LINEAR ALGEBRA - Solved Exercises 2

(Real and complex finite-dimensional vector spaces)

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A PARTIAL EVALUATION A

Exercise 1

A1 Prove that
$$\mathcal{B}=\{v_1=(1,0),\ v_2=(1,1)\}$$
is a basis in the vector space $\mathbb{R}^2$.

A2 Find the dimension of the space
$$\mathcal{V}=\{\, x=(x_1,x_2,x_3)\in\mathbb{R}^3 \mid x_1+x_2-x_3=0\,\}.$$

Exercise 2

A3 Find the eigenvalues of the linear operator
$$A:\mathbb{R}^3\longrightarrow\mathbb{R}^3,\quad A(x_1,x_2,x_3)=(x_2,x_1,2x_3),$$
and describe the corresponding eigenspaces.

A4 Find the eigenvalues of the linear operator
$$A:\mathbb{C}^2\longrightarrow\mathbb{C}^2,\quad A(z_1,z_2)=(-z_2,z_1),$$
and describe the corresponding eigenspaces.

Exercise 3

A5 Is the linear operator
$$A:\mathbb{R}^3\longrightarrow\mathbb{R}^3,$$
$$A(x_1,x_2,x_3)=(2x_1+x_2,2x_2,3x_3),$$
diagonalizable ?

A6 In $\mathbb{R}^4$, find $\dim(\mathcal{W}_1+\mathcal{W}_2)$, where
$$\mathcal{W}_1=\mathrm{span}\,\{(1,-1,0,2),(0,1,1,1),(1,0,1,3)\},$$
$$\mathcal{W}_2=\mathrm{span}\,\{(1,0,0,0),(1,1,1,1),(0,-1,0,2)\}.$$

A7 Find the eigenvalues of the operator A^{-1}, where
$$A:\mathbb{R}^3\longrightarrow\mathbb{R}^3,$$
$$A(x_1,x_2,x_3)=(x_2+x_3,x_1+x_3,x_1+x_2),$$
and describe the corresponding eigenspaces.

A8 Let $f(x)=x^{20}-2x^{-15}+1$ and
$$A:\mathbb{C}^2\longrightarrow\mathbb{C}^2,\quad A(z_1,z_2)=(-z_2,z_1).$$
Find the eigenvalues of $f(A)$, and describe the corresponding eigenspaces.

B PARTIAL EVALUATION B

Exercise 1

Solve the differential equations

B1 $$y''+2y'-3y=0.$$

B2 $$y''+2y'+y=0,$$

B3 $$y''+2y'+3y=0,$$

Exercise 2

B4 Prove that the linear operator
$$A:\mathbb{C}^3\longrightarrow\mathbb{C}^3,\quad A(x_1,x_2,x_3)=(x_3,x_2,x_1),$$
is self-adjoint, and find its eigenvalues/eigenvectors.

B5 Prove that the linear operator
$$U:\mathbb{C}^2\longrightarrow\mathbb{C}^2,\quad U(x_1,x_2)=\tfrac{1}{\sqrt{2}}(x_1+x_2,x_1-x_2),$$
is unitary, and find its eigenvalues/eigenvectors.

Exercise 3

Let $s\in\{1,2,3,...\}$, $d=2s+1$, and $\mathcal{H}$ be the complex Hilbert space of all the functions
$$\psi:\{-s,-s+1,...,s-1,s\}\longrightarrow\mathbb{C}$$
with the usual scalar product
$$\langle\varphi,\psi\rangle=\sum_{n=-s}^{s}\overline{\varphi(n)}\,\psi(n).$$

B6 Prove that the parity operator
$$\Pi:\mathcal{H}\to\mathcal{H}:\psi\mapsto\Pi\psi,\quad (\Pi\psi)(n)=\psi(-n),$$
is a self-adjoint operator.

B7 Find the eigenvalues of Π and the dimension of the corresponding eigenspaces.

B8 Prove that the Fourier transform
$$F:\mathcal{H}\to\mathcal{H}:\ \psi\mapsto F[\psi],$$
$$(F[\psi])(k)=\frac{1}{\sqrt{d}}\sum_{n=-s}^{s}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}kn}\psi(n),$$
is a unitary transform.

B9 Prove that the operators
$$Q:\mathcal{H}\to\mathcal{H}:\psi\mapsto Q\psi,\quad (Q\psi)(n)=n\,\psi(n),$$
$$P:\mathcal{H}\to\mathcal{H},\quad P=F^{\dagger}QF,$$
are self-adjoint, and find their eigenvalues and the corresponding eigenfunctions.

Some definitions, notations and results

1 Definition. $\{v_1,v_2,...,v_n\}$ is a **linearly independent** set of vectors if $\boxed{\begin{array}{c}\alpha_1v_1+\alpha_2v_2+...+\alpha_nv_n=0\\ \Downarrow\\ \alpha_1=\alpha_2=...=\alpha_n=0\end{array}}$

2 Definition. In a vector space over the field $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$,
$$\boxed{\mathrm{span}\{v_1,v_2,...,v_n\}\overset{\mathrm{def}}{=}\{\alpha_1v_1+\alpha_2v_2+...+\alpha_nv_n \mid \alpha_k\in\mathbb{K}\}}$$

3 Definition. $\mathfrak{B}=\{v_1,v_2,...,v_n\}$ is a **basis** of $\mathcal{V}$ if $\boxed{\begin{array}{c}\mathcal{V}=\mathrm{span}\,\mathfrak{B}\\ \text{and}\\ \mathfrak{B}\text{ linearly indep.}\end{array}}$

4 Definition. **Dimension** of a vector space over $\mathbb{K}$ is
$$\boxed{\dim_{\mathbb{K}}\mathcal{V}\overset{\mathrm{def}}{=}\begin{cases}\begin{array}{c}\text{number}\\ \text{of vectors}\\ \text{of a basis}\end{array} & \text{if } \mathcal{V}\neq\{0\};\\ 0 & \text{if } \mathcal{V}=\{0\}.\end{cases}}$$

5 Definition. The **sum** *of the subspaces* $\mathcal{W}_1,\mathcal{W}_2\subseteq\mathcal{V}$ is
$$\boxed{\mathcal{W}_1+\mathcal{W}_2\overset{\mathrm{def}}{=}\left\{w_1+w_2\ \middle|\ \begin{array}{l}w_1\in\mathcal{W}_1\\ w_2\in\mathcal{W}_2\end{array}\right\}}$$

6 Definition. Linear operator $A:\mathcal{V}\to\mathcal{V}$ is **diagonalizable** if $\boxed{\text{there exists an basis of } \mathcal{V} \text{ consisting of eigenvectors of } A.}$

7 Definition In a Hilbert space $\mathcal{H}$:
An operator $A:\mathcal{H}\to\mathcal{H}$ is a **self-adjoint operator** if $\boxed{\langle Ax,y\rangle=\langle x,Ay\rangle}$ $\forall x,y\in\mathcal{H}$.

8 Definition. In a Hilbert space $\mathcal{H}$:
An operator $A:\mathcal{H}\to\mathcal{H}$ is a **unitary operator** if $\boxed{\langle Ax,Ay\rangle=\langle x,y\rangle}$ $\forall x,y\in\mathcal{H}$.

9 Definition. In the case of vector space over $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$,

λ is an **eigenvalue** of a linear operator $A:\mathcal{V}\to\mathcal{V}$ if
- $\lambda\in\mathbb{K}$ and
- there exists $v\in\mathcal{V}$, $v\neq0$ (**eigenvector** corresp. to λ) such that $\boxed{Av=\lambda\,v}$.

The **eigenspace** corresponding to λ is $\boxed{\{\,x \mid Ax=\lambda x\,\}}$.

10 Theorem. In the case of vector space over $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$,

$\boxed{\lambda\text{ is an eigenvalue of a linear operator } A:\mathcal{V}\to\mathcal{V}}$ $\Leftrightarrow$
- $\lambda\in\mathbb{K}$ and
- λ is a root of the equation
$$\begin{vmatrix}a_1^1-\lambda & a_2^1 & \cdots & a_n^1\\ a_1^2 & a_2^2-\lambda & \cdots & a_n^2\\ \vdots & \vdots & \ddots & \vdots\\ a_1^n & a_2^n & \cdots & a_n^n-\lambda\end{vmatrix}=0$$

SOLUTIONS for **PARTIAL EVALUATION A**

$\boxed{A1}$

$\mathbb{R}^2=\{(x_1,x_2)\,|\,x_1,x_2\in\mathbb{R}\}=\{(x_1-x_2)v_1+x_2v_2\,|\,x_1,x_2\in\mathbb{R}\}$
$\quad=\{\alpha_1v_1+\alpha_2v_2\,|\,\alpha_1,\alpha_2\in\mathbb{R}\}=\mathrm{span}\{v_1,v_2\}.$

$\alpha_1(1,0)+\alpha_2(1,1)=(0,0)\ \Rightarrow\ \alpha_1=\alpha_2=0\ \Rightarrow\ v_1,v_2$ lin. ind.

$\boxed{A2}$

$$\mathcal{V}=\{\,(x_1,x_2,x_1+x_2)\ |\ x_1,x_2\in\mathbb{R}\,\}=\{\,x_1(1,0,1)+x_2(0,1,1)\ |\ x_1,x_2\in\mathbb{R}\,\}=\mathrm{span}\{\,(1,0,1),(0,1,1)\,\}.$$

$$\alpha_1(1,0,1)+\alpha_2(0,1,1)=(0,0,0)\ \Rightarrow\ \alpha_1=\alpha_2=0.$$

Consequently, $\{\,(1,0,1),(0,1,1)\,\}$is a basis, and $\dim\mathcal{V}=2$.

$\boxed{A3}$ In the canonical basis

$$A=\begin{pmatrix}0&1&0\\1&0&0\\0&0&2\end{pmatrix}.\quad \begin{vmatrix}0-\lambda&1&0\\1&0-\lambda&0\\0&0&2-\lambda\end{vmatrix}=0\ \Rightarrow\ \begin{array}{l}\lambda_1=-1\\ \lambda_2=1\\ \lambda_3=2.\end{array}$$

$\{x\,|\,Ax=-x\}=\{(x_1,x_2,x_3)\,|\,(x_2,x_1,2x_3)=(-x_1,-x_2,-x_3)\}$
$\quad=\{(x_1,-x_1,0)\,|\,x_1\in\mathbb{R}\,\}=\mathrm{span}\{\,(1,-1,0)\,\}.$

$\{x\,|\,Ax=x\}=\{(x_1,x_2,x_3)\,|\,(x_2,x_1,2x_3)=(x_1,x_2,x_3)\}$
$\quad=\{(x_1,x_1,0)\,|\,x_1\in\mathbb{R}\,\}=\mathrm{span}\{\,(1,1,0)\,\}.$

$\{x\,|\,Ax=2x\}=\{(x_1,x_2,x_3)\,|\,(x_2,x_1,2x_3)=(2x_1,2x_2,2x_3)\}$
$\quad=\{(0,0,x_3)\,|\,x_3\in\mathbb{R}\,\}=\mathrm{span}\{\,(0,0,1)\,\}.$

$\boxed{A4}$

In the canonical basis

$$A=\begin{pmatrix}0&-1\\1&0\end{pmatrix}.\quad \begin{vmatrix}0-\lambda&-1\\1&0-\lambda\end{vmatrix}=0\ \Rightarrow\ \begin{array}{l}\lambda_1=\mathrm{i}\\ \lambda_2=-\mathrm{i}\end{array}$$

$\{z\,|\,Az=\mathrm{i}z\}=\{(z_1,z_2)\,|\,(-z_2,z_1)=(\mathrm{i}z_1,\mathrm{i}z_2)\}$
$\quad=\{(z_1,-\mathrm{i}z_1)\,|\,z_1\in\mathbb{C}\,\}=\mathrm{span}\{\,(1,-\mathrm{i})\,\}.$

$\{z\,|\,Az=-\mathrm{i}z\}=\{(z_1,z_2)\,|\,(-z_2,z_1)=(-\mathrm{i}z_1,-\mathrm{i}z_2)\}$
$\quad=\{(z_1,\mathrm{i}z_1)\,|\,z_1\in\mathbb{C}\,\}=\mathrm{span}\{\,(1,\mathrm{i})\,\}.$

$\boxed{A5}$ In the canonical basis

$$A=\begin{pmatrix}2&1&0\\0&2&0\\0&0&3\end{pmatrix}.\quad \begin{vmatrix}2-\lambda&1&0\\0&2-\lambda&0\\0&0&3-\lambda\end{vmatrix}=0\ \Rightarrow\ \begin{array}{l}\lambda_1=\lambda_2=2\\ \lambda_3=3.\end{array}$$

$\{x\,|\,Ax=2x\}=\{(x_1,x_2,x_3)\,|\,(2x_1+x_2,2x_2,3x_3)=(2x_1,2x_2,2x_3)\}$
$\quad=\{(x_1,0,0)\,|\,x_1\in\mathbb{R}\,\}=\mathrm{span}\{\,(1,0,0)\,\}.$

$\{x\,|\,Ax=3x\}=\{(x_1,x_2,x_3)\,|\,(2x_1+x_2,2x_2,3x_3)=(3x_1,3x_2,3x_3)\}$
$\quad=\{(0,0,x_3)\,|\,x_3\in\mathbb{R}\,\}=\mathrm{span}\{\,(0,0,1)\,\}.$

There exist only two linearly independent eigenvectors of A.
The operator A is not diagonalizable.

$\boxed{A6}$

$$\left.\begin{array}{l}(1,0,1,3)=(1,-1,0,2)+(0,1,1,1)\\ (1,1,1,1)=(0,1,1,1)+(1,0,0,0)\\ (0,-1,0,2)=(1,0,1,3)-(1,1,1,1)=(1,-1,0,2)-(1,0,0,0)\end{array}\right\}\Rightarrow$$

$\Rightarrow\ \mathcal{W}_1+\mathcal{W}_2=\mathrm{spam}\{(1,-1,0,2),(0,1,1,1),(1,0,0,0)\}.$

$\alpha(1,-1,0,2)+\beta(0,1,1,1)+\gamma(1,0,0,0)=(0,0,0)\ \Rightarrow\alpha=\beta=\gamma=0.$

$\{(1,-1,0,2),(0,1,1,1),(1,0,0,0)\}$ is a basis $\Rightarrow\ \dim(\mathcal{W}_1+\mathcal{W}_2)=3.$

$\boxed{A7}$ In the canonical basis

$$A=\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}.\quad \begin{vmatrix}0-\lambda&1&1\\1&0-\lambda&1\\1&1&0-\lambda\end{vmatrix}=0\ \Rightarrow\ \begin{array}{l}\lambda_1=\lambda_2=-1\\ \lambda_3=2.\end{array}$$

$\{x|Ax=-x\}=\{(x_1,x_2,x_3)|(x_2+x_3,x_1+x_3,x_1+x_2)=(-x_1,-x_2,-x_3)\}$
$\quad=\{(x_1,x_2,-x_1-x_2)\,|\,x_1,x_2\in\mathbb{R}\,\}$
$\quad=\mathrm{span}\{\,(1,-1,0)\,(0,1,-1)\}.$

$\{x\,|\,Ax=x\}=\{(x_1,x_2,x_3)\,|\,(x_2+x_3,x_1+x_3,x_1+x_2)=(2x_1,2x_2,2x_3)\}$
$\quad=\{(x_1,x_1,x_1)\,|\,x_1\in\mathbb{R}\,\}=\mathrm{span}\{\,(1,1,1)\,\}.$

For $\lambda\neq0$, $Ax=\lambda x\ \Leftrightarrow\ A^{-1}x=\frac{1}{\lambda}x$. Consequently, A^{-1} has the same eigenspaces as A and the eigenvalues $\frac{1}{\lambda_1}=\frac{1}{\lambda_2}=-1,\ \frac{1}{\lambda_3}=\frac{1}{2}$.

$\boxed{A8}$

The eigenvalues and eigenspaces of A are presented above (see A4).
Since $Ax=\lambda x\ \Rightarrow\ f(A)x=f(\lambda)x$, the eigenvalues of $f(A)$ are

$f(i)=2-2\mathrm{i}$ with the eigenspace $\mathrm{span}\{\,(1,-\mathrm{i})\,\}$,
$f(-i)=2+2\mathrm{i}$ with the eigenspace $\mathrm{span}\{\,(1,\mathrm{i})\,\}$.

SOLUTIONS for **PARTIAL EVALUATION B**

$\boxed{B1}$

$$\lambda^2+2\lambda-3=0\ \Rightarrow\ \begin{array}{l}\lambda_1=1\\ \lambda_2=-3\end{array}\ \Rightarrow\ y(x)=C_1\mathrm{e}^{x}+C_2\mathrm{e}^{-3x}.$$

$\boxed{B2}$

$$\lambda^2+2\lambda+1=0\ \Rightarrow\ \lambda_1=\lambda_2=-1\ \Rightarrow\ y(x)=C_1\mathrm{e}^{-x}+C_2x\mathrm{e}^{-x}.$$

$\boxed{B3}$

$$\lambda^2+2\lambda+3=0\ \Rightarrow\ \lambda_{1,2}=-1\pm\mathrm{i}\sqrt{2}$$
$$\Rightarrow\ y(x)=C_1\mathrm{e}^{-x}\cos\sqrt{2}x+C_2\mathrm{e}^{-x}\sin\sqrt{2}x.$$

$\boxed{B4}$

$$A=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\ \text{satisfies } A^\dagger=A.\quad \begin{vmatrix}0-\lambda&0&1\\0&1-\lambda&0\\1&0&0-\lambda\end{vmatrix}=0$$

$$\Rightarrow\ \begin{array}{ll}\lambda_1=-1, & \{x\ |\ Ax=-x\}=\mathrm{span}\{(1,0,-1)\},\\ \lambda_2=\lambda_3=1, & \{x\ |\ Ax=x\}=\mathrm{span}\{(0,1,0),(1,0,1)\}.\end{array}$$

$\boxed{B5}$

$$U=\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\end{pmatrix}\ \text{satisfies } UU^\dagger=\mathbb{I}.\quad \begin{vmatrix}\frac{1}{\sqrt{2}}-\lambda&\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}-\lambda\end{vmatrix}=0$$

$$\Rightarrow\ \begin{array}{ll}\lambda_1=-1, & \{x\ |\ Ux=-x\}=\mathrm{span}\{(1,-\sqrt{2}-1)\},\\ \lambda_2=1, & \{x\ |\ Ux=x\}=\mathrm{span}\{(1,\sqrt{2}-1)\}.\end{array}$$

$\boxed{B6}$ By using the change of variable $m=-n$, we get

$$\langle\Pi\psi,\varphi\rangle=\sum_{n=-s}^{s}\overline{\Pi\psi(n)}\,\varphi(n)=\sum_{n=-s}^{s}\overline{\psi(-n)}\,\varphi(n)=$$
$$=\sum_{m=-s}^{s}\overline{\psi(m)}\,\varphi(-m)=\sum_{m=-s}^{s}\overline{\psi(m)}\,\Pi\varphi(m)=\langle\psi,\Pi\varphi\rangle.$$

$\boxed{B7}$ We have $\Pi^2\psi=\psi$, for any $\psi\in\mathcal{H}$. For $\psi\neq0$,

$\Pi\psi=\lambda\psi\ \Rightarrow\ \Pi^2\psi=\lambda^2\psi\ \Rightarrow\ \psi=\lambda^2\psi\ \Rightarrow\ \lambda^2=1\ \Rightarrow\ \lambda_{1,2}=\pm1.$

Eigenspace $\{\psi\ |\ \Pi\psi=\psi\}=\{\psi\ |\ \psi(-n)=\psi(n)\}$ has dim. $s+1$.
An even ψ depends on $s+1$ parameters: $\psi(0),\psi(1),...,\psi(s)$.
Eigenspace $\{\psi|\ \Pi\psi=-\psi\}=\{\psi|\psi(-n)=-\psi(n)\}$ has dim. s.
An odd ψ depends on s parameters: $\psi(-0)=-\psi(0)\Rightarrow\psi(0)=0$.

$\boxed{B8}$

$$\begin{aligned}\langle F[\psi],F[\varphi]\rangle&=\sum_{k=-s}^{s}\overline{F[\psi](k)}\,F[\varphi](k)\\ &=\frac{1}{d}\sum_{k=-s}^{s}\sum_{n=-s}^{s}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kn}\,\overline{\psi(n)}\sum_{m=-s}^{s}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}km}\,\varphi(m)\\ &=\frac{1}{d}\sum_{n=-s}^{s}\sum_{m=-s}^{s}\sum_{k=-s}^{s}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k(n-m)}\,\overline{\psi(n)}\,\varphi(m)\\ &=\sum_{n=-s}^{s}\sum_{m=-s}^{s}\delta_{nm}\,\overline{\psi(n)}\,\varphi(m)\\ &=\sum_{n=-s}^{s}\overline{\psi(n)}\,\varphi(n)=\langle\psi,\varphi\rangle\end{aligned}$$

because

$$\sum_{k=-s}^{s}\left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(n-m)}\right)^k=\begin{cases}d & \text{for}\quad n=m\\ \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(n-m)(-s)}\,\frac{1-\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(n-m)d}}{1-\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(n-m)}}=0 & \text{for}\quad n\neq m.\end{cases}$$

$\boxed{B9}$

$$\langle Q\psi,\varphi\rangle=\sum_{n=-s}^{s}\overline{(Q\psi)(n)}\,\varphi(n)=\sum_{n=-s}^{s}\overline{n\psi(n)}\,\varphi(n)$$
$$=\sum_{n=-s}^{s}\overline{\psi(n)}\,n\varphi(n)=\sum_{n=-s}^{s}\overline{\psi(n)}\,(Q\varphi)(n)=\langle\psi,Q\varphi\rangle.$$

$P^\dagger=(F^\dagger QF)^\dagger=F^\dagger Q^\dagger F=F^\dagger QF=P.$

For each $m\in\{-s,-s+1,...,s-1,s\}$ the function

$$\delta_m:\{-s,-s+1,...,s-1,s\}\to\mathbb{C},\ \delta_m(n)=\begin{cases}1 & \text{for}\quad n=m\\ 0 & \text{for}\quad n\neq m\end{cases}$$

is an eigenfunction of Q corresponding to the eigenvalue m:

$$(Q\delta_m)(n)=n\delta_m(n)=m\delta_m(n),\ \text{for any } n,$$

that is $\quad Q\delta_m=m\delta_m$.

Q can not have more distinct eigenvalues than $d=\dim\mathcal{H}$.
The function $F^\dagger\delta_m$ satisfies the relation

$$PF^\dagger\delta_m=F^\dagger QFF^\dagger\delta_m=F^\dagger Q\delta_m=m\,F^\dagger\delta_m,$$

that is, an eigenfunction of P corresponding to m.
P can not have more distinct eigenvalues than $d=\dim\mathcal{H}$.