

1 Improper Integrals

Exercise 1.1 Test the following integrals for convergence:

$$a) I = \int_1^{\infty} \frac{1}{x^2(1+e^x)} dx; \quad b) I = \int_1^{\infty} \frac{\sin x}{x^3} dx; \quad c) I = \int_1^{\infty} \frac{\sin x}{x} dx.$$

Exercise 1.2 Evaluate the integral

$$I = \int_0^1 \frac{1}{\sqrt{x}} dx.$$

Exercise 1.3 Evaluate the integral

$$I = \int_0^{\infty} x e^{-x} dx.$$

Exercise 1.4 Evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx.$$

Exercise 1.5 Study the nature of the integral

$$I = \int_1^{\infty} e^{-x^2} dx.$$

Exercise 1.6 Evaluate the integral

$$I = \int_e^{\infty} \frac{1}{x \ln x} dx.$$

Exercise 1.7 Show that the improper integral

$$I = \int_0^1 \frac{1}{\sqrt{1-x^2}} dx$$

is convergent.

Exercise 1.8 Determine whether the improper integral

$$I = \int_{-\infty}^{\infty} \frac{x^2}{9+x^6} dx$$

is convergent or divergent.

Exercise 1.9 Determine the nature of the integral

$$I = \int_0^1 \frac{1}{\sqrt{x(1+e^x)}} dx.$$

Exercise 1.10 Test the following improper integrals for convergence:

$$1) I = \int_0^{\infty} \frac{1}{\sqrt{x+1}} dx; \quad 2) I = \int_0^{\infty} 3e^x dx; \quad 3) I = \int_1^{\infty} \frac{dx}{x^5}; \quad 4) I = \int_1^{\infty} \frac{dx}{x^4(1+e^x)}.$$

Exercise 1.11 Discuss the nature of the following improper integrals:

$$1) I = \int_1^{\infty} \frac{x+1}{\sqrt{x^3}} dx; \quad 2) I = \int_1^{\infty} \frac{\sin x}{x^5} dx; \quad 3) I = \int_1^{\infty} \frac{\cos x}{x} dx.$$

2 Line Integrals

Exercise 2.1 Compute the integral

$$I = \int_{\gamma} x y \, dl,$$

where γ is the quarter of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ lying in the first quadrant ($x, y \geq 0$).

Exercise 2.2 Compute

$$I = \int_{\gamma} (x + y) \, dl,$$

where γ is the line segment joining the points $A(1, 1)$ and $B(2, 3)$.

Exercise 2.3 Compute the integral

$$I = \int_{\gamma} -x^2 y \, dx + x y^2 \, dy,$$

where γ is the circle $x^2 + y^2 = 9$ (positively oriented).

Exercise 2.4 Compute the line integral $\int_{\gamma} (x + y - z) \, dl$, where γ is the line segment from $(0, 0, 0)$ to $(1, 1, 1)$.

Exercise 2.5 Evaluate the integral

$$I = \int_{\gamma} y \, dx + (x + z) \, dy - y \, dz$$

where γ is the line segment joining the points $(0, 1, 1)$ and $(1, 1, 0)$.

Exercise 2.6 Compute, directly and by using Green's formula, the integral

$$I = \int_{\gamma} -x^2 y \, dx + x y^2 \, dy,$$

where γ is the circle $x^2 + y^2 = 9$.

Exercise 2.7 Compute, directly and by using Green's formula, the integral

$$I = \int_{\gamma} y^2 \, dx + x^2 \, dy,$$

where γ is the boundary of the domain

$$D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4, x, y \geq 0\}.$$

Exercise 2.8 Evaluate (directly and, also, by using Green's formula) the integral

$$I = \int_{\gamma} (x + y) \, dx + x y \, dy,$$

where γ is the boundary of the domain

$$D = \left\{ (x, y) \in \mathbb{R}^2 \mid \frac{x^2}{4} + y^2 \leq 1, y \geq 0 \right\}$$

3 Multiple Integrals

Exercise 3.1 Evaluate each of the following iterated integrals:

- a) $\int_0^2 \int_{-1}^1 (6x^2 + 2y) \, dy \, dx;$
b) $\int_0^{\pi/2} \int_0^{\pi} (x \sin y + y \cos x) \, dy \, dx.$

Exercise 3.2 Draw the domain of integration and, then, change the order of integration in the integral

$$I = \int_0^1 \int_{x^2}^x (x + y) \, dy \, dx.$$

Exercise 3.3 Sketch the region of integration and change the order of integration in the integral

$$I = \int_{-2}^2 \int_{x^2}^{8-x^2} (2x + y) \, dy \, dx.$$

Exercise 3.4 Let

$$D = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 4, y \leq x, x \geq 0\}.$$

Draw the domain D and compute the integral

$$I = \iint_D y \, dx \, dy.$$

Exercise 3.5 Using polar coordinates, evaluate the area of the finite domain bounded by the Oy -axis, the line $y = 3$, and a quarter of the circle of radius 3 with center at the point $(3, 0)$. Sketch the domain of integration.

Exercise 3.6 Compute the following integrals:

$$1) \quad I = \iint_D \frac{x^2}{1+y^2} \, dx \, dy, \quad (D) : \begin{cases} x = 0, & x = 1 \\ y = 0, & y = 1 \end{cases}$$

$$2) \quad I = \iint_D \frac{x^2}{y^2} \, dx \, dy, \quad (D) : \begin{cases} y = x, \\ y = \frac{1}{x}, \\ x \in [1, 2] \end{cases}$$

$$3) \quad I = \iint_D (x^2 + y) \, dx \, dy, \quad (D) : \begin{cases} y = x^2, \\ y^2 = x \end{cases}$$

$$4) \quad I = \iint_D (x - y) \, dx \, dy, \quad (D) : \begin{cases} y = 2 - x^2, \\ y = 2x - 1 \end{cases}$$

Exercise 3.7 Evaluate the following double integrals:

$$1) \quad I = \iint_D dx \, dy, \quad D = \left\{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4, x^2 + \frac{y^2}{4} \geq 1, x \geq 0 \right\};$$

$$2) \quad I = \iint_D dx \, dy, \quad D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 2, x \leq y^2, x \geq -y^2, y \leq 0\};$$

$$3) \quad I = \iint_D xy^2 \, dx \, dy, \quad (D) : \begin{cases} y^2 = 2px, \\ x = \frac{p}{2}, p > 0. \end{cases}$$

Exercise 3.8 Compute the following double integrals:

$$1) \quad I = \iint_D (x^2 + y^2)^2 \, dx \, dy, \quad D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1, x, y \geq 0\};$$

$$2) \quad \mu(D) = \iint_D dx \, dy, \quad D = \{(x, y) \in \mathbb{R}^2 \mid a^2 \leq x^2 + y^2 \leq b^2, y \geq x, b > a > 0\}.$$

Exercise 3.9 Evaluate the following integrals:

$$1) \quad I = \iint_D \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \, dx \, dy, \quad D = \left\{ (x, y) \in \mathbb{R}^2 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1, x, y \geq 0, a, b > 0 \right\}$$

$$2) \quad I = \iint_D (x^2 + y^2) \, dx \, dy, \quad (D) : \begin{cases} x^2 + y^2 = a^2 \\ y = x\sqrt{3}, y\sqrt{3} = x \end{cases}$$

$$3) \quad I = \iint_D \sin \sqrt{x^2 + y^2} \, dx \, dy, \\ D = \{(x, y) \in \mathbb{R}^2 \mid \pi^2 \leq x^2 + y^2 \leq 4\pi^2\}.$$

Exercise 3.10 Compute the area of the following domain (Bernoulli's lemniscate):

$$D = \{(x, y) \in \mathbb{R}^2 \mid (x^2 + y^2)^2 \leq x^2 - y^2\}.$$

Exercise 3.11 Compute the area of the following domains:

$$D = \{(x, y) \in \mathbb{R}^2 \mid (x^2 + y^2)^2 \leq a^2 x^2 - b^2 y^2, a, b > 0\};$$

$$D = \left\{ (x, y) \in \mathbb{R}^2 \mid \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right)^2 \leq \frac{x^2}{a^2} - \frac{y^2}{b^2}, x \geq 0, a, b > 0 \right\}.$$

Exercise 3.12 Evaluate the integral

$$I = \iint_D (x^2 + y^2)^2 \, dx \, dy,$$

where

$$D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4, x \geq 0\}.$$

Exercise 3.13 Evaluate the integral

$$I = \iint_D (x + y) \, dx \, dy,$$

where

$$D = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 4, y \geq 0\}.$$

Exercise 3.14 Compute

$$I = \iiint_D dx \, dy \, dz,$$

where

$$D = \left\{ (x, y, z) \in \mathbb{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1, a, b, c > 0 \right\}.$$

Exercise 3.15 Compute the volume of the solid D lying inside the sphere $x^2 + y^2 + z^2 = 64$ and outside the sphere $x^2 + y^2 + z^2 = 1$.

Exercise 3.16 Compute the volume of the solid D bounded by $x^2 + y^2 = 1$, $x^2 + y^2 = 9$, $z = 0$ and $z = 4$. Draw the domain of integration.

Exercise 3.17 Find the volume of the solid ball (D): $x^2 + y^2 + z^2 \leq 4$.

Exercise 3.18 Find the volume inside a solid hemisphere of radius 3 and center at $(0, 0, 0)$ lying in the upper half space $z \geq 0$.

Exercise 3.19 Find the volume of the domain

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq a^2, x, y, z \geq 0, a > 0\}.$$

Exercise 3.20 Compute the volume of the following region:

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid (x^2 + y^2 + z^2)^2 \leq az, a > 0\}.$$

Exercise 3.21 Find the volume of the following domain:

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid (x^2 + y^2 + z^2)^2 \leq x^2 + y^2\}.$$