DIFFERENTIAL EQUATIONS - Solved Exercises 2

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EXERCISES

Exercise 1

Find the general solution of the equations:

1a $y'' - 2y' - 3y = 0.$

1b $y'' - 2y' + y = 0.$

1c $y'' - 2y' + 2y = 0.$

1d $y''' + y'' + y' + y = 0.$

Exercise 2

Solve the equations

2a $y' - 2xy = 0.$

2b $y' - 2xy = x.$

2c $y'' - (\frac{1}{x} + 2x)y' = 0.$

Exercise 3

Solve by two methods

3 $(x+y)dx + (x-y)dy = 0.$

Exercise 4

Solve by two methods

4 $\begin{cases} y_1' = 2y_2 \\ y_2' = 2y_3 \\ y_3' = -2y_1 \end{cases}$

Exercise 5

Find the equation of the form

5 $y'' + \alpha(x)y' + \beta(x)y = 0$

admitting the solutions $y_1(x)=x$ and $y_2(x)=x^2$.

Exercise 6

Solve the equation $xy' - 2y = 0$

6a - by considering it as a separable equation;

6b - by considering it as an Euler equation;

6c - by using the method of integrating factor;

6d - by using the method of power series.

SOME DEFINITIONS, THEOREMS and REMARKS

T1 Theorem (*Linear equations with constant coefficients*).

The space of all the real solutions of

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = 0 \qquad (1)$$

where $a_0,\ldots,a_n \in \mathbb{R}$, is a real vector space of dimension n.

D1 Definition. The polynomial

$$P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$$

is called the *characteristic polynomial* of the equation (1).

T2 Theorem (Particular solutions).

$$\boxed{y(x) = e^{\lambda x} \text{ is a solution of (1)}} \Leftrightarrow \boxed{P(\lambda) = 0}$$

D2 Definition (*Complex exponential*).

$$e^{(\alpha+\beta i)x} = e^{\alpha x}\cos\beta x + i\, e^{\alpha x}\sin\beta x$$

T3 Theorem. General solution of $a_0y'' + a_1y' + a_2y = 0.$

$P(\lambda) = a_0\lambda^2 + a_1\lambda + a_2$ has the roots $\lambda_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0\, a_2}}{2a_0}$.

General solution:

- $\lambda_1 \neq \lambda_2 \in \mathbb{R} \Rightarrow y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}.$
- $\lambda_1 = \lambda_2 = \lambda \Rightarrow y(x) = C_1 e^{\lambda x} + C_2\, x\, e^{\lambda x}.$
- $\lambda_{1,2} = \alpha \pm \beta i \Rightarrow y(x) = C_1 e^{\alpha x}\cos\beta x + C_2 e^{\alpha x}\sin\beta x.$

T4 Theorem (Euler's equation).

$$\boxed{a_0x^n y^{(n)} + \ldots + a_{n-1}xy' + a_n y = 0} \xrightarrow[x=e^t]{\text{change}} \text{linear equation with constant coefficients}$$

T5 Theorem (*Primitives* of a continuous function)

$f:(a,b)\to\mathbb{R}$ continuous $\Rightarrow$ Primitives of f are: $F:(a,b)\to\mathbb{R}$,

$$\boxed{F(x) = \int_{x_0}^{x} f(t)\,dt + C}$$

$x_0 \in (a,b)$ is fixed

We have $F'(x) = f(x)$, that is

$$\boxed{\frac{d}{dx}\left(\int_{x_0}^{x} f(t)\,dt\right) = f(x)}$$

for any $x \in (a,b)$.

T6 Theorem (*Separable equations*).

The solution $y(x)$ of $\boxed{y' = f(x)\,g(y)}$ is defined by

$$\boxed{\int_{y_0}^{y} \frac{1}{g(u)}du = \int_{x_0}^{x} f(t)\,dt + C}$$

x_0, y_0 constants, $g(y_0) \neq 0$.

T7 Linear equation.

$\boxed{y' = f(x)y}$ has the general solution $\boxed{y(x) = C\, e^{\int_{x_0}^{x} f(t)dt}}$

T8 Method of the variation of parameter.

$\boxed{y' = f(x)y + g(x)}$ admits a particular solution of the form $y_p(x) = C(x)\, e^{\int_{x_0}^{x} f(t)dt}$.

T9 Linear non-homogeneous equation.

$$\boxed{\text{general solution of } y' = f(x)y + g(x)} = \boxed{\text{general solution of } y' = f(x)y} + \boxed{\text{a particular solution of } y' = f(x)y + g(x)}$$

T10 Theorem (*Exact equations*).

$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ in a simply connected domain D $\Rightarrow$ The function $F: D \to \mathbb{R}$,

$$F(x,y) = \int_\gamma Pdx + Qdy,$$

where $\gamma:[a,b] \to D$ is an arbitrary path connecting a fixed point (x_0,y_0) with (x,y), defines a function satisfying the relation

$$P(x,y)dx + Q(x,y)dy = dF$$

R3 Remark.

In D, the equation $P(x,y)dx + Q(x,y)dy = 0$ can be written as $dF = 0$, and its solution is described implicitly by $F(x,y) = C$.

R4 Remark. By denoting $Y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$,

$\begin{cases} y_1' = a_{11}y_1 + a_{12}y_2 \\ y_2' = a_{21}y_1 + a_{22}y_2 \end{cases}$ can be written as $\boxed{Y' = AY}$

T11 Theorem.

The space of all the real solutions of $Y' = AY$, where $a_{ij} \in \mathbb{R}$, is a real vector space of dimension 2.

D4 Definition. The polynomial

$$P(\lambda) = \begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = \lambda^2 - (a_{11}+a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21}$$

is called the *characteristic polynomial* of $Y' = AY$.

T12 Theorem (Particular non-null solutions)

$$Y(x) = \begin{pmatrix} p \\ q \end{pmatrix} e^{\lambda x} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ satisfies } Y' = AY \Leftrightarrow \begin{cases} P(\lambda) = 0 \quad \text{and} \\ A\begin{pmatrix} p \\ q \end{pmatrix} = \lambda \begin{pmatrix} p \\ q \end{pmatrix}. \end{cases}$$

T13 Theorem. If λ_1 and λ_2 are the solutions of $P(\lambda) = 0$, then:

$$\left.\begin{array}{l} \bullet\ \lambda_1,\ \lambda_2 \in \mathbb{R} \\ A\begin{pmatrix} p_j \\ q_j \end{pmatrix} = \lambda_j \begin{pmatrix} p_j \\ q_j \end{pmatrix} \\ \text{linearly independent} \end{array}\right\} \Rightarrow Y(x) = C_1 \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} e^{\lambda_1 x} + C_2 \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} e^{\lambda_2 x}.$$

$$\left.\begin{array}{l} \bullet\ \lambda_{1,2} = \alpha \pm \beta i \notin \mathbb{R} \\ A\begin{pmatrix} p \\ q \end{pmatrix} = (\alpha+\beta i)\begin{pmatrix} p \\ q \end{pmatrix} \end{array}\right\} \Rightarrow Y(x) = C_1\, \mathfrak{Re}\left\{\begin{pmatrix} p \\ q \end{pmatrix} e^{(\alpha+\beta i)x}\right\} + C_2\, \mathfrak{Im}\left\{\begin{pmatrix} p \\ q \end{pmatrix} e^{(\alpha+\beta i)x}\right\}.$$

SOLUTIONS

Exercise 1. We use T2 and T3.

1a $\lambda^2-2\lambda-3=0 \Rightarrow \lambda_1=-1,\ \lambda_2=3.$
$T3 \Rightarrow y(x)=C_1\,e^{-x}+C_2\,e^{3x}.$

1b $\lambda^2-2\lambda+1=0 \Rightarrow \lambda_1=\lambda_2=1.$
$T3 \Rightarrow y(x)=C_1\,e^{x}+C_2\,x\,e^{x}.$

1c $\lambda^2-2\lambda+2=0 \Rightarrow \lambda_{1,2}=1\pm i.$
$T3 \Rightarrow y(x)=C_1\,e^{x}\cos x+C_2\,e^{x}\sin x.$

1d $(\lambda^2+1)(\lambda+1)=0 \Rightarrow \lambda_1=-1,\ \lambda_{2,3}=\pm i.$
$T3 \Rightarrow y(x)=C_1\,e^{-x}+C_2\cos x+C_3\sin x.$

Exercise 2 We use T7.

2a The equation can be written successively as follows:
$y'=2xy,\quad \frac{y'}{y}=2x,\quad (\ln y)'=2x,\quad \ln y=x^2+\ln C,$
$\ln\left(\frac{y}{C}\right)=x^2,\quad \frac{y}{C}=e^{x^2},\quad y(x)=C\,e^{x^2}.$

2b We use T8 and T9. Looking for a particular solution of the form $y_p(x)=C(x)\,e^{x^2}$ we get successively:
$C'\,e^{x^2}+2x\,C\,e^{x^2}-2x\,C\,e^{x^2}=x,\qquad C'=x\,e^{-x^2},$
$C(x)=\int x\,e^{-x^2}dx=-\frac{1}{2}\,e^{-x^2}$, and consequently $y_p(x)=-\frac{1}{2}.$
T9 $\Rightarrow$ the general solution of $y'-2xy=x$ is $y(x)=C\,e^{x^2}-\frac{1}{2}.$

2c The equation satisfied by $z=y'$ is $z'-(\frac{1}{x}+2x)z=0$, and it can be written successively:
$\frac{z'}{z}=\frac{1}{x}+2x,\qquad \ln z=\ln x+x^2+\ln C_1,\qquad z(x)=C_1\,x\,e^{x^2}.$
Consequently, $y'(x)=C_1\,x\,e^{x^2}$ and $y(x)=\frac{1}{2}C_1\,e^{x^2}+C_2.$

Exercise 3

3a *Method 1.* The equation is exact in $D=\mathbb{R}^2$,
$\frac{\partial(x+y)}{\partial y}=1=\frac{\partial(x-y)}{\partial x}.$
By using T10 and the path
$\gamma:[0,1]\to\mathbb{R}^2,\quad \gamma(t)=(xt,yt)$
connecting $(0,0)$ with (x,y), we get
$$\begin{aligned}F(x,y)&=\int_\gamma (x+y)dx+(x-y)dy\\&=\int_0^1[(xt+yt)x+(xt-yt)y]\,dt\\&=\left((x^2+2xy-y^2)\frac{t^2}{2}\right)\Big|_{t=0}^{t=1}\\&=\frac{1}{2}(x^2+2xy-y^2).\end{aligned}$$
The equation can be written as (see R3)
$d(x^2+2xy-y^2)=0,$
and its solution is described by
$x^2+2xy-y^2=C.$

3b *Method 2.* The equation can be written as
$y'=\frac{y+x}{y-x}\quad$ or $\quad y'=\frac{\frac{y}{x}+1}{\frac{y}{x}-1}.$
It is a homogeneous equation. By using the change of function $z(x)=\frac{y(x)}{x}$, that is $y(x)=x\,z(x)$, we get
$z+x\,z'=\frac{z+1}{z-1},\qquad \int\frac{1}{x}\,dx=\int\frac{z-1}{1+2z-z^2}\,dz+\ln C_1$
$\ln x=-\frac{1}{2}\ln(1+2z-z^2)+\ln C_1,$
whence $\quad\frac{x}{C_1}=\frac{1}{\sqrt{1+2z-z^2}},\qquad \sqrt{1+2z-z^2}=\frac{C_1}{x},$
$1+2\frac{y}{x}-\frac{y^2}{x^2}=\frac{C_1^2}{x^2}$, that is $x^2+2xy-y^2=C.$

Exercise 4

4a *Method 1.* We have
$$\begin{cases}y_1'=2y_2\\y_2'=2y_3\\y_3'=-2y_1\end{cases}\Rightarrow\begin{cases}y_1''=2y_2'\\y_2'=2y_3\\y_3'=-2y_1\end{cases}\Rightarrow\begin{cases}y_1''=4y_3\\y_3'=-2y_1\end{cases}\Rightarrow\begin{cases}y_1'''=4y_3'\\y_3'=-2y_1\end{cases}$$
whence $\quad y_1'''+8y_1=0.$
$\lambda^3+8=0 \Rightarrow \lambda_1=-2,\quad \lambda_{2,3}=1\pm i\sqrt{3}.$

Consequently
$y_1(x)=C_1e^{-2x}+C_2e^{x}\cos\sqrt{3}x+C_3e^{x}\sin\sqrt{3}x,$
$y_2(x)=\frac{1}{2}y_1'=\dots$
$y_3(x)=\frac{1}{2}y_2'=\dots$

4b *Method 2.* The system can be written as $Y'=AY$, where.
$$Y(x)=\begin{pmatrix}y_1(x)\\y_2(x)\\y_3(x)\end{pmatrix},\ A=\begin{pmatrix}0&2&0\\0&0&2\\-2&0&0\end{pmatrix},\ \begin{vmatrix}-\lambda&2&0\\0&-\lambda&2\\-2&0&-\lambda\end{vmatrix}=0$$
$\Rightarrow \lambda^3+8=0 \Rightarrow \lambda_1=-2,\quad \lambda_{2,3}=1\pm i\sqrt{3}$ and
$$A\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}=-2\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}\text{ satisfied by }\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}=\begin{pmatrix}1\\-1\\1\end{pmatrix},$$
$$A\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}=(1+i\sqrt{3})\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}\text{ satisfied by }\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}=\begin{pmatrix}2\\1+i\sqrt{3}\\-1+i\sqrt{3}\end{pmatrix}.$$
We use T13. The solution is
$$Y(x)=C_1\begin{pmatrix}1\\-1\\1\end{pmatrix}e^{-2x}+C_2\mathfrak{Re}\left\{\begin{pmatrix}2\\1+i\sqrt{3}\\-1+i\sqrt{3}\end{pmatrix}e^{(1+i\sqrt{3})x}\right\}+C_3\mathfrak{Im}\{\dots\}.$$

Exercise 5.

The functions $\alpha(x)$ and $\beta(x)$ must satisfy
$$\begin{cases}\alpha(x)+x\,\beta(x)=0\\2+2x\,\alpha(x)+x^2\,\beta(x)=0\end{cases}\sim\begin{cases}\alpha(x)=-x\,\beta(x)\\2-x^2\,\beta(x)=0\end{cases}\sim\begin{cases}\alpha(x)=-\frac{2}{x}\\\beta(x)=\frac{2}{x^2}.\end{cases}$$

Exercise 6.

6a We get $x\,y'-2y=0,\quad \frac{y'}{y}=\frac{2}{x},\quad (\ln y)'=\frac{2}{x},$
$\ln y=2\ln x+\ln C,\qquad \ln\frac{y}{C}=\ln x^2,\qquad \frac{y}{C}=x^2,\quad y(x)=C\,x^2.$

6b We use the change of variable/function $y(x)\mapsto z(t)$,
$x=e^t,\qquad t=\ln x,\qquad y(e^t)=z(t),\qquad y(x)=z(\ln x).$
Since
$\frac{d}{dx}=\frac{d}{dt}\,\frac{dt}{dx}=\frac{1}{x}\,\frac{d}{dt}=e^{-t}\,\frac{d}{dt},$
the equation satisfied by z is
$z'-2z=0$
and we get $\frac{z'}{z}=2,\quad (\ln z)'=2,\quad \ln z=2t+\ln C,\quad z(t)=C\,e^{2t}.$
Consequently, we obtain
$y(x)=Ce^{2\ln x},\qquad y(x)=Ce^{\ln x^2},\qquad y(x)=C\,x^2.$

6c Written as $\frac{2}{x}\,dx-\frac{1}{y}\,dy=0,$ the equation is exact
$\frac{\partial}{\partial y}\left(\frac{2}{x}\right)=0=\frac{\partial}{\partial x}\left(-\frac{1}{y}\right).$
By integrating along a path connecting $(1,1)$ with (x,y) parallel to the axes of coordinates, we get
$$F(x,y)=\int_1^x\frac{2}{t}\,dt+\int_1^y\frac{-1}{t}\,dt=2\ln t|_1^x-\ln t|_1^y=\ln x^2-\ln y.$$
The solutions are described by the relation
$\ln x^2-\ln y=\ln C_1$
equivalent to $\quad y(x)=\frac{x^2}{C_1}\quad$ and $\quad y(x)=C\,x^2.$

6d A power series
$y(x)=a_0+a_1\,x+a_2\,x^2+a_3\,x^3+\dots+a_n\,x^n+\cdots$
convergent in a certain interval (a,b), satisfies the equation $x\,y'=2\,y$ if and only if
$a_1\,x+2\,a_2\,x^2+3\,a_3\,x^3+\dots+n\,a_n\,x^n+\dots=$
$=2\,a_0+2\,a_1\,x+2\,a_2\,x^2+2\,a_3\,x^3+\dots+2\,a_n\,x^n+\cdots$
that is, for
$$\begin{cases}2\,a_0=0\\2\,a_1=a_1\\2\,a_3=3a_3\\\dots\dots\dots\dots\\2\,a_n=na_n\\\dots\dots\dots\dots\end{cases}\quad\text{that is, for } a_0=a_1=a_3=\dots a_n=\dots=0.$$
Consequently, y satisfies $xy'=2y$ if and only if $y(x)=a_2\,x^2.$