LINEAR ALGEBRA - Solved Exercises 3

(Real and complex finite-dimensional vector spaces)

Nicolae Cotfas, version 05 Feb. 2023 (for future updates see https://unibuc.ro/user/nicolae.cotfas/)

A PARTIAL EVALUATION A

Exercise 1

A1 In $\mathbb{R}^3$, find the dimension of the subspace
$$\mathcal{V}=\{\ x=(x_1,x_2,x_3)\in\mathbb{R}^3\ |\ x_1-x_2-x_3=0\ \}.$$

A2 In $\mathbb{R}^3$, find the dimension of the subspace
$$\mathcal{W}=\left\{\ x=(x_1,x_2,x_3)\in\mathbb{R}^3\ \middle|\ \begin{array}{l} x_1+x_2+x_3=0\\ x_1-x_2+x_3=0\end{array}\right\}.$$

Exercise 2

A3 Find the eigenvalues of the linear operator
$$A:\mathbb{R}^2\longrightarrow\mathbb{R}^2,\quad A(x_1,x_2)=(x_2,x_1).$$

A4 Describe the eigenspaces of the linear operator
$$A:\mathbb{R}^2\longrightarrow\mathbb{R}^2,\quad A(x_1,x_2)=(x_2,x_1).$$

Exercise 3

A5 Prove that $\{\mathbb{I},\sigma_x,\sigma_y,\sigma_z\}$, where
$$\mathbb{I}=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ \sigma_y=\begin{pmatrix}0&-\mathrm{i}\\ \mathrm{i}&0\end{pmatrix},\ \sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix},$$
is a basis in the complex vector space
$$\mathcal{M}=\left\{\ A=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}\ \middle|\ a_{jk}\in\mathbb{C}\ \right\}.$$

A6 Find the coordinates of $M=\begin{pmatrix}2&1-\mathrm{i}\\1+\mathrm{i}&0\end{pmatrix}$.

Exercise 4

A7 In the case of the real vector space $\mathbb{R}^2$, prove that
$$A:\mathbb{R}^2\longrightarrow\mathbb{R}^2,\quad A(x_1,x_2)=(x_1-x_2,x_1+x_2)$$
is not diagonalizable

A8 In the case of the real vector space $\mathbb{R}^2$, prove that
$$B:\mathbb{R}^2\longrightarrow\mathbb{R}^2,\quad B(x_1,x_2)=(x_1+x_2,x_2)$$
is not diagonalizable

A9 In the case of the complex vector space $\mathbb{C}^2$, prove that
$$A:\mathbb{C}^2\longrightarrow\mathbb{C}^2,\quad A(x_1,x_2)=(x_1-x_2,x_1+x_2)$$
is diagonalizable. Find the operator $A^n:\mathbb{C}^2\longrightarrow\mathbb{C}^2$.

Some definitions, notations and results

1 **Definition**. $\{v_1,v_2,...,v_n\}$ is a **linearly independent** set of vectors if $\boxed{\begin{array}{c}\alpha_1v_1+\alpha_2v_2+...+\alpha_nv_n=0\\ \Downarrow\\ \alpha_1=\alpha_2=...=\alpha_n=0\end{array}}$

2 **Definition**. In a vector space over the field $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$,
$$\boxed{\mathrm{span}\{v_1,v_2,...,v_n\}\overset{\text{def}}{=}\{\alpha_1v_1+\alpha_2v_2+...+\alpha_nv_n\ |\ \alpha_k\in\mathbb{K}\}}$$

3 **Definition**. $\mathfrak{B}=\{v_1,v_2,...,v_n\}$ is a **basis** of $\mathcal{V}$ if $\boxed{\begin{array}{c}\mathcal{V}=\mathrm{span}\,\mathfrak{B}\\ \text{and}\\ \mathfrak{B}\text{ linearly indep.}\end{array}}$

4 **Definition**. **Dimension** of a vector space over $\mathbb{K}$ is $\boxed{\dim_{\mathbb{K}}\mathcal{V}\overset{\text{def}}{=}\begin{cases}\begin{array}{c}\text{number}\\ \text{of vectors}\\ \text{of a basis}\end{array} & \text{if } \mathcal{V}\neq\{0\};\\ 0 & \text{if } \mathcal{V}=\{0\}.\end{cases}}$

5 **Definition**. The **sum** *of the subspaces* $\mathcal{W}_1,\mathcal{W}_2\subseteq\mathcal{V}$ is $\boxed{\mathcal{W}_1+\mathcal{W}_2\overset{\text{def}}{=}\left\{w_1+w_2\ \middle|\ \begin{array}{l}w_1\in\mathcal{W}_1\\ w_2\in\mathcal{W}_2\end{array}\right\}}$

6 **Definition**. Linear operator $A:\mathcal{V}\to\mathcal{V}$ is **diagonalizable** if $\boxed{\begin{array}{c}\text{there exists an basis of } \mathcal{V}\\ \text{consisting of eigenvectors of } A.\end{array}}$

7 **Definition** In a Hilbert space $\mathcal{H}$: An operator $A:\mathcal{H}\to\mathcal{H}$ is a **self-adjoint operator** if $\boxed{\langle Ax,y\rangle=\langle x,Ay\rangle}$ $\forall x,y\in\mathcal{H}$.

B PARTIAL EVALUATION B

Exercise 1

Let us consider in $\mathbb{R}^3$ the subspace
$$\mathcal{H}=\{\ x=(x_1,x_2,x_3)\in\mathbb{R}^3\ \ |\ \ x_2+x_3=0\ \}.$$

B1 Find a basis of $\mathcal{H}$.

B2 Find an orthonormal basis of $\mathcal{H}$.

B3 Describe the orthogonal complement of $\mathcal{H}$ by indicating a basis.

B4 Find the orthogonal projection of $x=(1,2,3)$ on $\mathcal{H}$.

Exercise 2

B5 Prove that the linear operator
$$A:\mathbb{C}^3\longrightarrow\mathbb{C}^3,\quad A(x_1,x_2,x_3)=(x_3,-x_2,x_1),$$
is self-adjoint, and find its eigenvalues/eigenvectors.

B6 Prove that the linear operator
$$U:\mathbb{C}^2\longrightarrow\mathbb{C}^2,\quad U(x_1,x_2)=\tfrac{1}{\sqrt{2}}(x_1-x_2,x_1+x_2),$$
is unitary, and find its eigenvalues/eigenvectors.

Exercise 3

On the Hilbert space
$$\mathbb{C}^3=\left\{x=\begin{pmatrix}x_0\\x_1\\x_2\end{pmatrix}\middle|\ x_0,\ x_1,\ x_2\in\mathbb{C}\right\}\ \text{ with } \langle x,y\rangle=\sum_{n=0}^{2}\overline{x_n}\,y_n,$$
we consider the linear operator $F:\mathbb{C}^3\longrightarrow\mathbb{C}^3$,
$$F\begin{pmatrix}x_0\\x_1\\x_2\end{pmatrix}=\frac{1}{\sqrt{3}}\begin{pmatrix}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}}&1&\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\\ 1&1&1\\ \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}&1&\mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}}\end{pmatrix}\begin{pmatrix}x_0\\x_1\\x_2\end{pmatrix}$$
where $\mathrm{e}^{\mathrm{i}t}=\cos t+\mathrm{i}\sin t$.

B7 Compute F^2, F^3, F^4 and F^{2023}.

B8 Prove that F is a unitary operator.

B9 Prove that $\lambda_0=1$ is an eigenvalue of F.

B10 Prove that $\lambda_1=-\mathrm{i}$ is an eigenvalue of F.

B11 Prove that $\lambda_2=-1$ is an eigenvalue of F.

B12 Find an orthonormal basis in $\mathbb{C}^3$ containing only eigenvectors of F.

B13 Indicate the spectral decomposition of F.

8 **Definition**. In a Hilbert space $\mathcal{H}$: An operator $A:\mathcal{H}\to\mathcal{H}$ is a **unitary operator** if $\boxed{\langle Ax,Ay\rangle=\langle x,y\rangle}$ $\forall x,y\in\mathcal{H}$.

9 **Definition**. In the case of vector space over $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$,

λ is an **eigenvalue** of a linear operator $A:\mathcal{V}\to\mathcal{V}$ if
- $\lambda\in\mathbb{K}$ and
- there exists $v\in\mathcal{V}$, $v\neq0$ (**eigenvector** corresp. to λ) such that $\boxed{Av=\lambda\,v}$.

The **eigenspace** corresponding to λ is $\boxed{\{\ x\ |\ Ax=\lambda x\ \}}$.

10 **Theorem**. In the case of vector space over $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$,

$\boxed{\lambda \text{ is an eigenvalue of a linear operator } A:\mathcal{V}\to\mathcal{V}}$ $\Leftrightarrow$
- $\lambda\in\mathbb{K}$ and
- λ is a root of the equation
$$\begin{vmatrix}a_1^1-\lambda&a_2^1&\cdots&a_n^1\\ a_1^2&a_2^2-\lambda&\cdots&a_n^2\\ \vdots&\vdots&\ddots&\vdots\\ a_1^n&a_2^n&\cdots&a_n^n-\lambda\end{vmatrix}=0$$

■ SOLUTIONS for PARTIAL EVALUATION A

$\boxed{A1}$ $\mathcal{V}=\{(x_2+x_3,x_2,x_3)|x_{2,3}\in\mathbb{R}\}$
$=\{x_2(1,1,0)+x_3(1,0,1)|x_{2,3}\in\mathbb{R}\}=\text{span}\{(1,1,0),(1,0,1)\}$.
Since $(1,1,0)$ and $(1,0,1)$ are lin. independent, it follows that $\dim\mathcal{V}=2$ (a plane in $\mathbb{R}^3$).

$\boxed{A2}$ $\mathcal{W}=\{(-x_3,0,x_3)|x_3\in\mathbb{R}\}=\text{span}\{(-1,0,1)\}$
$\Downarrow$
$\dim\mathcal{W}=1$ (a line in $\mathbb{R}^3$).

$\boxed{A3}$ In the canonical basis $\{(1,0),(0,1)\}$, the matrix of A is

$$A=\begin{pmatrix}0&1\\1&0\end{pmatrix}.\qquad \begin{vmatrix}-\lambda&1\\1&-\lambda\end{vmatrix}=0\ \Rightarrow\ \lambda=\pm1.$$

$\boxed{A4}$ $\{\,(x_1,x_2)\mid A(x_1,x_2)=1\,(x_1,x_2)\,\}=\text{span}\{(1,1)\}$,
$\{\,(x_1,x_2)\mid A(x_1,x_2)=-1\,(x_1,x_2)\,\}=\text{span}\{(1,-1)\}$.

$\boxed{A5}$

$$\alpha_0\mathbb{I}+\alpha_1\sigma_x+\alpha_2\sigma_y+\alpha_3\sigma_z=0\quad\Rightarrow\quad\alpha_0=\alpha_1=\alpha_2=\alpha_3=0.$$

As $\mathcal{M}\cong\mathbb{C}^4$, we have $\dim\mathcal{M}=4$, so that $\{\mathbb{I},\sigma_x,\sigma_y,\sigma_z\}$ is a *maximal* set of linear independent "vectors", hence a basis.

$\boxed{A6}$ $M=\mathbb{I}+\sigma_x+\sigma_y+\sigma_z$.

$\boxed{A7}$ In the canonical basis $\{(1,0),(0,1)\}$, the matrix of A is

$$A=\begin{pmatrix}1&-1\\1&1\end{pmatrix}.\qquad\begin{vmatrix}1-\lambda&-1\\1&1-\lambda\end{vmatrix}=0\Rightarrow\lambda=1\pm\mathrm{i}\notin\mathbb{R}.$$

So the linear operator A defined on the real vector space $\mathbb{R}^2$ has no eigenvalues; in particular it cannot be diagonalizable.

$\boxed{A8}$ In the canonical basis $\{(1,0),(0,1)\}$, the matrix of B is

$$B=\begin{pmatrix}1&1\\0&1\end{pmatrix}.\qquad\begin{vmatrix}1-\lambda&1\\0&1-\lambda\end{vmatrix}=0\Rightarrow\lambda_1=\lambda_2=1.$$

The allgebraic multiplicity of the root 1 is 2.
But $\{\,(x_1,x_2)\mid B(x_1,x_2)=1\,(x_1,x_2)\,\}=\text{span}\{(1,0)\}$, so geometric multiplicty is $1\neq2\Rightarrow B$ it is not diagonalizable.

$\boxed{A9}$ In the canonical basis $\{(1,0),(0,1)\}$, the matrix of A is

$$A=\begin{pmatrix}1&-1\\1&1\end{pmatrix}.\qquad\begin{vmatrix}1-\lambda&-1\\1&1-\lambda\end{vmatrix}=0\Rightarrow\lambda=1\pm\mathrm{i}\in\mathbb{C}.$$

$\{\,(x_1,x_2)\mid A(x_1,x_2)=(1+\mathrm{i})\,(x_1,x_2)\,\}=\text{span}\{(1,-\mathrm{i})\}$
$\{\,(x_1,x_2)\mid A(x_1,x_2)=(1-\mathrm{i})\,(x_1,x_2)\,\}=\text{span}\{(1,\mathrm{i})\}$.
This shows that all the roots of the characteristic polynomial belong to $\mathbb{C}$, and for each of them, the algebraic and geometric multiplicities are equal (to 1). Therefore A is diagonalizable. More exactly,

$$S^{-1}AS=\begin{pmatrix}1+\mathrm{i}&0\\0&1-\mathrm{i}\end{pmatrix},\quad\text{where}\quad S=\begin{pmatrix}1&1\\-\mathrm{i}&\mathrm{i}\end{pmatrix}$$

is the change of basis matrix from the canonical basis $\{(1,0),(0,1)\}$ to the basis $\{(1,-\mathrm{i}),(1,\mathrm{i})\}$. Consequently,

$$A^2=S\begin{pmatrix}1+\mathrm{i}&0\\0&1-\mathrm{i}\end{pmatrix}S^{-1}S\begin{pmatrix}1+\mathrm{i}&0\\0&1-\mathrm{i}\end{pmatrix}S^{-1}.$$

and

$$A^n=S\begin{pmatrix}1+\mathrm{i}&0\\0&1-\mathrm{i}\end{pmatrix}^nS^{-1}=S\begin{pmatrix}(1+\mathrm{i})^n&0\\0&(1-\mathrm{i})^n\end{pmatrix}S^{-1}.$$

where $(1\pm\mathrm{i})^n=(\sqrt{2}\mathrm{e}^{\pm\frac{\pi\mathrm{i}}{4}})^n=(\sqrt{2})^n\mathrm{e}^{\pm\frac{n\pi\mathrm{i}}{4}},\ S^{-1}=\begin{pmatrix}\frac12&\frac{\mathrm{i}}{2}\\\frac12&-\frac{\mathrm{i}}{2}\end{pmatrix}$.

■ SOLUTIONS for PARTIAL EVALUATION B

$\boxed{B1}$

$$\begin{aligned}\text{Since }\mathcal{H}&=\{\ (x_1,x_2,x_3)\ \mid\ x_2+x_3=0\ \}\\&=\{\ (x_1,x_2,-x_2)\ \mid\ x_1,\,x_2\in\mathbb{R}\ \}\\&=\{\ x_1(1,0,0)+x_2(0,1,-1)\ \mid\ x_1,\,x_2\in\mathbb{R}\ \}\\&=\text{span}\{\ (1,0,0),\,(0,1,-1)\ \}\end{aligned}$$

and $\alpha(1,0,0)+\beta(0,1,-1)=(0,0,0)\ \Rightarrow\ \alpha=\beta=0$,
the set $\{v_1=(1,0,0),\,v_2=(0,1,-1)\ \}$ is a basis in $\mathcal{H}$.

$\boxed{B2}$
Since $\langle v_1,v_2\rangle=0$, the basis $\{v_1,\,v_2\}$ is orthogonal.
The set $\left\{u_1=\frac{v_1}{||v_1||}=(1,0,0),\,u_2=\frac{v_2}{||v_2||}=\left(0,\frac{1}{\sqrt2},-\frac{1}{\sqrt2}\right)\right\}$ is an orthonormal basis in $\mathcal{H}$.

$\boxed{B3}$

$$\begin{aligned}\text{Since }\mathcal{H}^\perp&=\{x=(x_1,x_2,x_3)\ \mid\ \langle x,y\rangle=0,\text{ for any }y\in\mathcal{H}\}\\&=\{\ x=(x_1,x_2,x_3)\ \mid\ \langle x,v_1\rangle=0,\ \langle x,v_2\rangle=0\ \}\\&=\{\ x=(x_1,x_2,x_3)\ \mid\ x_1=0,\ x_2-x_3=0\ \}\\&=\{\ (0,x_2,x_2)\mid x_2\in\mathbb{R}\}=\{\ \alpha(0,1,1)\mid\alpha\in\mathbb{R}\}\end{aligned}$$

$\{(0,1,1)\}$is a basis in the one-dimensional space $\mathcal{H}^\perp$.

$\boxed{B4}$ In Dirac's notation, the orthogonal projector on $\mathcal{H}$ is

$$P=|u_1\rangle\langle u_1|+|u_2\rangle\langle u_2|.$$

Consequently, the orthogonal projection of $x=(1,2,3)$ is

$$\begin{aligned}P|x\rangle&=|u_1\rangle\langle u_1|x\rangle+|u_2\rangle\langle u_2|x\rangle\\&=|u_1\rangle-\tfrac{1}{\sqrt2}|u_2\rangle=(1,0,0)-\left(0,\tfrac12,-\tfrac12\right)=\left(1,-\tfrac12,\tfrac12\right).\end{aligned}$$

$\boxed{B5}$

$$A=\begin{pmatrix}0&0&1\\0&-1&0\\1&0&0\end{pmatrix}\text{ satisfies }A^\dagger=A.\qquad\begin{vmatrix}0-\lambda&0&1\\0&-1-\lambda&0\\1&0&0-\lambda\end{vmatrix}=0$$

$$\Rightarrow\begin{array}{ll}\lambda_1=1,&\{x\mid Ax=x\}=\text{span}\{(1,0,1)\},\\\lambda_2=\lambda_3=-1,&\{x\mid Ax=-x\}=\text{span}\{(0,1,0),(1,0,-1)\}.\end{array}$$

$\boxed{B6}$

$$U=\begin{pmatrix}\frac{1}{\sqrt2}&-\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\end{pmatrix}\text{ satisfies }UU^\dagger=\mathbb{I}.\qquad\begin{vmatrix}\frac{1}{\sqrt2}-\lambda&-\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&\frac{1}{\sqrt2}-\lambda\end{vmatrix}=0$$

$$\Rightarrow\begin{array}{ll}\lambda_1=\frac{1}{\sqrt2}+\frac{\mathrm{i}}{\sqrt2},&\{x\mid Ux=\lambda_1x\}=\text{span}\{(1,-\mathrm{i})\},\\\lambda_2=\frac{1}{\sqrt2}-\frac{\mathrm{i}}{\sqrt2},&\{x\mid Ux=\lambda_2x\}=\text{span}\{(1,\mathrm{i})\}.\end{array}$$

$\boxed{B7}$

$$F^2=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix},\ F^3=\frac{1}{\sqrt3}\begin{pmatrix}\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}&1&\mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}}\\1&1&1\\\mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}}&1&\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\end{pmatrix}\equiv F^\dagger,$$

$$F^4=(F^2)^2=\mathbb{I}\ \text{ and }\ F^{2023}=(F^4)^{505}F^3=F^3=F^\dagger.$$

$\boxed{B8}$ We have $FF^\dagger=\mathbb{I}$.

$\boxed{B9}$ We have to prove that there exists $v_0\neq0$ such that $Fv_0=v_0$. The equations of the system

$$\begin{cases}\left(-\frac12-\mathrm{i}\frac{\sqrt3}{2}\right)x_0+x_1+\left(-\frac12+\mathrm{i}\frac{\sqrt3}{2}\right)x_2=\sqrt3\,x_0\\ x_0+x_1+x_2=\sqrt3\,x_1\\ \left(-\frac12+\mathrm{i}\frac{\sqrt3}{2}\right)x_0+x_1+\left(-\frac12-\mathrm{i}\frac{\sqrt3}{2}\right)x_2=\sqrt3\,x_2\end{cases}$$

are not linearly independent because

$$\begin{vmatrix}-\frac12-\sqrt3-\mathrm{i}\frac{\sqrt3}{2}&1&-\frac12+\mathrm{i}\frac{\sqrt3}{2}\\1&1-\sqrt3&1\\-\frac12+\mathrm{i}\frac{\sqrt3}{2}&1&-\frac12-\sqrt3-\mathrm{i}\frac{\sqrt3}{2}\end{vmatrix}=0.$$

We get

$$\begin{cases}\left(-\frac12-\mathrm{i}\frac{\sqrt3}{2}\right)x_0+x_1+\left(-\frac12+\mathrm{i}\frac{\sqrt3}{2}\right)x_2=\sqrt3\,x_0\\ x_0+x_1+x_2=\sqrt3\,x_1\end{cases}\Rightarrow v_0=\begin{pmatrix}1\\1+\sqrt3\\1\end{pmatrix}.$$

$\boxed{B10,11}$

In a similar way, we get $v_1=\begin{pmatrix}-1\\0\\1\end{pmatrix},\ v_2=\begin{pmatrix}1\\1-\sqrt3\\1\end{pmatrix}$.

$\boxed{B12}$ An orthonormal eigenbasis corresponding to F is

$$\left\{u_0=\frac{v_0}{||v_0||}),\ u_1=\frac{v_1}{||v_1||},u_2=\frac{v_2}{||v_2||}\right\}$$

$$=\left\{u_0=\begin{pmatrix}\frac12\sqrt{1-\frac{1}{\sqrt3}}\\\frac{1}{\sqrt2}\sqrt{1+\frac{1}{\sqrt3}}\\\frac12\sqrt{1-\frac{1}{\sqrt3}}\end{pmatrix},u_1=\begin{pmatrix}-\frac{1}{\sqrt2}\\0\\\frac{1}{\sqrt2}\end{pmatrix},u_2=\begin{pmatrix}\frac12\sqrt{1+\frac{1}{\sqrt3}}\\-\frac{1}{\sqrt2}\sqrt{1-\frac{1}{\sqrt3}}\\\frac12\sqrt{1+\frac{1}{\sqrt3}}\end{pmatrix}\right\}.$$

$\boxed{B13}$ The spectral decomposition of F is

$$F=|u_0\rangle\langle u_0|-\mathrm{i}|u_1\rangle\langle u_1|-|u_2\rangle\langle u_2|.$$