 FORMAT(A,I3,A,I3,A,F10.7,A,F9.7)
20  CONTINUE
10 CONTINUE
END
```

```
INTEGER FUNCTION DELTA(Ia,Ja,Da)
INTEGER :: Ia, Ja, Da
DELTA=0
IF(MOD(Ia-Ja,Da)==0) DELTA=1
RETURN
END

REAL FUNCTION WIGNER(Nb,Kb,Sa,Sb,Sc,Rb,Db)
INTEGER :: Nb,Kb,Lb,Mb, Db
REAL :: Sa, Sb, Sc, Rb
WIGNER=0
DO 70 Lb=-100,100
DO 80 Mb=-100,100
WIGNER=WIGNER+(-1)**(Lb*Mb)*EXP(-Rb*(Sb*(Nb+Lb*Db*0.5)**2-2*Sc*(Nb+Lb*Db*0.5)*(Kb+Mb*Db*0.5)
                                                                      +Sa*(Kb+Mb*Db*0.5)**2))
80  CONTINUE
70  CONTINUE
RETURN
END
```

$\boxed{6}$

The numerical data from Table 4 are obtained by using

```
d = 9;    nu = 2;    mu = 3;    m = 1;    l = 2
w[q_, p_, nuu_]:= Exp[-(2 Pi/d) (q^2 + p^2)/nuu];     j=(d-1)/2
Cw[nuu_] :=N[Sum[(-1)^(alpha beta) w[alpha/2, beta/2, nuu],{alpha,-50,50}, {beta,-50,50}]]
DiscreteWigner[n_,k_,nuu_]:=N[Sum[(-1)^(alpha beta) w[n+alpha d/2,k+beta d/2,nuu],
                                             {alpha,-50,50},{beta,-50,50}]/Cw[nuu]]
f[mm_,ll_]:=Sum[Exp[-4 Pi I (k mm-n ll)/d]DiscreteWigner[n,k,nu]DiscreteWigner[n+mm,k+ll,mu],
                                                            {n,-j,j},{k,-j,j}]
f[m,l]-f[l,m]
```

$\boxed{7}$

The matrix $(|\langle e_n|\mathbf{g}_1\rangle|)$ has been obtained by using the program

```
d=11;        kappa=1;      j=(d-1)/2
gaus[m_]:=N[Sum[Exp[-kappa Pi (a d+m)^2/d],{a,-50,50}]]
gauss:=Table[gaus[m]/Sqrt[Sum[(gaus[n])^2,{n,-j,j}]],{m,-j,j}]
Q=Table[N[n DiscreteDelta[n-m]],{n,-j,j},{m,-j,j}]
P=Table[N[(1/d) Sum[k Exp[2 Pi I(n-m)k/d],{k,-j,j}]],{n,-j,j},{m,-j,j}]
Id=Table[N[DiscreteDelta[n-m]],{n,-j,j},{m,-j,j}]
QPPQ=N[Q.P-P.Q-I(d/(2 Pi))Id]
MatrixForm[Table[{n,Im[Eigenvalues[QPPQ][[n]]] I},{n,1,d}]]
T=Table[Eigenvectors[QPPQ][[d-n+1]][[k]],{k,1,d},{n,1,d}]
MatrixForm[Abs[MatrixPower[T,-1].Transpose[{gauss}]]]
```

and the matrix $(|\langle e_n\,|\varrho_\sigma|\,e_m\rangle|)$ for $\sigma=\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}$ has been obtained by using the program

```
d=11;     sigma={{2,1},{1,1}};       j=(d-1)/2
sigma11=sigma[[1]][[1]]; sigma22=sigma[[2]][[2]]; sigma12=sigma[[1]][[2]];
Eigenvalues[{{sigma11,sigma12+I},{sigma12-I,sigma22}}]
w[q_,p_]:= Exp[-(2 Pi/d) (sigma22 q^2-2 sigma12 q p+sigma11 p^2)/Det[sigma]]
Cw=N[Sum[(-1)^(alpha beta) w[alpha/2,beta/2],{alpha,-20,20},{beta,-20,20} ]]
DiscreteWigner[n_,k_]:=N[Sum[(-1)^(alpha beta) w[n+alpha d/2,k+beta d/2],
                                                     {alpha,-20,20},{beta,-20,20}]/Cw]
rho:=Table[Sum[Exp[2 Pi I k (a-b)/d] DiscreteDelta[Mod[2 n-a-b,d]] DiscreteWigner[n,k],
                                                 {n,-j,j},{k,-j,j}],{a,-j,j},{b,-j,j}]
Q=Table[N[n DiscreteDelta[n-m]],{n,-j,j},{m,-j,j}]
P=Table[N[(1/d) Sum[k Exp[2 Pi I (n-m) k/d],{k,-j,j}]],{n,-j,j},{m,-j,j}]
Id=Table[N[DiscreteDelta[n-m]],{n,-j,j},{m,-j,j}]
QPPQ=N[Q.P-P.Q-I(d/(2 Pi))Id]
MatrixForm[Table[{n,Im[Eigenvalues[QPPQ][[n]]] I}, {n,1,d}]]
T=Table[Eigenvectors[QPPQ][[d-n+1]][[k]],{k,1,d},{n,1,d}]
MatrixForm[Abs[MatrixPower[T,-1].rho.T]]
```

$\boxed{8}$

The numerical data presented at $\boxed{B16}$ have been obtained by using the program

```
d = 31; kappa = 1; s = (d - 1)/2
gaus[m_] := N[Sum[Exp[-kappa Pi (a d + m)^2/d], {a, -50, 50}]]
gauss := Table[gaus[m]/Sqrt[Sum[(gaus[n])^2, {n, -s, s}]], {m, -s, s}]
Q := Table[N[n DiscreteDelta[n - m]], {n, -s, s}, {m, -s, s}]
P := Table[N[(1/d) Sum[k Exp[2 Pi I (n - m) k/d], {k, -s, s}]], {n, -s, s}, {m, -s, s}]
a := Sqrt[Pi/d] (Q + I P)
MatrixForm[a . Transpose[{gauss}]]
```

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