Overview on $\boxed{\textbf{UNITARY IRREDUCIBLE REPRESENTATIONS OF } SU(2)}$

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- The Lie algebra of the group $SU(2)=\left\{g \;\middle|\; \begin{array}{c} g^\dagger=g^{-1} \\ \det g=1 \end{array}\right\}=\left\{g=\begin{pmatrix} a & b \\ -\bar{b} & \bar{a}\end{pmatrix} \;\middle|\; \begin{array}{c} a,b\in\mathbb{C} \\ |a|^2+|b|^2=1 \end{array}\right\}$

is the real 3D vector space $su(2)=\left\{u \;\middle|\; \begin{array}{c} u^\dagger=-u \\ \mathrm{tr}\, u=0 \end{array}\right\}=\left\{u=\begin{pmatrix} \mathrm{i}z & -\mathrm{i}x+y \\ -\mathrm{i}x-y & -\mathrm{i}z\end{pmatrix} \;\middle|\; x,y,z\in\mathbb{R}\right\}$

endowed with the additional operation $[u,v]=uv-vu$.

- If we identify the vector spaces $\mathbb{R}^3$ and $su(2)$ in the way

$$(x,y,z)\equiv\begin{pmatrix} \mathrm{i}z & -\mathrm{i}x+y \\ -\mathrm{i}x-y & -\mathrm{i}z\end{pmatrix},$$

$O(3)$ is the group of transformations $R: su(2)\longrightarrow su(2)$ satisfying the condition

$$\det(Ru)=\det u,$$

and the mapping $\quad SU(2)\longrightarrow SO(3): \; g\mapsto R_g, \qquad R_g u=gug^\dagger$

is a morphism of groups with the kernel formed by $\pm\mathbb{I}$. The one-dimensional subgroups

$$g_x(t)=\begin{pmatrix} \cos\frac{t}{2} & -\mathrm{i}\,\sin\frac{t}{2} \\ -\mathrm{i}\,\sin\frac{t}{2} & \cos\frac{t}{2}\end{pmatrix},\quad g_y(t)=\begin{pmatrix} \cos\frac{t}{2} & \sin\frac{t}{2} \\ -\sin\frac{t}{2} & \cos\frac{t}{2}\end{pmatrix},\quad g_z(t)=\begin{pmatrix} \mathrm{e}^{\mathrm{i}\frac{t}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\frac{t}{2}}\end{pmatrix}$$

of $SU(2)$ with the infinitesimal generators

$$I_x=\begin{pmatrix} 0 & -\frac{\mathrm{i}}{2} \\ -\frac{\mathrm{i}}{2} & 0\end{pmatrix},\qquad I_y=\begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0\end{pmatrix},\qquad I_z=\begin{pmatrix} \frac{\mathrm{i}}{2} & 0 \\ 0 & -\frac{\mathrm{i}}{2}\end{pmatrix}$$

correspond to the rotations around Ox, Oy, Oz, respectively.

- The self-adjoint operators $J_x^{(j)}$, $J_y^{(j)}$, $J_z^{(j)}:\mathbb{C}^{2j+1}\longrightarrow\mathbb{C}^{2j+1}$ defined by

$$\boxed{\begin{array}{l} J_z^{(j)}|\delta_m\rangle=m|\delta_m\rangle \\ (J_x^{(j)}\pm\mathrm{i}\,J_y^{(j)})|\delta_m\rangle=\sqrt{(j\mp m)(j\pm m+1)}\,|\delta_{m\pm1}\rangle \end{array}} \qquad (*)$$

satisfy the relations $\quad [J_x^{(j)},J_y^{(j)}]=\mathrm{i}\,J_z^{(j)}, \qquad [J_y^{(j)},J_z^{(j)}]=\mathrm{i}\,J_x^{(j)}, \qquad [J_z^{(j)},J_x^{(j)}]=\mathrm{i}\,J_y^{(j)}.$

- In the case $j=\frac{1}{2}$, the operators defined by (*) are

$$J_x=\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0\end{pmatrix}=\mathrm{i}\,I_x,\quad J_y=\begin{pmatrix} 0 & \frac{\mathrm{i}}{2} \\ -\frac{\mathrm{i}}{2} & 0\end{pmatrix}=\mathrm{i}\,I_y,\quad J_z=\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2}\end{pmatrix}=\mathrm{i}\,I_z$$

and $\qquad g_x(t)=\mathrm{e}^{tI_x}=\mathrm{e}^{-\mathrm{i}tJ_x}, \qquad g_y(t)=\mathrm{e}^{tI_y}=\mathrm{e}^{-\mathrm{i}tJ_y}, \qquad g_z(t)=\mathrm{e}^{tI_z}=\mathrm{e}^{-\mathrm{i}tJ_z}.$

- In terms of the Euler angles,

$$\boxed{\begin{array}{l} SU(2)=\left\{\mathrm{e}^{-\mathrm{i}\alpha J_z}\mathrm{e}^{-\mathrm{i}\beta J_x}\mathrm{e}^{-\mathrm{i}\gamma J_z}\;\middle|\; 0\leq\alpha<2\pi,\; 0\leq\beta\leq\pi,\; -2\pi\leq\gamma<2\pi\right\} \\ \quad =\left\{\begin{pmatrix} \cos\frac{\beta}{2}\,\mathrm{e}^{\mathrm{i}\frac{\alpha+\gamma}{2}} & -\mathrm{i}\,\sin\frac{\beta}{2}\,\mathrm{e}^{\mathrm{i}\frac{\alpha-\gamma}{2}} \\ -\mathrm{i}\,\sin\frac{\beta}{2}\,\mathrm{e}^{-\mathrm{i}\frac{\alpha-\gamma}{2}} & \cos\frac{\beta}{2}\,\mathrm{e}^{-\mathrm{i}\frac{\alpha+\gamma}{2}}\end{pmatrix} \;\middle|\; \begin{array}{r} 0\leq\alpha<2\pi, \\ 0\leq\beta\leq\pi, \\ -2\pi\leq\gamma<2\pi \end{array}\right\}, \end{array}}$$

and the mapping

$$\boxed{D^j: SU(2)\longrightarrow\mathcal{L}(\mathbb{C}^{2j+1}),\quad D^j\left(\mathrm{e}^{-\mathrm{i}\alpha J_z}\mathrm{e}^{-\mathrm{i}\beta J_x}\mathrm{e}^{-\mathrm{i}\gamma J_z}\right)=\mathrm{e}^{-\mathrm{i}\alpha J_z^{(j)}}\mathrm{e}^{-\mathrm{i}\beta J_x^{(j)}}\mathrm{e}^{-\mathrm{i}\gamma J_z^{(j)}}} \qquad (**)$$

is an irreducible $(2j+1)$-dimensional unitary representation of $SU(2)$.

- If $d=2j+1$ is odd, then $D^j(-g)=D^j(g)$ and consequently, the relation (**) defines an irreducible representation of $SO(3)$.