Pure and mixed discrete variable Gaussian states

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Abstract. A Gaussian function of discrete variable can be associated in a natural way to a Gaussian function of continuous variable in each odd-dimensional Hilbert space. The discrete Fourier transform and discrete Wigner distribution of such a function can be expressed in terms of the corresponding continuous counterpart. By extending this continuous-discrete correspondence up to all the continuous variable Gaussian states, we define some new (to our knowledge) pure and mixed discrete variable Gaussian states, and investigate some of their properties.

1. Introduction

There exist several experimental possibilities to obtain quantum systems with a finite-dimensional Hilbert space. New mathematical models are necessary for an adequate description of these systems.

We consider only quantum systems with an odd-dimensional ($d=2s+1$) Hilbert space $\mathcal{H}$. The space $\mathcal{H}$ can be regarded as the space of all the functions

$$\psi:\{-s,-s+1,...,s-1,s\}\to\mathbb{C} \tag{1}$$

with the inner product

$$\langle\varphi|\psi\rangle=\sum_{n=-s}^{s}\overline{\varphi(n)}\,\psi(n). \tag{2}$$

Each function $\psi\in\mathcal{H}$ can be extended up to a periodic function $\psi:\mathbb{Z}\to\mathbb{C}$ of period d, and $\mathcal{H}$ can also be regarded as the space of all the functions $\psi:\mathbb{Z}_d\to\mathbb{C}$.
The canonical basis $\{\delta_{-s},\ \delta_{-s+1},...,\delta_{s-1},\delta_s\}$, where

$$\delta_m(n)=\delta_{mn}^{[d]}=\left\{\begin{array}{lll} 1 & \text{if} & n=m \ \text{ modulo } d\\ 0 & \text{if} & n\neq m \ \text{ modulo } d\end{array}\right. \tag{3}$$

is an orthonormal basis. By using Dirac's notation $|m\rangle$ instead of δ_m, we have

$$\langle m|k\rangle=\delta_{mk}^{[d]},\qquad \sum_{m=-s}^{s}|m\rangle\langle m|=\mathbb{I}, \tag{4}$$

where $\mathbb{I}\!:\!\mathcal{H}\!\rightarrow\!\mathcal{H},\ \ \mathbb{I}\psi\!=\!\psi,$ is the identity operator.
The discrete *Fourier transform* [1]

$$\mathfrak{F}\!:\!\mathcal{H}\!\rightarrow\!\mathcal{H}\!:\!\psi\mapsto\mathfrak{F}[\psi],, \qquad \mathfrak{F}[\psi](k)\!=\!\tfrac{1}{\sqrt{d}}\sum_{n=-s}^{s}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}kn}\,\psi(n) \tag{5}$$

is a unitary transform.
The self-adjoint operator ($-s\leq\mathring{n}\leq s$ is the representative modulo d of n)

$$\hat{\mathfrak{q}}\!:\!\mathcal{H}\!\rightarrow\!\mathcal{H}\!:\!\psi\mapsto\hat{\mathfrak{q}}\psi, \qquad (\hat{\mathfrak{q}}\psi)(n)\!=\!\mathring{n}\,\psi(n) \tag{6}$$

corresponds to the *coordinate operator* $\hat{q}$, and

$$\hat{\mathfrak{p}}\!:\!\mathcal{H}\!\rightarrow\!\mathcal{H}, \qquad \hat{\mathfrak{p}}\!=\!\mathfrak{F}^{\dagger}\hat{\mathfrak{q}}\mathfrak{F} \tag{7}$$

corresponds to the *momentum operator* $\hat{p}$ from the continuous case.
The operators (the addition is modulo d)

$$\begin{array}{ll}\mathcal{H}\!\rightarrow\!\mathcal{H}\!:\!\psi\!\mapsto\!\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k\hat{\mathfrak{q}}}\psi, & \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k\hat{\mathfrak{q}}}\psi(n)\!=\!\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kn}\psi(n),\\ \mathcal{H}\!\rightarrow\!\mathcal{H}\!:\!\psi\!\mapsto\!\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}n\hat{\mathfrak{p}}}\psi, & \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}n\hat{\mathfrak{p}}}\psi(m)\!=\!\psi(m\!-\!n),\end{array} \tag{8}$$

are the *translation operators*, and

$$\mathfrak{D}(n,k)\!:\!\mathcal{H}\!\rightarrow\!\mathcal{H}, \qquad \begin{array}{l}\mathfrak{D}(n,k)\!=\!\mathrm{e}^{-\frac{\pi\mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k\hat{\mathfrak{q}}}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}n\hat{\mathfrak{p}}},\\ \qquad\qquad\ =\!\mathrm{e}^{\frac{\pi\mathrm{i}}{d}nk}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}n\hat{\mathfrak{p}}}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k\hat{\mathfrak{q}}},\\ \mathfrak{D}(n,k)\psi(m)\,=\,\mathrm{e}^{-\frac{\pi\mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}km}\,\psi(m\!-\!n)\end{array} \tag{9}$$

are the *displacement operators*. The *parity operators*

$$\Pi(n,k)\!:\!\mathcal{H}\!\rightarrow\!\mathcal{H}, \qquad \begin{array}{l}\Pi(n,k)\!=\!\mathfrak{D}(n,k)\,\Pi\,\mathfrak{D}^{\dagger}(n,k)\\ \Pi(n,k)\psi(m)\,=\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}2k(n-m)}\,\psi(2n\!-\!m)\end{array} \tag{10}$$

where $\Pi\psi(n)=\psi(-n)$, form an orthogonal basis in the real Hilbert space

$$\mathcal{A}(\mathcal{H})\!=\!\{\,A\!:\!\mathcal{H}\!\rightarrow\!\mathcal{H}\ |\ \ A^{\dagger}\!=\!A\,\} \tag{11}$$

of all the self-adjoint operators, namely

$$\langle\Pi(n,k),\Pi(m,\ell)\rangle\!=\!\mathrm{tr}(\Pi(n,k)\,\Pi(m,\ell))\!=\!d\,\delta_{nm}\,\delta_{k\ell}. \tag{12}$$

Any density operator $\varrho\!:\!\mathcal{H}\!\rightarrow\!\mathcal{H}$ can be written as [2, 3]

$$\varrho\!=\!\sum_{n,k}\mathfrak{W}_{\varrho}(n,k)\,\Pi(n,k), \tag{13}$$

where (the addition is modulo d)

$$\begin{array}{l}\mathfrak{W}_{\varrho}\!:\!\{-s,-s\!+\!1,...,s\!-\!1,s\}\times\{-s,-s\!+\!1,...,s\!-\!1,s\}\longrightarrow\mathbb{R},\\ \mathfrak{W}_{\varrho}(n,k)\!=\!\frac{1}{d}\ \mathrm{tr}(\varrho\,\Pi(n,k))\!=\!\frac{1}{d}\sum\limits_{m=-s}^{s}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\,\langle n\!+\!m|\varrho|n\!-\!m\rangle\end{array} \tag{14}$$

is the discrete *Wigner function* of ϱ. The Wigner function of a pure state $\varrho\!=\!|\psi\rangle\langle\psi|$ is

$$\mathfrak{W}_{\psi}(n,k)\!=\!\frac{1}{d}\langle\psi|\Pi(n,k)|\psi\rangle\!=\!\frac{1}{d}\sum_{m=-s}^{s}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\,\psi(n\!+\!m)\,\overline{\psi(n\!-\!m)}. \tag{15}$$

More details and proofs can be found, for example, in [14, 15].

2. Gaussian functions of one discrete variable

For any $\kappa \in (0,\infty)$, the real function

$$g_\kappa : \mathbb{R} \to \mathbb{R}, \quad g_\kappa(q) = \mathrm{e}^{-\frac{\kappa}{2\hbar}q^2} = \mathrm{e}^{-\frac{\kappa\pi}{h}q^2} \tag{16}$$

is a *Gaussian function of continuous variable.* Its Fourier transform $\mathcal{F}[g_\kappa] : \mathbb{R} \to \mathbb{R}$,

$$\mathcal{F}[g_\kappa](p) = \frac{1}{\sqrt{h}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{2\pi\mathrm{i}}{h}pq}\, g_\kappa(q)\, dq = \frac{1}{\sqrt{\kappa}} \mathrm{e}^{-\frac{\pi}{\kappa h}p^2}, \tag{17}$$

is also a Gaussian function, and

$$\mathcal{F}[g_\kappa] = \tfrac{1}{\sqrt{\kappa}} g_{\kappa^{-1}}. \tag{18}$$

The Wigner function of g_κ is $\mathcal{W}_{g_\kappa} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$,

$$\begin{aligned} \mathcal{W}_{g_\kappa}(q,p) &= \tfrac{2}{h} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{4\pi\mathrm{i}}{h}px}\, g_\kappa(q+x)\, \overline{g_\kappa(q-x)}\, dx, \\ &= \sqrt{\tfrac{2}{\kappa h}}\, \mathrm{e}^{-\frac{2\pi}{h}\begin{pmatrix} q & p \end{pmatrix} \begin{pmatrix} \kappa^{-1} & 0 \\ 0 & \kappa \end{pmatrix}^{-1} \begin{pmatrix} q \\ p \end{pmatrix}}. \end{aligned} \tag{19}$$

The function $\mathfrak{g}_\kappa : \{-s, -s+1, ..., s-1, s\} \to \mathbb{R}$,

$$\begin{aligned} \mathfrak{g}_\kappa(n) &= \sum_{\alpha=-\infty}^{\infty} \mathrm{e}^{-\frac{\kappa\pi}{d}(n+\alpha d)^2} \\ &= \sum_{\alpha=-\infty}^{\infty} g_\kappa\Big((n+\alpha d)\sqrt{\tfrac{h}{d}}\Big) \end{aligned} \tag{20}$$

can be regarded as a *Gaussian function of discrete variable.*
Its discrete Fourier transform $\mathfrak{F}[\mathfrak{g}_\kappa] : \{-s, -s+1, ..., s-1, s\} \to \mathbb{R}$,

$$\begin{aligned} \mathfrak{F}[\mathfrak{g}_\kappa](k) &= \tfrac{1}{\sqrt{d}} \sum_{n=-s}^{s} \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}kn}\, \mathfrak{g}_\kappa(n) \\ &= \tfrac{1}{\sqrt{\kappa}} \sum_{\alpha=-\infty}^{\infty} \mathrm{e}^{-\frac{\pi}{\kappa d}(k+\alpha d)^2} \\ &= \tfrac{1}{\sqrt{\kappa}} \sum_{\alpha=-\infty}^{\infty} g_{\kappa^{-1}}\Big((k+\alpha d)\sqrt{\tfrac{h}{d}}\Big) \\ &= \sum_{\alpha=-\infty}^{\infty} \mathcal{F}[g_\kappa]\Big((k+\alpha d)\sqrt{\tfrac{h}{d}}\Big) \end{aligned} \tag{21}$$

is also a Gaussian function of discrete variable (see [12]), and

$$\mathfrak{F}[\mathfrak{g}_\kappa] = \tfrac{1}{\sqrt{\kappa}}\ \mathfrak{g}_{\kappa^{-1}}. \tag{22}$$

The discrete Wigner function of $\mathfrak{g}_\kappa$ is

$$\begin{aligned} &\mathfrak{W}_{\mathfrak{g}_\kappa} : \{-s, -s+1, ..., s-1, s\} \times \{-s, -s+1, ..., s-1, s\} \longrightarrow \mathbb{R}, \\ &\mathfrak{W}_{\mathfrak{g}_\kappa}(n,k) = \tfrac{1}{d} \sum_{m=-s}^{s} \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\, \mathfrak{g}_\kappa(n+m)\, \overline{\mathfrak{g}_\kappa(n-m)}. \end{aligned} \tag{23}$$

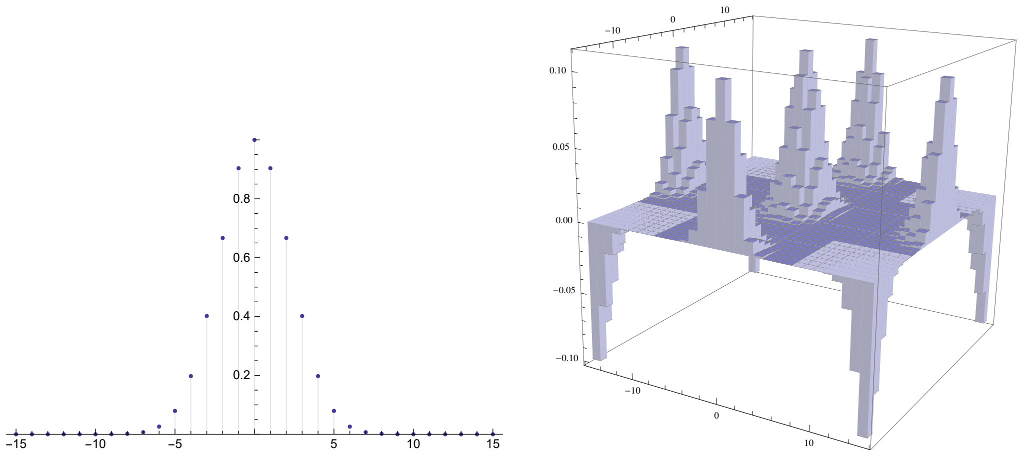

Figure 1. The Gaussian function $\mathfrak{g}_1$ and its Wigner function in the case $d\!=\!31$.

The explicit formula obtained in [12] can be written as

$$\mathfrak{W}_{\mathfrak{g}_\kappa}(n,k)=C\sum_{\alpha,\beta=-\infty}^{\infty}(-1)^{\alpha\beta}\ w\Big(n+\alpha\tfrac{d}{2},k+\beta\tfrac{d}{2}\Big)\,, \tag{24}$$

where C is a normalizing factor and

$$w(q,p)=\mathrm{e}^{-\frac{2\pi}{d}(\,q\ \ p\,)\begin{pmatrix}\kappa^{-1} & 0\\ 0 & \kappa\end{pmatrix}^{-1}\begin{pmatrix}q\\ p\end{pmatrix}}, \tag{25}$$

that is

$$\mathfrak{W}_{\mathfrak{g}_\kappa}(n,k)=C\sqrt{\tfrac{\kappa h}{2}}\sum_{\alpha,\beta=-\infty}^{\infty}(-1)^{\alpha\beta}\mathcal{W}_{g_\kappa}\Big(\big(n+\alpha\tfrac{d}{2}\big)\sqrt{\tfrac{h}{d}},\,\big(k+\beta\tfrac{d}{2}\big)\sqrt{\tfrac{h}{d}}\Big)\,. \tag{26}$$

In the next sections, we show that this continuous-discrete correspondence can be extended up to the Wigner function of the continuous variable Gaussian states, and used in order to define a discrete counterpart of these pure or discrete states.

3. Pure and mixed single-mode discrete variable Gaussian states

In the case of a continuous variable quantum system, a Gaussian state ρ is a state with a Gaussian Wigner function of the form [4-10]

$$\mathcal{W}_\rho(q,p)=\frac{2}{h\sqrt{\det\sigma}}\,\mathrm{e}^{-\frac{2\pi}{h}(\,q-\tilde{q}\ \ p-\tilde{p}\,)\begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{pmatrix}^{-1}\begin{pmatrix}q-\tilde{q}\\ p-\tilde{p}\end{pmatrix}}, \tag{27}$$

where $\tilde{q},\ \tilde{p}\!\in\!\mathbb{R}$ are two parameters, and the covariance matrix

$$\sigma=\begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{pmatrix} \tag{28}$$

is a real $2\!\times\!2$ symmetric positive-definite matrix satisfying the condition

$$\sigma+\mathrm{i}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\geq 0. \tag{29}$$

In the case $\tilde{q}\!=\!\tilde{p}\!=\!0$, we shall write ρ_σ instead of ρ. Inspired by the relations (24) and (25), we propose the definition.

Definition 1. In the case of a discrete variable quantum system, a state ϱ is a *single-mode discrete variable Gaussian state* if there exists a real $2{\times}2$ symmetric positive-definite matrix

$$\sigma=\begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{pmatrix} \tag{30}$$

such that the discrete Wigner function of ϱ is

$$\mathfrak{W}_\varrho(n,k)=C\sum_{\alpha,\beta=-\infty}^{\infty}(-1)^{\alpha\beta}\ w_\sigma\left(n+\alpha\tfrac{d}{2},k+\beta\tfrac{d}{2}\right), \tag{31}$$

where

$$w_\sigma(q,p)=\mathrm{e}^{-\frac{2\pi}{d}\,(\,q-\tilde{q}\quad p-\tilde{p}\,)\begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{pmatrix}^{-1}\begin{pmatrix}q-\tilde{q}\\ p-\tilde{p}\end{pmatrix}}, \tag{32}$$

$\tilde{q}$, $\tilde{p}\in\mathbb{R}$ are two parameters and C is a normalizing constant.

In the case $\tilde{q}=\tilde{p}=0$, we write ϱ_σ instead of ϱ. We do not know what conditions σ has to satisfy in order to have

$$\varrho=\sum_{n,k}\mathfrak{W}_{\varrho_\sigma}(n,k)\,\Pi(n,k)\geq 0. \tag{33}$$

Our numerical simulations suggest that (29) may be a sufficient condition.

Theorem 1. *For any $\kappa\in(0,\infty)$, the pure state*

$$\mathbf{g}_\kappa=\frac{1}{\sqrt{\langle\mathfrak{g}_\kappa,\mathfrak{g}_\kappa\rangle}}\,\mathfrak{g}_\kappa \tag{34}$$

is a discrete variable Gaussian state with $\sigma=\sigma_\kappa$, where

$$\sigma_\kappa=\begin{pmatrix}\kappa^{-1} & 0\\ 0 & \kappa\end{pmatrix}. \tag{35}$$

Proof. Direct consequence of the relations (24) and (25). □

Theorem 2. *For any $\kappa\in(0,\infty)$ and any n_0, $k_0\in\{-s,-s+1,...,s-1,s\}$, the pure state*

$$\psi=\mathfrak{D}(n_0,k_0)\mathbf{g}_\kappa \tag{36}$$

is a discrete variable Gaussian state with $\sigma=\sigma_\kappa$.

Proof. We have

$$\psi(m)=\mathfrak{D}(n_0,k_0)\mathbf{g}_\kappa(m)=\mathrm{e}^{-\frac{\pi\mathrm{i}}{d}n_0k_0}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k_0m}\,\mathbf{g}_\kappa(m-n_0) \tag{37}$$

and

$$\begin{aligned}\mathfrak{W}_\psi(n,k)&=\tfrac{1}{d}\sum_{m=-s}^{s}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k_0(n+m)}\,\mathbf{g}_\kappa(n+m-n_0)\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}k_0(n-m)}\,\mathbf{g}_\kappa(n-m-n_0)\\ &=\tfrac{1}{d}\sum_{m=-s}^{s}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\,\mathrm{e}^{\frac{4\pi\mathrm{i}}{d}k_0m}\,\mathbf{g}_\kappa(n+m-n_0)\,\mathbf{g}_\kappa(n-m-n_0)\\ &=\tfrac{1}{d}\sum_{m=-s}^{s}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}(k-k_0)m}\,\mathbf{g}_\kappa(n-n_0+m)\,\mathbf{g}_\kappa(n-n_0-m)\\ &=\mathfrak{W}_{\mathbf{g}_\kappa}(n-n_0,k-k_0).\end{aligned} \tag{38}$$

□

Theorem 3. *If ϱ is a discrete variable Gaussian state, then $\mathfrak{D}(n_0,k_0)\varrho\mathfrak{D}^\dagger(n_0,k_0)$ is also a discrete variable Gaussian state, for any $n_0,\ k_0\in\{-s,-s+1,...,s-1,s\}$.*

Proof. Since $\mathfrak{D}^\dagger(n_0,k_0)\Pi(n,k)\mathfrak{D}(n_0,k_0)=\Pi(n-n_0,k-k_0)$, we get

$$\begin{aligned}\mathfrak{W}_{\mathfrak{D}(n_0,k_0)\varrho\mathfrak{D}^\dagger(n_0,k_0)}(n,k) &= \tfrac{1}{d}\,\mathrm{tr}(\mathfrak{D}(n_0,k_0)\varrho\mathfrak{D}^\dagger(n_0,k_0)\Pi(n,k))\\ &= \tfrac{1}{d}\,\mathrm{tr}(\varrho\mathfrak{D}^\dagger(n_0,k_0)\Pi(n,k)\mathfrak{D}(n_0,k_0))\\ &= \tfrac{1}{d}\,\mathrm{tr}(\varrho\Pi(n-n_0,k-k_0))\\ &= \mathfrak{W}_\varrho(n-n_0,k-k_0). \qquad \square\end{aligned} \tag{39}$$

Theorem 4. *The Fourier transform $\mathfrak{F}\varrho\mathfrak{F}^\dagger$ of a discrete variable Gaussian state ϱ is also a discrete variable Gaussian state. If the matrix corresponding to ϱ is σ, then the matrix corresponding to $\mathfrak{F}\varrho\mathfrak{F}^\dagger$ is $\Omega\sigma\Omega^T$, where*

$$\Omega=\left(\begin{array}{cc} 0 & 1\\ -1 & 0\end{array}\right). \tag{40}$$

Proof. Since $\mathfrak{F}^\dagger\mathfrak{D}(n,k)\mathfrak{F}=\mathfrak{D}(-k,n)$, we have $\mathfrak{F}^\dagger\Pi(n,k)\mathfrak{F}=\Pi(-k,n)$ and consequently

$$\begin{aligned}\mathfrak{W}_{\mathfrak{F}\varrho\mathfrak{F}^\dagger}(n,k) &= \tfrac{1}{d}\,\mathrm{tr}(\mathfrak{F}\varrho\mathfrak{F}^\dagger\Pi(n,k))\\ &= \tfrac{1}{d}\,\mathrm{tr}(\varrho\mathfrak{F}^\dagger\Pi(n,k)\mathfrak{F})\\ &= \tfrac{1}{d}\,\mathrm{tr}(\varrho\Pi(-k,n))\\ &= \mathfrak{W}_\varrho(-k,n)=\mathfrak{W}_\varrho(\Omega^{-1}(n,k)). \qquad \square\end{aligned} \tag{41}$$

The last two theorem show that Fourier and displacement transforms preserve the nature of discrete variable Gaussian states, that is, they are *discrete Gaussian transforms.*

The discrete variable Gaussian state

$$|0,0\rangle=|\mathbf{g}_1\rangle \tag{42}$$

corresponds to the *vacuum state*, and the discrete variable Gaussian states

$$|n,k\rangle=\mathfrak{D}(n,k)|0,0\rangle \tag{43}$$

satisfying the resolution of the identity [14, 15]

$$\tfrac{1}{d}\sum_{n,k=-s}^{s}|n,k\rangle\langle n,k|=\mathbb{I} \tag{44}$$

correspond to the *canonical coherent states* from the continuous variable case.
Any pure state $\psi\in\mathcal{H}$ admits the representation

$$|\psi\rangle=\mathbb{I}|\psi\rangle=\tfrac{1}{d}\sum_{n,k=-s}^{s}|n,k\rangle\langle n,k|\psi\rangle \tag{45}$$

as a linear superposition of discrete variable Gaussian states.

Theorem 5. *Each continuous variable Gaussian state ρ_σ is the limit of a sequence of discrete variable Gaussian states ϱ_σ, namely*

$$\rho_\sigma=\lim_{d\to\infty}\varrho_\sigma. \tag{46}$$

Proof. Since $\lim_{q^2+p^2\to\infty} w_\sigma(q,p)=0$, for d large enough, we have

$$w_\sigma\big(n+\alpha\tfrac{d}{2}, k+\beta\tfrac{d}{2}\big)\approx 0 \quad \text{for} \quad (\alpha,\beta)\neq(0,0), \tag{47}$$

and consequently, up to a normalizing constant C,

$$\mathfrak{W}_{\varrho_\sigma}(n,k)\approx C\,\mathcal{W}_{\rho_\sigma}\left(n\sqrt{\tfrac{h}{d}}, k\sqrt{\tfrac{h}{d}}\right). \tag{48}$$

The set

$$\left\{ \left(n\sqrt{\tfrac{h}{d}}, k\sqrt{\tfrac{h}{d}}\right) \;\middle|\; \begin{array}{l} d=2s+1\in\{3,5,7,...\} \\ n,k\in\{-s,-s+1,...,s-1,s\} \end{array} \right\} \tag{49}$$

being dense in the phase space $\mathbb{R}^2$, the function $\mathcal{W}_{\rho_\sigma}:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is determined by the functions $\mathfrak{W}_{\varrho_\sigma}:\{-s,-s+1,...,s-1,s\}\times\{-s,-s+1,...,s-1,s\}\longrightarrow\mathbb{R}$.

Table 1. Dependence of the purity on the dimension d.

σ	$\frac{1}{\sqrt{\det\sigma}}$	d	$\mathrm{tr}\,\varrho_\sigma^2$
$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	0.5	3	0.52865
		5	0.50099
		7	0.50003
		9	0.50000
$\begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$	0.7071	3	0.6632
		5	0.7009
		7	0.7064
		9	0.7070
$\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 6 \end{pmatrix}$	0.5773	3	0.7020
		5	0.5948
		7	0.5795
		9	0.5776
$\begin{pmatrix} 7 & -\pi \\ -\pi & 5 \end{pmatrix}$	0.19948	3	0.3389
		5	0.2332
		7	0.2080
		9	0.2017

Theorem 6. *For any covariance matrix σ, we have*

$$\lim_{d\to\infty}\mathrm{tr}\,\varrho_\sigma^2=\frac{1}{\sqrt{\det\sigma}}. \tag{50}$$

Proof. For d large enough, we have $\mathfrak{W}_{\varrho_\sigma}(n,k)\approx C\,\mathcal{W}_{\rho_\sigma}\left(n\sqrt{\tfrac{h}{d}}, k\sqrt{\tfrac{h}{d}}\right)$, and consequently

$$\mathrm{tr}\,\varrho_\sigma^2=d\sum_{n,k=-s}^{s}\mathfrak{W}_{\varrho_\sigma}^2(n,k)\approx d\frac{\sum\limits_{n,k=-s}^{s}\mathcal{W}_{\rho_\sigma}^2\left(n\sqrt{\frac{h}{d}}, k\sqrt{\frac{h}{d}}\right)}{\left(\sum\limits_{n,k=-s}^{s}\mathcal{W}_{\rho_\sigma}\left(n\sqrt{\frac{h}{d}}, k\sqrt{\frac{h}{d}}\right)\right)^2}. \tag{51}$$

By considering a partition of the rectangle $\left[-\frac{d}{2}\sqrt{\frac{h}{d}}, \frac{d}{2}\sqrt{\frac{h}{d}}\right]\times\left[-\frac{d}{2}\sqrt{\frac{h}{d}}, \frac{d}{2}\sqrt{\frac{h}{d}}\right]$ into d^2 squares of area $\frac{h}{d}$ and regarding the integrals as limits of Riemann sums, we get

$$\mathrm{tr}\,\varrho_\sigma^2\approx h\frac{\sum\limits_{n,k=-s}^{s}\frac{h}{d}\,\mathcal{W}_{\rho_\sigma}^2\left(n\sqrt{\frac{h}{d}},k\sqrt{\frac{h}{d}}\right)}{\left(\sum\limits_{n,k=-s}^{s}\frac{h}{d}\,\mathcal{W}_{\rho_\sigma}\left(n\sqrt{\frac{h}{d}},k\sqrt{\frac{h}{d}}\right)\right)^2}\xrightarrow{d\to\infty}\frac{h\int_{\mathbb{R}^2}W_{\rho_\sigma}^2(q,p)\,dq\,dp}{\left(\int_{\mathbb{R}^2}W_{\rho_\sigma}(q,p)\,dq\,dp\right)^2}=\mathrm{tr}\,\rho_\sigma^2=\frac{1}{\sqrt{\det\sigma}}.\ \square \tag{52}$$

Table 2. Some discrete variable Gaussian states ϱ_σ with $\det\sigma = 1$.

σ	d	Spectrum of the density operator ϱ_σ
$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$	3	$1.,\ -7.26023*10^{-17},\ 4.48467*10^{-17}$
	5	$1.,\ 1.38888*10^{-16},\ 8.37904*10^{-17},\ -5.87708*10^{-17},\ -6.19389*10^{-18}$
	7	$1.,\ 1.58*10^{-16}, -7.84*10^{-17}, -6.73*10^{-17},\ 5.24*10^{-17},\ 3.19*10^{-17},\ -3.10*10^{-17}$
$\begin{pmatrix} 1 & 3 \\ 3 & 10 \end{pmatrix}$	3	$1.,\ 4.47648*10^{-17},\ 1.68388*10^{-17}$
	5	$1.,\ -9.69875*10^{-17},\ 8.67892*10^{-17},\ -3.28385*10^{-17},\ 1.97939*10^{-17}$
	7	$1.,\ 3.21*10^{-16},\ 9.10*10^{-17}, -8.99*10^{-17},\ 4.99*10^{-17}, -3.57*10^{-17},\ 2.98*10^{-17}$
$\begin{pmatrix} 3 & \sqrt{5} \\ \sqrt{5} & 2 \end{pmatrix}$	3	$1.,\ 2.17738*10^{-16},\ 3.20619*10^{-17}$
	5	$1.,\ 2.70799*10^{-16},\ 1.12761*10^{-16},\ -9.79558*10^{-17},\ -2.03061*10^{-17}$
	7	$1.,\ 2.37*10^{-16}, -1.31*10^{-16},\ 1.17*10^{-16},\ -8.50*10^{-17}, -2.36*10^{-17}, -1.01*10^{-18}$
$\begin{pmatrix} 3 & -\sqrt{5} \\ -\sqrt{5} & 2 \end{pmatrix}$	3	$1.,\ 2.28273*10^{-16},\ 7.70378*10^{-17}$
	5	$1.,\ 2.85697*10^{-16},\ -8.77195*10^{-17},\ 8.2357*10^{-17},\ -4.52847*10^{-17}$
	7	$1.,\ 2.63*10^{-16}, -1.46*10^{-16},\ 1.30*10^{-16},\ -4.82*10^{-17}, -1.63*10^{-17}, -4.67*10^{-19}$

Theorem 7. *If* $\det\sigma = 1$, *then the discrete Gaussian state* ϱ_σ *is a pure state:*

$$\det\sigma = 1 \quad\Rightarrow\quad \mathrm{tr}\,\varrho_\sigma^2 = 1. \tag{53}$$

Proof. If $\det\sigma = 1$, then there exist $\kappa \in (0,\infty)$ and a canonical transformation (rotation)

$$\begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \tag{54}$$

of the phase space such that

$$\sigma = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \kappa^{-1} & 0 \\ 0 & \kappa \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{55}$$

Consequently

$$\mathcal{W}_{\rho_\sigma}(q,p) = \tfrac{2}{h}\mathrm{e}^{-\frac{2\pi}{h}\left(\kappa q'^2 + \kappa^{-1}p'^2\right)} = \mathcal{W}_{\rho_{\sigma_\kappa}}(q',p') \tag{56}$$

but, we know that the discrete Wigner function $\mathfrak{W}_{\varrho_{\sigma_\kappa}}$ corresponding to $\mathcal{W}_{\rho_{\sigma_\kappa}}$ represents a pure state. $\square$

The continuous variable Gaussian state with the covariance matrix

$$I_\nu = \begin{pmatrix} \nu & 0 \\ 0 & \nu \end{pmatrix} \qquad \text{with} \quad \nu > 1, \tag{57}$$

is the thermal state [5, 9]

$$\rho_{I_\nu} = \tfrac{2}{\nu+1} \sum_{n=0}^{\infty} \left(\tfrac{\nu-1}{\nu+1}\right)^n |\Psi_n\rangle\langle\Psi_n|, \tag{58}$$

where Ψ_n is the Hermite-Gauss function

$$\Psi_n(q)=\frac{1}{\sqrt{n!\,2^n\sqrt{\pi}}}H_n(q)\mathrm{e}^{-\frac{q^2}{2}} \tag{59}$$

in Dirac notation. Particularly, this means that the spectrum of ρ_{I_ν} is formed by the numbers

$$\frac{1}{\sum\limits_{k=0}^{\infty}\left(\frac{\nu-1}{\nu+1}\right)^k}\left(\frac{\nu-1}{\nu+1}\right)^n, \qquad \text{where} \;\; n\in\{0,1,2,3,...\}. \tag{60}$$

The spectrum of the discrete Gaussian state with the covariance I_ν is well-approximated by the set of numbers

$$N_n=\frac{1}{\sum\limits_{k=0}^{d-1}\left(\frac{\nu-1}{\nu+1}\right)^k}\left(\frac{\nu-1}{\nu+1}\right)^n\!, \;\text{where}\; n\in\{0,1,2,3,...,d-1\}. \tag{61}$$

For example, in the case $\nu=2$, $d=7$, the spectrum is

$$0.6667,\;0.2219,\;0.0751,\;0.0229,\;0.0105,\;0.0017,\;0.0010, \tag{62}$$

and the numbers N_0, N_1, ..., N_6 are

$$0.6669,\;0.2223,\;0.0741,\;0.0247,\;0.0082,\;0.0027, 0.0009. \tag{63}$$

Definition 2. We call *discrete thermal state* any discrete variable Gaussian state with a covariance matrix of the form (57).

The pure state

$$\lim_{\nu\to 1}\varrho_{I_\nu}=|\mathbf{g}_1\rangle\langle\mathbf{g}_1|, \tag{64}$$

represents a discrete counterpart of the vacuum.
Since $\mathfrak{W}_{\varrho_{I_\nu}}(n,-k)=\mathfrak{W}_{\varrho_{I_\nu}}(n,k)=\mathfrak{W}_{\varrho_{I_\nu}}(k,n)$, we have

$$\overline{\langle\delta_a|\varrho_{I_\nu}|\delta_b\rangle}=\langle\delta_a|\varrho_{I_\nu}|\delta_b\rangle=\langle\delta_b|\varrho_{I_\nu}|\delta_a\rangle. \tag{65}$$

Consequently, in $\mathcal{H}$ there exists an orthonormal basis $\{|\psi_n\rangle\}$ formed by eigenfunctions of ϱ_{I_ν} with real eigenvalues. In the investigated cases [14], the eigenfunctions of ϱ_{I_ν} are almost independent on ν. Thus, the real eigenfunctions $|\psi_n\rangle$, considered in the increasing order of the number of sign alternations, can be regarded as a discrete version of the Hermite-Gauss functions, and

$$H=\sum_{n=0}^{d-1}\left(n+\tfrac{1}{2}\right)|\psi_n\rangle\langle\psi_n| \tag{66}$$

as the Hamiltonian of a d-dimensional counterpart of the harmonic quantum oscillator.

4. Gaussian functions of two discrete variables

Let $\tau=\left(\begin{smallmatrix} a & b\\ b & c\end{smallmatrix}\right)$ be a 2×2 real symmetric positive-definite matrix. The real function

$$g_\tau:\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R},\quad g_\tau(q_1,q_2)=\mathrm{e}^{-\frac{\pi}{h}(q_1\;\,q_2)\left(\begin{smallmatrix} a & b\\ b & c\end{smallmatrix}\right)\left(\begin{smallmatrix} q_1\\ q_2\end{smallmatrix}\right)} \tag{67}$$

is a *Gaussian function of continuous variable.* Its Fourier transform $\mathcal{F}[g_\tau]:\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$,

$$\begin{aligned}\mathcal{F}[g_\tau](p_1,p_2)&=\tfrac{1}{h}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{h}(p_1q_1+p_2q_2)}\,g_\tau(q_1,q_2)\,dq_1dq_2\\&=\tfrac{1}{\sqrt{\det\tau}}\;\mathrm{e}^{-\frac{\pi}{h}(p_1\;\;p_2)\left(\begin{smallmatrix}a&b\\b&c\end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix}p_1\\p_2\end{smallmatrix}\right)},\end{aligned}\tag{68}$$

is also a Gaussian function of continuous variable, and

$$\mathcal{F}[g_\tau]=\frac{1}{\sqrt{\det\tau}}\,g_{\tau^{-1}}.\tag{69}$$

The Wigner function of g_τ is $\mathcal{W}_{g_\tau}:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$,

$$\begin{aligned}\mathcal{W}_{g_\tau}(q_1,q_2,p_1,p_2)&=\tfrac{4}{h^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{h}(p_1x_1+p_2x_2)}\,g_\tau(q_1+x_1,q_2+x_2)\,\overline{g_\tau(q_1-x_1,q_2-x_2)}\,dx_1dx_2,\\&=\tfrac{2}{h\sqrt{\det\,\tau}}\,g_{2\tau}(q_1,q_2)\;g_{2\tau^{-1}}(p_1,p_2)\\&=\tfrac{2}{h\sqrt{\det\,\tau}}\;\mathrm{e}^{-\frac{2\pi}{h}\left\{(\,q_1\;\;q_2\;\;p_1\;\;p_2\,)\begin{pmatrix}\tau^{-1}&\begin{matrix}0&0\\0&0\end{matrix}\\\begin{matrix}0&0\\0&0\end{matrix}&\tau\end{pmatrix}^{-1}\begin{pmatrix}q_1\\q_2\\p_1\\p_2\end{pmatrix}\right\}}.\end{aligned}\tag{70}$$

The function $\mathfrak{g}_\tau:\{-s,-s+1,...,s-1,s\}\times\{-s,-s+1,...,s-1,s\}\longrightarrow\mathbb{R}$,

$$\begin{aligned}\mathfrak{g}_\tau(n_1,n_2)&=\sum_{\alpha_1,\alpha_2=-\infty}^{\infty}\mathrm{e}^{-\frac{\pi}{d}(n_1+\alpha_1d\;\;n_2+\alpha_2d)\left(\begin{smallmatrix}a&b\\b&c\end{smallmatrix}\right)\left(\begin{smallmatrix}n_1+\alpha_1d\\n_2+\alpha_2d\end{smallmatrix}\right)}\\&=\sum_{\alpha_1,\alpha_2=-\infty}^{\infty}g_\tau\left((n_1+\alpha_1d)\sqrt{\tfrac{h}{d}},(n_2+\alpha_2d)\sqrt{\tfrac{h}{d}}\right)\end{aligned}\tag{71}$$

can be regarded as a *Gaussian function of two discrete variables.*
Its discrete Fourier transform $\mathfrak{F}[\mathfrak{g}_\tau]:\{-s,-s+1,...,s-1,s\}\times\{-s,-s+1,...,s-1,s\}\longrightarrow\mathbb{R}$,

$$\begin{aligned}\mathfrak{F}[\mathfrak{g}_\tau](k_1,k_2)&=\tfrac{1}{d}\sum_{n_1=-s}^{s}\sum_{n_2=-s}^{s}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}(k_1n_1+k_2n_2)}\,\mathfrak{g}_\tau(n_1,n_2)\\&=\tfrac{1}{\sqrt{\det\tau}}\sum_{\beta_1,\beta_2=-\infty}^{\infty}g_{\tau^{-1}}\left((k_1+\beta_1d)\sqrt{\tfrac{h}{d}},(k_2+\beta_2d)\sqrt{\tfrac{h}{d}}\right)\\&=\sum_{\beta_1,\beta_2=-\infty}^{\infty}\mathcal{F}[g_\tau]\left((k_1+\beta_1d)\sqrt{\tfrac{h}{d}},(k_2+\beta_2d)\sqrt{\tfrac{h}{d}}\right).\end{aligned}\tag{72}$$

is also a Gaussian function of discrete variable (see [13]), and

$$\mathfrak{F}[\mathfrak{g}_\tau]=\frac{1}{\sqrt{\det\tau}}\,\mathfrak{g}_{\tau^{-1}}.\tag{73}$$

The discrete Wigner function of $\mathfrak{g}_\tau$ is

$$\begin{aligned}&\mathfrak{W}_{\mathfrak{g}_\tau}:\{-s,-s+1,...,s-1,s\}\times\{-s,-s+1,...,s-1,s\}\times\{-s,-s+1,...,s-1,s\}\times\{-s,-s+1,...,s-1,s\}\to\mathbb{R},\\&\mathfrak{W}_{\mathfrak{g}_\tau}(n_1,n_2,k_1,k_2)=\tfrac{1}{d^2}\sum_{m_1=-s}^{s}\sum_{m_2=-s}^{s}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}(k_1m_1+k_2m_2)}\,\mathfrak{g}_\tau(n_1+m_1,n_2+m_2)\,\overline{\mathfrak{g}_\tau(n_1-m_1,n_2-m_2)}.\end{aligned}\tag{74}$$

The explicit formula obtained in [13] can be written as

$$\begin{aligned}\mathfrak{W}_{\mathfrak{g}_\tau}(n_1,n_2,k_1,k_2)=C&\sum_{\alpha_1,\alpha_2=-\infty}^{\infty}\sum_{\beta_1,\beta_2=-\infty}^{\infty}(-1)^{\alpha_1\beta_1+\alpha_2\beta_2}\\&\times w\left(n_1+\alpha_1\tfrac{d}{2},n_2+\alpha_2\tfrac{d}{2},k_1+\beta_1\tfrac{d}{2},k_2+\beta_2\tfrac{d}{2}\right),\end{aligned}\tag{75}$$

where C is a normalizing factor, and

$$w(q_1,q_2,p_1,p_2)=\mathrm{e}^{-\frac{2\pi}{d}\left\{(\,q_1\;\;q_2\;\;p_1\;\;p_2\,)\begin{pmatrix} \tau^{-1} & & 0 & 0\\ & & 0 & 0\\ 0 & 0 & & \\ 0 & 0 & & \tau \end{pmatrix}^{-1}\begin{pmatrix} q_1\\ q_2\\ p_1\\ p_2\end{pmatrix}\right\}}. \tag{76}$$

5. Pure and mixed two-mode discrete variable Gaussian states

Inspired by the relations (75) and (76), we propose the following definition.

Definition 3. In the case of a two-mode discrete variable quantum system, a state ϱ *is a two-mode discrete variable Gaussian state* if there exists a real 4×4 symmetric positive-definite matrix

$$\sigma=\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14}\\ \sigma_{12} & \sigma_{22} & \sigma_{23} & \sigma_{24}\\ \sigma_{13} & \sigma_{23} & \sigma_{33} & \sigma_{34}\\ \sigma_{14} & \sigma_{24} & \sigma_{34} & \sigma_{44}\end{pmatrix} \tag{77}$$

such that the discrete Wigner function of ϱ is

$$\begin{aligned}\mathfrak{W}_\varrho(n_1,n_2,k_1,k_2)=C\sum_{\alpha_1,\alpha_2=-\infty}^{\infty}\;\sum_{\beta_1,\beta_2=-\infty}^{\infty}(-1)^{\alpha_1\beta_1+\alpha_2\beta_2}\\ \times w_\sigma\!\left(n_1+\alpha_1\tfrac{d}{2},\,n_2+\alpha_2\tfrac{d}{2},\,k_1+\beta_1\tfrac{d}{2},\,k_2+\beta_2\tfrac{d}{2}\right),\end{aligned} \tag{78}$$

where C is a normalizing factor,

$$w_\sigma(q_1,q_2,p_1,p_2)=\mathrm{e}^{-\frac{2\pi}{d}\left\{(\,q_1-\tilde q_1\;\;q_2-\tilde q_2\;\;p_1-\tilde p_1\;\;p_2-\tilde p_2\,)\;\sigma^{-1}\begin{pmatrix} q_1-\tilde q_1\\ q_2-\tilde q_2\\ p_1-\tilde p_1\\ p_2-\tilde p_2\end{pmatrix}\right\}} \tag{79}$$

and $\tilde q_1,\ \tilde q_2,\ \tilde p_1,\ \tilde p_2\in\mathbb{R}$ are parameters.

In the case $\tilde q_1=\tilde q_2=\tilde p_1=\tilde p_2=0$, we write ϱ_σ instead of ϱ.

6. On the position-momentum commutation relation

In the continuous case, the usual position-momentum commutation relation

$$[\hat q,\hat p]=\mathrm{i}\frac{h}{2\pi} \tag{80}$$

is satisfied in a subspace dense in $L^2(\mathbb{R})$. The discrete counterpart

$$[\hat{\mathfrak{q}},\hat{\mathfrak{p}}]=\mathrm{i}\frac{d}{2\pi} \tag{81}$$

is not satisfied, but for d large enough, most of the eigenvalues of the operator $[\hat{\mathfrak{q}},\hat{\mathfrak{p}}]-\mathrm{i}\frac{d}{2\pi}$

are almost null. For example, in the cases $d\!=\!11$ and $d\!=\!61$, the eigenvalues are:

$$
\begin{array}{c|rrrrr}
\underline{\text{Case } d=11} & & & \underline{\text{Case } d=61} & & \\
 & -4.05333*10^{-15}\mathrm{i}, & -3.36139*10^{-14}\mathrm{i}, & -6.82471*10^{-14}\mathrm{i}, & -2.30564*10^{-11}\mathrm{i}, & -0.00513798\,\mathrm{i}\\
-1.4{*}10^{-8}\mathrm{i} & 9.31866*10^{-15}\mathrm{i}, & 3.14251*10^{-14}\mathrm{i}, & 6.89526*10^{-14}\mathrm{i}, & 1.41605*10^{-10}\mathrm{i}, & 0.0198162\,\mathrm{i}\\
7.9{*}10^{-7}\mathrm{i} & 7.48845*10^{-15}\mathrm{i}, & 3.68406*10^{-14}\mathrm{i}, & 7.0783*10^{-14}\mathrm{i}, & -8.34352*10^{-10}\mathrm{i}, & -0.0733599\,\mathrm{i}\\
2{*}10^{-5}\mathrm{i} & -1.17774*10^{-15}\mathrm{i}, & -3.41122*10^{-14}\mathrm{i}, & -7.25619*10^{-14}\mathrm{i}, & 4.73159*10^{-9}\mathrm{i}, & 0.260404\,\mathrm{i}\\
3.3{*}10^{-4}\mathrm{i} & -3.78568*10^{-15}\mathrm{i}, & -3.71574*10^{-14}\mathrm{i}, & 8.03083*10^{-14}\mathrm{i}, & -2.58237*10^{-8}\mathrm{i}, & -0.88457\,\mathrm{i}\\
-3.9{*}10^{-3}\mathrm{i} & 1.14028*10^{-14}\mathrm{i}, & 4.1543*10^{-14}\mathrm{i}, & -8.1415*10^{-14}\mathrm{i}, & 1.35662*10^{-7}\mathrm{i}, & 2.87316\,\mathrm{i}\\
3.4{*}10^{-2}\mathrm{i} & -1.69386*10^{-14}\mathrm{i}, & -4.48723*10^{-14}\mathrm{i}, & 8.70762*10^{-14}\mathrm{i}, & -6.86077*10^{-7}\mathrm{i}, & -8.86191\,\mathrm{i}\\
-0.24\,\mathrm{i} & -2.31158*10^{-14}\mathrm{i}, & 4.08535*10^{-14}\mathrm{i}, & -8.9346*10^{-14}\mathrm{i}, & 3.34023*10^{-6}\mathrm{i}, & 26.1438\,\mathrm{i}\\
1.33\,\mathrm{i} & 2.23703*10^{-14}\mathrm{i}, & 4.78884*10^{-14}\mathrm{i}, & -9.94767*10^{-14}\mathrm{i}, & -0.0000156547\,\mathrm{i}, & -70.9696\,\mathrm{i}\\
-5.45\,\mathrm{i} & 2.76156*10^{-14}\mathrm{i}, & -4.97772*10^{-14}\mathrm{i}, & 1.13415*10^{-13}\mathrm{i}, & 0.0000706183\,\mathrm{i}, & 189.607\,\mathrm{i}\\
19.99\,\mathrm{i} & 2.73091*10^{-14}\mathrm{i}, & -4.63059*10^{-14}\mathrm{i}, & -5.72182*10^{-13}\mathrm{i}, & -0.000306537\,\mathrm{i}, & -404.509\,\mathrm{i}\\
-34.92\,\mathrm{i} & -3.32093*10^{-14}\mathrm{i}, & 6.80431*10^{-14}\mathrm{i}, & 3.61784*10^{-12}\mathrm{i}, & 0.00127992\,\mathrm{i}, & 944.717\,\mathrm{i}\\
 & & & & & -1270.53\,\mathrm{i}
\end{array}
\tag{82}
$$

Let λ_1, λ_2, ..., λ_d be the eigenvalues of $[\hat{\mathfrak{q}},\hat{\mathfrak{p}}]-\mathrm{i}\frac{d}{2\pi}$, considered in the increasing order of their modulus ($|\lambda_1|\leq|\lambda_2|\leq...\leq|\lambda_d|$), and let φ_1, φ_2, ... , φ_d be the corresponding eigenfunctions. If we expand a discrete variable Gaussian state in terms of the eigenbasis $\{\varphi_k\}$, the most significant coefficients seem to be those corresponding to the functions φ_k with small $|\lambda_k|$. For example, in the case $d\!=\!11$, the matrix $(|\langle\varphi_n|\mathbf{g}_1\rangle|)$ of $\mathbf{g}_1$ in the eigenbasis $\{\varphi_k\}$ is

$$
\left(\begin{array}{c}
0.9999\\ 1{*}10^{-10}\\ 2{*}10^{-11}\\ 2{*}10^{-13}\\ 0.0079\\ 4{*}10^{-15}\\ 1{*}10^{-15}\\ 2{*}10^{-16}\\ 0.0004\\ 6{*}10^{-18}\\ 2{*}10^{-17}
\end{array}\right)
\tag{83}
$$

and for $\sigma\!=\!\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)$, the matrix $(|\langle\varphi_n\,|\varrho_\sigma|\,\varphi_m\rangle|)$ of ϱ_σ is

$$
\left(\begin{array}{ccccccccccc}
0.8954 & 6{*}10^{-11} & 0.2837 & 1{*}10^{-12} & 0.1059 & 1{*}10^{-14} & 0.0393 & 3{*}10^{-16} & 0.0164 & 3{*}10^{-17} & 0.0084\\
1{*}10^{-10} & 5{*}10^{-18} & 5{*}10^{-11} & 1{*}10^{-17} & 1{*}10^{-11} & 8{*}10^{-18} & 7{*}10^{-12} & 1{*}10^{-17} & 3{*}10^{-12} & 1{*}10^{-17} & 1{*}10^{-12}\\
0.2837 & 2{*}10^{-11} & 0.0899 & 3{*}10^{-13} & 0.0335 & 4{*}10^{-15} & 0.0124 & 1{*}10^{-16} & 0.0052 & 1{*}10{-}17 & 0.0026\\
2{*}10^{-13} & 1{*}10^{-17} & 7{*}10^{-14} & 3{*}10^{-18} & 2{*}10^{-14} & 3{*}10^{-17} & 1{*}10^{-14} & 4{*}10^{-17} & 4{*}10^{-15} & 1{*}10^{-17} & 2{*}10^{-15}\\
0.1059 & 7{*}10^{-12} & 0.0335 & 1{*}10^{-13} & 0.0125 & 1{*}10^{-15} & 0.0046 & 4{*}10^{-17} & 0.0019 & 3{*}10^{-18} & 0.0010\\
1{*}10^{-15} & 1{*}10^{-17} & 5{*}10^{-16} & 2{*}10^{-17} & 2{*}10^{-16} & 1{*}10^{-17} & 8{*}10^{-17} & 1{*}10^{-17} & 3{*}10^{-17} & 1{*}10^{-17} & 7{*}10^{-18}\\
0.0393 & 2{*}10^{-12} & 0.0124 & 5{*}10^{-14} & 0.0046 & 5{*}10^{-16} & 0.0017 & 2{*}10^{-17} & 0.0007 & 6{*}10^{-18} & 0.0003\\
2{*}10^{-16} & 1{*}10^{-17} & 8{*}10^{-17} & 4{*}10^{-17} & 3{*}10^{-17} & 7{*}10^{-18} & 2{*}10^{-17} & 4{*}10^{-18} & 8{*}10^{-18} & 3{*}10^{-17} & 5{*}10^{-18}\\
0.0164 & 1{*}10^{-12} & 0.0052 & 2{*}10^{-14} & 0.0019 & 2{*}10^{-16} & 0.0007 & 1{*}10^{-17} & 0.0003 & 8{*}10^{-19} & 0.0001\\
2{*}10^{-17} & 1{*}10^{-17} & 8{*}10^{-18} & 1{*}10^{-17} & 2{*}10^{-18} & 1{*}10^{-17} & 5{*}10^{-18} & 3{*}10^{-17} & 1{*}10^{-18} & 4{*}10^{-17} & 5{*}10^{-18}\\
0.0084 & 6{*}10^{-13} & 0.0026 & 1{*}10^{-14} & 0.0010 & 1{*}10^{-16} & 0.00037 & 2{*}10^{-18} & 0.0001 & 5{*}10^{-18} & 0.0000
\end{array}\right).
\tag{84}
$$

For $\varepsilon\!>\!0$, let d_ε be such that

$$\{\lambda_1,\,\lambda_2,\,...\,,\,\lambda_{d_\varepsilon}\}=\{\,\lambda_k\,\mid\,\ |\lambda_k|\!<\!\varepsilon\,\},\tag{85}$$

and let

$$\mathcal{H}_\varepsilon=\mathrm{span}\{\,\varphi_1,\,\varphi_2,\,...\,,\,\varphi_{d_\varepsilon}\}.\tag{86}$$

If ε is small enough, then we can consider that

$$[\hat{\mathfrak{q}},\hat{\mathfrak{p}}]\approx\mathrm{i}\frac{d}{2\pi}\quad\text{in}\quad\mathcal{H}_\varepsilon.\tag{87}$$

By following the analogy with the continuous case, we define

$$
\begin{array}{l}
\hat{\mathfrak{a}}=\sqrt{\frac{\pi}{d}}(\hat{\mathfrak{q}}+\mathrm{i}\hat{\mathfrak{p}}),\\
\hat{\mathfrak{a}}^\dagger=\sqrt{\frac{\pi}{d}}(\hat{\mathfrak{q}}-\mathrm{i}\hat{\mathfrak{p}}),
\end{array}
\qquad\text{similar to}\qquad
\begin{array}{l}
\hat{a}=\sqrt{\frac{\pi}{h}}(\hat{q}+\mathrm{i}\hat{p}),\\
\hat{a}^\dagger=\sqrt{\frac{\pi}{h}}(\hat{q}-\mathrm{i}\hat{p}),
\end{array}
\tag{88}
$$

The discrete variable Gaussian states seem to mainly belong to $\mathcal{H}_\varepsilon$.

In the case of continuous variable, the transformations $\hat{U}=\mathrm{e}^{-\frac{\mathrm{i}}{2}\hat{H}}$, where

$$\hat{H}=(\,\hat{a}^\dagger\quad \hat{a}\,)\begin{pmatrix} A & B\\ \bar{B} & A\end{pmatrix}\begin{pmatrix}\hat{a}\\ \hat{a}^\dagger\end{pmatrix}\qquad \text{with}\quad \begin{array}{l} A\in\mathbb{R},\\ B\in\mathbb{C},\end{array} \tag{89}$$

are Gaussian transforms, that is they preserve the Gaussian nature of the states. For $\hat{U}=\mathrm{e}^{-\frac{\mathrm{i}}{2}\hat{H}}$, there exists $S\in\mathrm{Sp}(2,\mathbb{R})$ such that

$$\mathrm{e}^{-\frac{\mathrm{i}}{2}\hat{H}}\rho_\sigma \mathrm{e}^{\frac{\mathrm{i}}{2}\hat{H}}=\rho_{S\sigma S^T}. \tag{90}$$

In the case of discrete variable, the relation $\lim\limits_{d\to\infty}\varrho_\sigma=\rho_\sigma$ suggests that for

$$\hat{\mathfrak{H}}=(\,\hat{\mathfrak{a}}^\dagger\quad \hat{\mathfrak{a}}\,)\begin{pmatrix} A & B\\ \bar{B} & A\end{pmatrix}\begin{pmatrix}\hat{\mathfrak{a}}\\ \hat{\mathfrak{a}}^\dagger\end{pmatrix}\qquad \text{with}\quad \begin{array}{l} A\in\mathbb{R},\\ B\in\mathbb{C},\end{array} \tag{91}$$

we must have

$$\mathrm{e}^{-\frac{\mathrm{i}}{2}\hat{\mathfrak{H}}}\varrho_\sigma \mathrm{e}^{\frac{\mathrm{i}}{2}\hat{\mathfrak{H}}}\approx\varrho_{S\sigma S^T}\qquad \text{for}\ \ d\ \ \text{large enough.} \tag{92}$$

Some numerical results concerning a phase shift and a single-mode squeezing transformation are presented in Table 2.

Table 3. Norm of $\mathrm{e}^{-\frac{\mathrm{i}}{2}\hat{\mathfrak{H}}}\varrho_\sigma \mathrm{e}^{\frac{\mathrm{i}}{2}\hat{\mathfrak{H}}}-\varrho_{S\sigma S^T}$ in two particular cases.

A	B	σ	d	$\left\Vert \mathrm{e}^{-\frac{\mathrm{i}}{2}\hat{\mathfrak{H}}}\varrho_\sigma \mathrm{e}^{\frac{\mathrm{i}}{2}\hat{\mathfrak{H}}}-\varrho_{S\sigma S^T}\right\Vert$
$\frac{\pi}{4}$	0	$\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}$	5	0.1701
			7	0.0932
			9	0.0489
			11	0.0256
			13	0.0135
			15	0.0071
0	$\frac{1}{2}\,\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$	$\begin{pmatrix}3 & 2\\ 2 & 2\end{pmatrix}$	5	0.1605
			7	0.1217
			9	0.1057
			11	0.0887
			13	0.0729
			15	0.0593

A possible explanation can be obtained by following the analogy with the continuous case. From the relation

$$[\hat{\mathfrak{a}},\hat{\mathfrak{a}}^\dagger]\approx 1 \tag{93}$$

satisfied in $\mathcal{H}_\varepsilon$, one gets the relations

$$\begin{array}{l}\left[\frac{\mathrm{i}}{2}\hat{\mathfrak{H}},\hat{\mathfrak{a}}\right]\approx\mathrm{i}\left(-A\,\hat{\mathfrak{a}}-B\,\hat{\mathfrak{a}}^\dagger\right)\\ \left[\frac{\mathrm{i}}{2}\hat{\mathfrak{H}},\hat{\mathfrak{a}}^\dagger\right]\approx\mathrm{i}\left(\bar{B}\,\hat{\mathfrak{a}}+A\,\hat{\mathfrak{a}}^\dagger\right)\end{array}\qquad \text{in}\quad \mathcal{H}_\varepsilon \tag{94}$$

which can be written together as

$$\left[\frac{\mathrm{i}}{2}\hat{\mathfrak{H}},\begin{pmatrix}\hat{\mathfrak{a}}\\ \hat{\mathfrak{a}}^\dagger\end{pmatrix}\right]\approx\mathrm{i}\begin{pmatrix}-A & -B\\ \bar{B} & A\end{pmatrix}\begin{pmatrix}\hat{\mathfrak{a}}\\ \hat{\mathfrak{a}}^\dagger\end{pmatrix}\qquad \text{in}\quad \mathcal{H}_\varepsilon. \tag{95}$$

By iteration, we obtain

$$\left[\tfrac{\mathrm{i}}{2}\hat{\mathfrak{H}},\left[\tfrac{\mathrm{i}}{2}\hat{\mathfrak{H}},\begin{pmatrix}\hat{\mathfrak{a}}\\ \hat{\mathfrak{a}}^\dagger\end{pmatrix}\right]\right]\approx \mathrm{i}^2\begin{pmatrix}-A & -B\\ \bar{B} & A\end{pmatrix}^2\begin{pmatrix}\hat{\mathfrak{a}}\\ \hat{\mathfrak{a}}^\dagger\end{pmatrix}\qquad \text{in}\quad \mathcal{H}_\varepsilon, \tag{96}$$

$$\left[\tfrac{\mathrm{i}}{2}\hat{\mathfrak{H}},\left[\tfrac{\mathrm{i}}{2}\hat{\mathfrak{H}},\left[\tfrac{\mathrm{i}}{2}\hat{\mathfrak{H}},\begin{pmatrix}\hat{\mathfrak{a}}\\ \hat{\mathfrak{a}}^\dagger\end{pmatrix}\right]\right]\right]\approx \mathrm{i}^3\begin{pmatrix}-A & -B\\ \bar{B} & A\end{pmatrix}^3\begin{pmatrix}\hat{\mathfrak{a}}\\ \hat{\mathfrak{a}}^\dagger\end{pmatrix},\ \text{etc.}\qquad \text{in}\quad \mathcal{H}_\varepsilon. \tag{97}$$

From the relation

$$\mathrm{e}^X Y \mathrm{e}^{-X} = Y + \frac{1}{1!}[X,Y] + \frac{1}{2!}[X,[X,Y]] + \frac{1}{3!}[X,[X,[X,Y]]] + \cdots \tag{98}$$

it follows that $\hat{\mathfrak{U}}=\mathrm{e}^{-\frac{\mathrm{i}}{2}\hat{\mathfrak{H}}}$ satisfies the relation

$$\hat{\mathfrak{U}}^\dagger\begin{pmatrix}\hat{\mathfrak{a}}\\ \hat{\mathfrak{a}}^\dagger\end{pmatrix}\hat{\mathfrak{U}}\approx \mathrm{e}^{\mathrm{i}\begin{pmatrix}-A & -B\\ \bar{B} & A\end{pmatrix}}\begin{pmatrix}\hat{\mathfrak{a}}\\ \hat{\mathfrak{a}}^\dagger\end{pmatrix}\qquad \text{in}\quad \mathcal{H}_\varepsilon. \tag{99}$$

Since

$$\begin{pmatrix}\hat{\mathfrak{a}}\\ \hat{\mathfrak{a}}^\dagger\end{pmatrix}=\sqrt{\tfrac{\pi}{d}}\begin{pmatrix}1 & \mathrm{i}\\ 1 & -\mathrm{i}\end{pmatrix}\begin{pmatrix}\hat{\mathfrak{q}}\\ \hat{\mathfrak{p}}\end{pmatrix}, \tag{100}$$

the previous relation can be written as

$$\hat{\mathfrak{U}}^\dagger\begin{pmatrix}\hat{\mathfrak{q}}\\ \hat{\mathfrak{p}}\end{pmatrix}\hat{\mathfrak{U}}\approx S\begin{pmatrix}\hat{\mathfrak{q}}\\ \hat{\mathfrak{p}}\end{pmatrix}\qquad \text{in}\quad \mathcal{H}_\varepsilon, \tag{101}$$

where

$$S=\tfrac{1}{2}\begin{pmatrix}1 & 1\\ -\mathrm{i} & \mathrm{i}\end{pmatrix}\mathrm{e}^{\mathrm{i}\begin{pmatrix}-A & -B\\ \bar{B} & A\end{pmatrix}}\begin{pmatrix}1 & \mathrm{i}\\ 1 & -\mathrm{i}\end{pmatrix}=\mathrm{e}^{-\frac{1}{2}\begin{pmatrix}(B-\bar{B})\mathrm{i} & -2A+B+\bar{B}\\ 2A+B+\bar{B} & -(B-\bar{B})\mathrm{i}\end{pmatrix}} \tag{102}$$

This matrix with real elements and $\det S=1$ belongs to the symplectic group $\mathrm{Sp}(2,\mathbb{R})$. For example, the phase shift $U=\mathrm{e}^{\frac{\mathrm{i}}{2}(\varphi\,\hat{a}^\dagger\hat{a}+\varphi\,\hat{a}\hat{a}^\dagger)}$ corresponds to

$$S_\varphi=\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix} \tag{103}$$

and the single-mode squeezing $U=\mathrm{e}^{\frac{s}{2}\left(\mathrm{e}^{\mathrm{i}\theta}(\hat{a}^\dagger)^2-\mathrm{e}^{-\mathrm{i}\theta}\,\hat{a}^2\right)}$ to

$$S_{s,\theta}=\begin{pmatrix}\cosh s+\cos\theta\ \sinh s & \sin\theta\ \sinh s\\ \sin\theta\ \sinh s & \cosh s-\cos\theta\ \sinh s\end{pmatrix}. \tag{104}$$

In $\mathcal{H}_\varepsilon$, we have $[\hat{\mathfrak{q}},[\hat{\mathfrak{q}},\hat{\mathfrak{p}}]]\approx 0$ and $[\hat{\mathfrak{p}},[\hat{\mathfrak{q}},\hat{\mathfrak{p}}]]\approx 0$. Consequently,

$$\begin{aligned}\mathfrak{D}(n,k)&=\mathrm{e}^{-\frac{\pi\mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k\hat{\mathfrak{q}}}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}n\hat{\mathfrak{p}}}\\ &=\mathrm{e}^{-\frac{\pi\mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{1}{2}\left[\frac{2\pi\mathrm{i}}{d}k\hat{\mathfrak{q}},-\frac{2\pi\mathrm{i}}{d}n\hat{\mathfrak{p}}\right]}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}(n\hat{\mathfrak{p}}-k\hat{\mathfrak{q}})}\\ &\approx\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}(n\hat{\mathfrak{p}}-k\hat{\mathfrak{q}})}\end{aligned} \tag{105}$$

and by denoting $S=\begin{pmatrix}s_{11} & s_{12}\\ s_{21} & s_{22}\end{pmatrix}$, we get

$$\begin{aligned}\hat{\mathfrak{U}}^\dagger\mathfrak{D}(n,k)\hat{\mathfrak{U}}&\approx\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}\hat{\mathfrak{U}}^\dagger(n\hat{\mathfrak{p}}-k\hat{\mathfrak{q}})\hat{\mathfrak{U}}}\\ &=\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}(n(s_{21}\hat{\mathfrak{q}}+s_{22}\hat{\mathfrak{p}})-k(s_{11}\hat{\mathfrak{q}}+s_{12}\hat{\mathfrak{p}}))}\\ &=\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}((s_{22}n-s_{12}k)\hat{\mathfrak{p}}-(-s_{21}n+s_{11}k)\hat{\mathfrak{q}}))}\\ &\approx\mathfrak{D}(S^{-1}(n,k)).\end{aligned} \tag{106}$$

Since

$$w_\sigma(S^{-1}(q,p)) = \mathrm{e}^{-(q\;\;p)(S^{-1})^T\sigma^{-1}S^{-1}\begin{pmatrix}q\\p\end{pmatrix}} = \mathrm{e}^{-(q\;\;p)(S\sigma S^T)^{-1}\begin{pmatrix}q\\p\end{pmatrix}} = w_{S\sigma S^T}(q,p) \tag{107}$$

and

$$\begin{array}{lllll} \Pi^2 = \mathbb{I} & \Pi\mathfrak{F} = \mathfrak{F}\Pi & \Pi\hat{\mathfrak{q}} = -\hat{\mathfrak{q}}\Pi & \Pi\hat{\mathfrak{a}} = -\hat{\mathfrak{a}}\Pi & \Pi\hat{\mathfrak{H}}\Pi = \hat{\mathfrak{H}} \\ \mathfrak{F}^2 = \Pi & \Pi\mathfrak{F}^\dagger = \mathfrak{F}^\dagger\Pi & \Pi\hat{\mathfrak{p}} = -\hat{\mathfrak{p}}\Pi & \Pi\hat{\mathfrak{a}}^\dagger = -\hat{\mathfrak{a}}^\dagger\Pi & \Pi\hat{\mathfrak{U}}\Pi = \hat{\mathfrak{U}} \end{array} \tag{108}$$

we get $\hat{\mathfrak{U}}^\dagger\Pi\hat{\mathfrak{U}} = \Pi$ and

$$\begin{aligned} \mathfrak{W}_{\hat{\mathfrak{U}}\varrho_\sigma\hat{\mathfrak{U}}^\dagger}(n,k) &= \tfrac{1}{d}\,\mathrm{tr}(\hat{\mathfrak{U}}\varrho_\sigma\hat{\mathfrak{U}}^\dagger\,\Pi(n,k)) = \tfrac{1}{d}\,\mathrm{tr}(\varrho_\sigma\hat{\mathfrak{U}}^\dagger\,\Pi(n,k)\,\hat{\mathfrak{U}}) \\ &= \tfrac{1}{d}\,\mathrm{tr}(\varrho_\sigma\hat{\mathfrak{U}}^\dagger\,\mathfrak{D}(n,k)\,\Pi\,\mathfrak{D}^\dagger(n,k)\,\hat{\mathfrak{U}}) \\ &= \tfrac{1}{d}\,\mathrm{tr}(\varrho_\sigma\hat{\mathfrak{U}}^\dagger\,\mathfrak{D}(n,k)\,\hat{\mathfrak{U}}\,\hat{\mathfrak{U}}^\dagger\,\Pi\,\hat{\mathfrak{U}}\,\hat{\mathfrak{U}}^\dagger\,\mathfrak{D}^\dagger(n,k)\,\hat{\mathfrak{U}}) \\ &\approx \tfrac{1}{d}\,\mathrm{tr}(\varrho_\sigma\,\mathfrak{D}(S^{-1}(n,k))\,\Pi\,\mathfrak{D}^\dagger(S^{-1}(n,k))) \\ &= \tfrac{1}{d}\,\mathrm{tr}(\varrho_\sigma\,\Pi(S^{-1}(n,k)) = \mathfrak{W}_{\varrho_\sigma}(S^{-1}(n,k)) \\ &= C\sum_{\alpha,\beta=-\infty}^{\infty}(-1)^{\alpha\beta}\;w_\sigma\big(S^{-1}(n,k)+(\alpha,\beta)\tfrac{d}{2}\big)\,. \end{aligned} \tag{109}$$

If d is large enough, then

$$w_\sigma\left(S^{-1}(n,k)+(\alpha,\beta)\frac{d}{2}\right)\approx 0 \quad \text{for} \quad (\alpha,\beta)\neq(0,0), \tag{110}$$

and we get the relation

$$\begin{aligned} \mathfrak{W}_{\hat{\mathfrak{U}}\varrho_\sigma\hat{\mathfrak{U}}^\dagger}(n,k) &\approx C\;w_\sigma(S^{-1}(n,k)) \\ &= C\;w_{S\sigma S^T}(n,k) \\ &\approx \mathfrak{W}_{\varrho_{S\sigma S^T}}(n,k) \end{aligned} \tag{111}$$

equivalent to

$$\hat{\mathfrak{U}}\varrho_\sigma\hat{\mathfrak{U}}^\dagger \approx \varrho_{S\sigma S^T} \qquad \text{in} \quad \mathcal{H}_\varepsilon. \tag{112}$$

7. Concluding remarks

The use of the discrete variable Gaussian states extends the class of the quantum states which can be described analytically. These particular states have some significant properties and may be useful in certain applications. They can also be used as finite-dimensional approximates for the continuous variable Gaussian states. Generally, in $\rho_\sigma = \lim_{d\to\infty}\varrho_\sigma$, the convergence is very fast.

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