COMPLEX ANALYSIS - Solved Problems 3

Nicolae Cotfas, version 04 June 2025 (for future updates see https://unibuc.ro/user/nicolae.cotfas/)

PROBLEMS

Problem 1

Compute:

1a $(2-\mathrm{i})^2+(\overline{1+2\mathrm{i}})+\frac{1-\mathrm{i}}{1+\mathrm{i}}+\mathrm{i}^7+|\sqrt{3}+\mathrm{i}|$.

1b $\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}+\mathrm{e}^{1+\mathrm{i}\frac{\pi}{3}}+\cos(\mathrm{i})$.

1c $(z^3+\frac{1}{z^3}+\cos(2z)+z^3\mathrm{e}^{\mathrm{i}z})'$.

Problem 2

Compute the residues of the functions

2a $f:\mathbb{C}\backslash\{\mathrm{i},-\mathrm{i}\}\to\mathbb{C},\quad f(z)=\frac{\mathrm{e}^z}{z^2+1}$

2b $g:\mathbb{C}\backslash\{1\}\to\mathbb{C},\quad g(z)=\frac{1}{(z-1)^3}\,\mathrm{e}^z$

in the corresponding singular points.

Problem 3

Compute the integrals

3a $I_0=\int_{\gamma_0}\bar{z}\,dz,\quad$ where $\gamma_0:[0,1]\to\mathbb{C},\quad \gamma_0(t)=t-1+t\mathrm{i}$.

3b $I_1=\int_{\gamma_1}\frac{\mathrm{e}^z}{(z-1)^3}\,dz,\quad$ where $\gamma_1:[0,2\pi]\to\mathbb{C},\quad \gamma_1(t)=2\mathrm{e}^{\mathrm{i}t}$.

Problem 4

Expand in a series of powers of $(z-\mathrm{i})$ the function

$$f:\mathbb{C}\backslash\{1\}\to\mathbb{C},\quad f(z)=\frac{1}{z-1}$$

4a - in the domain $\{\,z\mid\ |z-\mathrm{i}|<\sqrt{2}\,\}$,

4b - in the domain $\{\,z\mid\ |z-\mathrm{i}|>\sqrt{2}\,\}$.

Problem 5

Compute the integral

5 $$I=\int_0^{2\pi}\frac{\cos t}{2+\cos t}dt$$

by using the theorem of residues.

SOME DEFINITIONS AND THEOREMS

D1 Definition

$(x_1+y_1\mathrm{i})(x_2+y_2\mathrm{i})=(x_1x_2-y_1y_2)+(x_1y_2+x_2y_1)\mathrm{i}$

$\overline{x+y\mathrm{i}}=x-y\mathrm{i}$,

$|x+y\mathrm{i}|=\sqrt{x^2+y^2}$,

$|z_1-z_2|=$ distance between z_1 and z_2,

$|z|=$ distance between z and 0,

$\mathrm{e}^{\mathrm{i}t}\stackrel{\text{def}}{=}\cos t+\mathrm{i}\sin t$ (Euler's formula),

$\mathrm{e}^{x+y\mathrm{i}}=\mathrm{e}^x\cos y+\mathrm{i}\,\mathrm{e}^x\sin y$,

$z=|z|\,\mathrm{e}^{\mathrm{i}\arg z}$, where $-\pi<\arg z\le\pi$,

$\cos z=\frac{1}{2}(\mathrm{e}^{\mathrm{i}z}+\mathrm{e}^{-\mathrm{i}z})$,

$\sin z=\frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}z}-\mathrm{e}^{-\mathrm{i}z})$,

$\log z=\ln|z|+\mathrm{i}\arg z,\qquad \log_k z=\ln|z|+\mathrm{i}(\arg z+2k\pi)$.

D2 Definition Let $D\subseteq\mathbb{C}$ be an open set, and $z_0\in D$.

A function $f:D\longrightarrow\mathbb{C}$ is **complex-differentiable** ($\mathbb{C}$-differentiable) at z_0 $\stackrel{\text{def}}{\Longleftrightarrow}$ there exists and is finite $f'(z_0)\stackrel{\text{def}}{=}\lim\limits_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$.

D3 Definition

$f:D\to\mathbb{C}$ defined on an open set D is called $\mathbb{C}$-*differentiable* if (or *holomorphic function*) f is $\mathbb{C}$-differentiable at any point $z_0\in D$.

T1 Theorem

$(f\pm g)'=f'\pm g'$ $\qquad (z^n)'=nz^{n-1}$

$(fg)'=f'g+fg'$ $\qquad (\mathrm{e}^z)'=\mathrm{e}^z$

$\left(\frac{f}{g}\right)'=\frac{f'g-fg'}{g^2}$ $\qquad (\sin z)'=\cos z$

$(f(\varphi(z)))'=f'(\varphi(z))\,\varphi'(z)$ $\qquad (\cos z)'=-\sin z$.

T2 Theorem

$$\frac{1}{1-z}=1+z+z^2+z^3+\dots \text{ for } |z|<1$$

$$\mathrm{e}^z=1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots \text{ for any } z$$

D4 Definition

Let $D\subseteq\mathbb{C}$ be an open set,
$f:D\longrightarrow\mathbb{C}$ be a continuous function,
$\gamma:[a,b]\longrightarrow D$ be a path of class C^1.

The complex line integral of f along γ is $\int_\gamma f(z)\,dz=\int_a^b f(\gamma(t))\,\gamma'(t)dt$

T3 Theorem

$g:D\to\mathbb{C}$ is a primitive of f (that is $g'=f$) $\Rightarrow$ $\int_\gamma f(z)\,dz=g(\gamma(b))-g(\gamma(a))$

If $f:D\to\mathbb{C}$ admits a primitive and γ is closed, then $\int_\gamma f(z)\,dz=0$.

D5 Definition Let D be an open set and $f:D\to\mathbb{C}$ holomorphic.

$z_0\in\mathbb{C}\backslash D$ is *an isolated singular point* if there exists $r>0$ such that $\{\,z\mid 0<|z-z_0|<r\,\}\subset D$.

$z_0\in D$ is *a zero of multiplicity k* if $f(z_0)=f'(z_0)=\dots=f^{(k-1)}(z_0)=0$ and $f^{(k)}(z_0)\neq0$.

$z_0\in\mathbb{C}\backslash D$ is *a pole of order k of f* if z_0 is a zero of multiplicity k of $\frac{1}{f}$.

T4 Theorem Let D be an open set and $f:D\longrightarrow\mathbb{C}$ holomorphic.

z_0 is a an isolated singular point and $\{\,z\mid 0<|z-z_0|<r\,\}\subset D$ $\Rightarrow$ there exists a unique Laurent series such that $f(z)=\sum\limits_{n=-\infty}^{\infty}a_n\,(z-z_0)^n$, for z satisfying $0<|z-z_0|<r$.

The number $\mathbf{Res}_{z_0}f\stackrel{\text{def}}{=}a_{-1}$ is the **residue** of f in z_0.

T5 Theorem

If z_0 pole of order k, then around z_0 the function f admits an expansion of the form

$$f(z)=\frac{a_{-k}}{(z-z_0)^k}+\dots+\frac{a_{-1}}{(z-z_0)}+a_0+a_1(z-z_0)+\dots$$

and

$$\mathbf{Res}_{z_0}f=\frac{1}{(k-1)!}\lim_{z\to z_0}\left((z-z_0)^k\,f(z)\right)^{(k-1)}$$

D6 Definition (*Index* of a point z_0 with respect to a path).

For a closed path γ not passing through z_0,

$n(z_0,\gamma)\stackrel{\text{def}}{=}\frac{1}{2\pi\mathrm{i}}\int_\gamma\frac{1}{z-z_0}dz$ shows how many turns around z_0, γ makes.

D7 Definition. Let D be an open set .

A closed path $\gamma:[a,b]\to D$ is called *homotopic to zero in* D if γ can be continuosly deformed inside D up to a constant path (a path having as image just a point).

T6 Theorem of Residues. For an open set $D\subseteq\mathbb{C}$:

$f:D\longrightarrow\mathbb{C}$ holomorphic function
S set of isolated singular points
γ path homotopic to zero in $D\cup S$
$\Rightarrow$ $\int_\gamma f(z)dz=2\pi\mathrm{i}\sum\limits_{z\in S}n(z,\gamma)\mathbf{Res}_z f$

SOLUTIONS

Problem 1

$\boxed{1a}$ By using D1, we get

$$(2-\mathrm{i})^2=4-4\mathrm{i}-1=3-4\mathrm{i},$$
$$\overline{1+2\mathrm{i}}=1-2\mathrm{i},$$
$$\frac{1-\mathrm{i}}{1+\mathrm{i}}=\frac{(1-\mathrm{i})^2}{(1-\mathrm{i})(1+\mathrm{i})}=\frac{1-2\mathrm{i}-1}{2}=\frac{-2\mathrm{i}}{2}=-\mathrm{i},$$
$$\mathrm{i}^7=\mathrm{i}^4\,\mathrm{i}^3=-\mathrm{i},$$
$$|\sqrt{3}+\mathrm{i}|=\sqrt{3+1}=2,$$

whence

$$(2-\mathrm{i})^2+(\overline{1+2\mathrm{i}})+\frac{1-\mathrm{i}}{1+\mathrm{i}}+\mathrm{i}^7+|\sqrt{3}+\mathrm{i}|=6-8\mathrm{i}.$$

$\boxed{1b}$ By using D1, we get

$$\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}=\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}=\frac{\sqrt{3}}{2}+\frac{\mathrm{i}}{2},$$
$$\mathrm{e}^{1+\mathrm{i}\frac{\pi}{3}}=\mathrm{e}\,\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}=\mathrm{e}\,(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3})=-\frac{\mathrm{e}}{2}-\mathrm{i}\frac{\mathrm{e}\sqrt{3}}{2},$$
$$\cos(\mathrm{i})=\frac{\mathrm{e}^{-1}+\mathrm{e}}{2},$$

whence

$$\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}+\mathrm{e}^{1+\mathrm{i}\frac{\pi}{3}}+\cos(\mathrm{i})=\frac{\sqrt{3}-\mathrm{e}\sqrt{3}-\mathrm{e}^{-1}}{2}+\frac{\mathrm{i}}{2}.$$

$\boxed{1c}$ By using D1, we obtain

$$(z^3+\tfrac{1}{z^3}+\cos(2z)+z^3\mathrm{e}^{\mathrm{i}z})'=3z^2-\tfrac{3}{z^4}-2\sin 2z+3z^2\mathrm{e}^{\mathrm{i}z}+\mathrm{i}z^3\mathrm{e}^{\mathrm{i}z}.$$

Problem 2

$\boxed{2a}$ We use T5 and D1. The singular points are i and $-\mathrm{i}$. They are poles of order 1 of f, and consequently

$$\mathbf{Res}_{\mathrm{i}}f=\frac{1}{0!}\lim_{z\to\mathrm{i}}(z-\mathrm{i})f(z)$$
$$=\lim_{z\to\mathrm{i}}(z-\mathrm{i})\frac{\mathrm{e}^z}{(z-\mathrm{i})(z+\mathrm{i})}=\lim_{z\to\mathrm{i}}\frac{\mathrm{e}^z}{(z+\mathrm{i})}=-\frac{1}{2}\mathrm{i}\mathrm{e}^{\mathrm{i}},$$
$$\mathbf{Res}_{-\mathrm{i}}f=\frac{1}{0!}\lim_{z\to-\mathrm{i}}(z+\mathrm{i})f(z)$$
$$=\lim_{z\to-\mathrm{i}}(z+\mathrm{i})\frac{\mathrm{e}^z}{(z-\mathrm{i})(z+\mathrm{i})}=\lim_{z\to-\mathrm{i}}\frac{\mathrm{e}^z}{(z-\mathrm{i})}=\frac{1}{2}\mathrm{i}\mathrm{e}^{-\mathrm{i}}.$$

$\boxed{2b}$ We use T2 and T4. The only singular point is 1. Since it is a pole of order 3, we get

$$\mathbf{Res}_1 g=\frac{1}{2!}\lim_{z\to1}\Big((z-1)^3\frac{\mathrm{e}^z}{(z-1)^3}\Big)''=\frac{1}{2}\lim_{z\to1}(\mathrm{e}^z)''=\frac{1}{2}\lim_{z\to1}\mathrm{e}^z=\frac{\mathrm{e}}{2}.$$

Alternatively, $\mathbf{Res}_1 g$ can be obtained by using the Laurent series of g as the coefficient of $\frac{1}{z-1}$ from the expansion of g around the point 1, namely

$$g(z)=\frac{1}{(z-1)^3}\,\mathrm{e}^z=\frac{1}{(z-1)^3}\,\mathrm{e}^{z-1+1}=\frac{\mathrm{e}}{(z-1)^3}\,\mathrm{e}^{z-1}$$
$$=\frac{\mathrm{e}}{(z-1)^3}\left[1+\frac{z-1}{1!}+\frac{(z-1)^2}{2!}+\frac{(z-1)^3}{3!}+\frac{(z-1)^4}{4!}+\cdots\right]$$
$$=\ldots+\frac{\mathrm{e}}{2!}\frac{1}{z-1}+\ldots$$

Problem 3

$\boxed{3a}$ By using the definition D4 of the complex integral, we get

$$I_0=\int_{\gamma_0}\bar{z}\,dz=\int_0^1(\overline{t-1+t\mathrm{i}})(t-1+t\mathrm{i})'dt=\int_0^1(t-1-t\mathrm{i})(1+\mathrm{i})dt$$
$$=(1+\mathrm{i})\int_0^1(t-1-t\mathrm{i})dt=(1+\mathrm{i})\left(\frac{t^2}{2}-t-\frac{t^2}{2}\mathrm{i}\right)\Big|_0^1$$
$$=(1+\mathrm{i})\left(\tfrac{1}{2}-1-\tfrac{1}{2}\mathrm{i}\right)=-\tfrac{1}{2}(1+\mathrm{i})(1+\mathrm{i})=-\mathrm{i}.$$

$\boxed{3b}$ We use T5 and the theorem of residues T6. The only singular point of the function is $z_0=1$, and it is a pole of order 3. Consequently

$$\mathbf{Res}_1 g=\frac{1}{2!}\lim_{z\to1}\Big((z-1)^3\frac{\mathrm{e}^z}{(z-1)^3}\Big)''=\frac{1}{2}\lim_{z\to1}(\mathrm{e}^z)''=\frac{1}{2}\lim_{z\to1}\mathrm{e}^z=\frac{\mathrm{e}}{2}.$$

The index of 1 with restect to the circular path γ_1 is $n(1,\gamma_1)=1$ because γ turns once around $z_0=1$. In view of the theorem of residues

$$I_1=\int_{\gamma_1}\frac{\mathrm{e}^z}{(z-1)^3}dz=2\pi\mathrm{i}\,\mathbf{Res}_1\frac{\mathrm{e}^z}{(z-1)^3}=2\pi\mathrm{i}\;n(1,\gamma_1)\frac{\mathrm{e}}{2}=\mathrm{e}\pi\mathrm{i}.$$

Problem 4

$\boxed{4a}$ We have

$$f(z)=\frac{1}{z-1}=\frac{1}{z-\mathrm{i}-1+\mathrm{i}}=\frac{1}{\mathrm{i}-1}\,\frac{1}{1-\frac{1+\mathrm{i}}{2}(z-\mathrm{i})}.$$

Because

$$\left|\tfrac{1+\mathrm{i}}{2}(z-\mathrm{i})\right|<1\quad\Leftrightarrow\quad|z-\mathrm{i}|<\sqrt{2},$$

by using T2, we get

$$f(z)=-\frac{1+\mathrm{i}}{2}\left[1+\frac{1+\mathrm{i}}{2}(z-\mathrm{i})+\frac{(1+\mathrm{i})^2}{2^2}(z-\mathrm{i})^2+\cdots\right]$$
$$=-\frac{1+\mathrm{i}}{2}-\frac{(1+\mathrm{i})^2}{2^2}(z-\mathrm{i})-\frac{(1+\mathrm{i})^3}{2^3}(z-\mathrm{i})^2+\cdots$$

$\boxed{4b}$ We have

$$f(z)=\frac{1}{z-1}=\frac{1}{z-\mathrm{i}-1+\mathrm{i}}=\frac{1}{z-\mathrm{i}}\,\frac{1}{1-\frac{1-\mathrm{i}}{z-\mathrm{i}}}.$$

Because

$$\left|\frac{1-\mathrm{i}}{z-\mathrm{i}}\right|<1\quad\Leftrightarrow\quad|z-\mathrm{i}|>\sqrt{2},$$

by using T2, we get

$$f(z)=\frac{1}{z-\mathrm{i}}\left[1+\frac{1-\mathrm{i}}{z-\mathrm{i}}+\frac{(1-\mathrm{i})^2}{(z-\mathrm{i})^2}+\frac{(1-\mathrm{i})^3}{(z-\mathrm{i})^3}+\cdots\right]$$
$$=\frac{1}{z-\mathrm{i}}+\frac{1-\mathrm{i}}{(z-\mathrm{i})^2}+\frac{(1-\mathrm{i})^2}{(z-\mathrm{i})^3}+\frac{(1-\mathrm{i})^3}{(z-\mathrm{i})^4}+\cdots$$

Problem 5

We have

$$I=\int_0^{2\pi}\frac{2+\cos t-2}{2+\cos t}dt=\int_0^{2\pi}dt-2\int_0^{2\pi}\frac{1}{2+\cos t}dt$$

By using the definition D1, the expression

$$I_0=\int_0^{2\pi}\frac{1}{2+\cos t}dt$$

can be written as

$$I_0=\int_0^{2\pi}\frac{1}{2+\frac{1}{2}(\mathrm{e}^{\mathrm{i}t}+\mathrm{e}^{-\mathrm{i}t})}dt$$

or

$$I_0=\int_0^{2\pi}\frac{2}{4+\mathrm{e}^{\mathrm{i}t}+\mathrm{e}^{-\mathrm{i}t}}dt$$

or

$$I_0=\int_0^{2\pi}\frac{2}{\mathrm{i}\mathrm{e}^{\mathrm{i}t}}\,\frac{1}{4+\mathrm{e}^{\mathrm{i}t}+\mathrm{e}^{-\mathrm{i}t}}\,(\mathrm{e}^{\mathrm{i}t})'dt.$$

This formula, written as

$$I_0=\int_0^{2\pi}\frac{-2\mathrm{i}}{(\mathrm{e}^{\mathrm{i}t})^2+4\mathrm{e}^{\mathrm{i}t}+1}\,(\mathrm{e}^{\mathrm{i}t})'dt$$

is the complex integral

$$\int_\gamma\frac{-2\mathrm{i}}{z^2+4z+1}dz$$

of $f(z)=\frac{-2\mathrm{i}}{z^2+4z+1}$ along $\gamma:[0,2\pi]\to\mathbb{C},\quad\gamma(t)=\mathrm{e}^{\mathrm{i}t}$. It can be computed by using the theorem T6. The isolated singular points of f are

$$z_{1,2}=-2\pm\sqrt{4-1}=-2\pm\sqrt{3},$$

but only $z_1=-2+\sqrt{3}$ is inside the domain with the frontier γ. The point z_1 is a pole of order 1, and

$$\mathbf{Res}_{z_1}\frac{-2\mathrm{i}}{z^2+4z+1}=\lim_{z\to z_1}(z-z_1)\frac{-2\mathrm{i}}{(z-z_1)(z-z_2)}$$
$$=\lim_{z\to z_1}\frac{-2\mathrm{i}}{z-z_2}=\frac{-2\mathrm{i}}{z_1-z_2}=\frac{-2\mathrm{i}}{2\sqrt{3}}=\frac{-\mathrm{i}}{\sqrt{3}}.$$

Therefore,

$$I=2\pi-2I_0=2\pi-2\pi\mathrm{i}\,\mathbf{Res}_{z_1}f(z)$$
$$=2\pi-2\pi\mathrm{i}\,\frac{-\mathrm{i}}{\sqrt{3}}=2\pi-\frac{2\pi}{\sqrt{3}}.$$