Problems

Part I

Metric Spaces. Normed Spaces. Pre-Hilbertian Spaces

1) Show that on $\mathbb{R}$, the map $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ defined by $d(x, y) = |x - y|$ is a metric.

2) Prove that on $\mathbb{R}^n$ the following maps are distances:

$$d(x,y) = \Big(\sum_{i=1}^{n}(x_i - y_i)^2\Big)^{1/2} \text{(the Euclidean distance);}$$

$$d_1(x,y) = \sum_{i=1}^{n}|x_i - y_i| \quad \text{(the Manhattan distance);}$$

$$d_\infty(x,y) = \max_{1\le i\le n}|x_i - y_i| \quad \text{(the Chebyshev distance).}$$

3) Let $a, b \in \mathbb{R}, a < b$ and let

$$\mathcal{C}([a,b]) = \{f : [a,b] \to \mathbb{R} \,|\, f \text{ is continuous on } [a,b]\}.$$

Show that the map $d : \mathcal{C}([a,b]) \times \mathcal{C}([a,b]) \to \mathbb{R}_+$, defined by

$$d(f,g) = \max_{x\in[a,b]} |f(x) - g(x)|,$$

is a distance on $\mathcal{C}([a,b])$.

4) Prove that the map

$$d(f,g) = \int_a^b |f(x) - g(x)|\,\mathrm{d}x$$

is a metric on $\mathcal{C}([a,b])$.

5) Let (X, d) be a metric space and let $d_1 : X \times X \to \mathbb{R}_+$ defined by

$$d_1(x,y) = \ln(1 + d(x,y)),\ \forall x,y \in X.$$

Prove that d_1 defines a new metric on X.

6) Let $X = \mathbb{C}^n$, $K = \mathbb{C}$. Prove that the map $\langle\cdot,\cdot\rangle : X \times X \to \mathbb{C}$ defined by

$$\langle x,y\rangle = \sum_{i=1}^{n} x_i\overline{y_i}$$

is a scalar product on $\mathbb{C}^n$.

7) Let $X = \mathcal{C}([a,b]) = \{f : [a,b] \to \mathbb{R} \mid f \text{ continuous on } [a,b]\}$. Prove that the map $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ defined by

$$\langle f, g \rangle = \int_a^b f(x)g(x)\,\mathrm{d}x$$

is a scalar product on X.

8) Let $X = \mathbb{R}^n$. The following maps are norms on X:

$$a) \quad \|x\|_1 = \sum_{i=1}^{n} |x_i|,$$

$$b) \quad \|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2} \quad (\text{ \underline{the Euclidean norm}}),$$

$$c) \quad \|x\|_\infty = \max_{1 \le i \le n} |x_i|.$$

9) Show that in $\mathbb{R}^2$ we have

$$x \perp y \Leftrightarrow \|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

10) Let X be a real pre-Hilbertian space. Show that

$$\|x+y\| = \|x\| + \|y\| \Leftrightarrow \langle x, y \rangle = \|x\| \cdot \|y\|.$$

Series

1) Using the ratio and the root test, discuss the nature of the following series:

a) $\displaystyle\sum_{n\ge1} \frac{a^n}{n!}, \quad a > 0;$ b) $\displaystyle\sum_{n\ge1} \frac{a^n}{n^b}, \quad a, b > 0;$ c) $\displaystyle\sum_{n\ge2} \frac{1}{(\ln n)^n};$

d) $\displaystyle\sum_{n\ge1} \frac{n!}{n^n},$ e) $\displaystyle\sum_{n\ge1} (2+\sin n)^n \left(1 - \frac{2}{n}\right)^{n^2}.$

2) Compute the sum of the following series:

a) $\displaystyle\sum_{n\ge2} \frac{1}{n(n^2-1)};$ $\displaystyle\sum_{n\ge1} \frac{1}{n(n+1)(n+2)};$ c) $\displaystyle\sum_{n\ge1} \ln \frac{(n+1)^2}{n(n+2)}.$

3) Test the following series for convergence:

a) $\sum_{n\geq 0} \frac{1}{3^n+n}$; b) $\sum_{n\geq 1} \frac{\sqrt{n+1}-\sqrt{n}}{n}$; c) $\sum_{n\geq 1} \tan\frac{1}{n}$.

4) Decide upon the nature of the following series:

a) $\sum_{n\geq 1} \sin\frac{1}{n}$; b) $\sum_{n\geq 2} \frac{1}{n\ln n}$; c) $\sum_{n\geq 1} \frac{\sin n}{2^n}$.

5) Test the following series for convergence:

a) $\sum_{n\geq 1} \frac{(-1)^n}{(3n+1)^2}$; b) $\sum_{n\geq 1} \frac{1}{n+\sqrt{n+1}}$;

c) $\sum_{n\geq 1} \frac{n!}{a(a+1)\cdots(a+n)}$, $a>0$; d) $\sum_{n\geq 2} \frac{\sin\, n+1}{n(\ln\, n)^2}$.

6) Discuss, in terms of p and q, the nature of the following series:

$$\sum_{n\geq 2} \frac{1}{n^p(\ln n)^q}.$$

7) Decide upon the nature of the following series:

a) $\sum_{n\geq 2} \frac{\sqrt{n+2}-\sqrt{n-2}}{n^2}$; b) $\sum_{n\geq 1} \left(\frac{an+b}{cn+d}\right)^n$, $a,b,c,d>0$;

c) $\sum_{n\geq 1} \left(\sqrt{(n+1)(n+a)}-n\right)^n$, $a>0$; d) $\sum_{n\geq 1} \frac{1}{n!}\left(\frac{n}{a}\right)^n$, $a>0$. ■

Limits

1) Prove that $\nexists \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$.

2) Prove that $\exists \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$.

3) Show that $\nexists \lim_{(x,y)\to(0,0)} \frac{x-y}{x+y}$, but $\exists \lim_{x\to 0}(\lim_{y\to 0} f(x,y)) \neq \lim_{y\to 0}(\lim_{x\to 0} f(x,y))$.

4) Prove that $\exists \lim_{(x,y)\to(0,0)} (x+y)\sin\frac{1}{x}\sin\frac{1}{y}$, but we don't have iterated limits.

5) Show that $\nexists \lim_{(x,y)\to(0,0)} \frac{x\sin\frac{1}{x}+y}{x+y}$.

6) Compute the directional limits in the direction $\overrightarrow{u} = (u_1, u_2) \neq \overrightarrow{0}$, at the point $(0, 0)$, for

$$f(x,y) = \frac{3xy}{x^2 + y^2}.$$

7) Compute the directional limits, at the point $(0, 0)$, in the direction $\overrightarrow{u} = (u_1, u_2) \neq \overrightarrow{0}$ for

$$f(x,y) = \frac{2x^2 y}{x^2 + y^2}.$$

8) Prove that $\exists \lim\limits_{(x,y)\to(0,0)} (x^2 + y^2) \sin \dfrac{1}{x^2 + y^2}$.

9) Let

$$f(x,y) = \begin{cases} x \sin \dfrac{1}{y} + y \sin \dfrac{1}{x}, & (x,y) \neq (0,0), \\ 0, \quad (x,y) = (0,0). \end{cases}$$

Prove that $\exists \lim\limits_{(x,y)\to(0,0)} f(x,y)$, but $\nexists$ the iterated limits.

10) Let

$$f(x,y,z) = \begin{cases} \dfrac{x^2 - 2y^2 + z^2}{x^2 + y^2 + z^2}, & (x,y,z) \neq (0,0,0), \\ 0, \quad (x,y,z) = (0,0,0). \end{cases}$$

Show that $\nexists \lim\limits_{(x,y,z)\to(0,0,0)} f(x,y,z)$.

11) Let $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ defined as follows:

$$f(x,y) = \frac{x^2 y^2}{x^2 y^2 + (x - y)^4}.$$

Prove that $\nexists \lim\limits_{(x,y)\to(0,0)} f(x,y)$, but $\exists$ the iterated limits.

12) Compute the directional limits at the point $(\pi/4, 1)$ in the direction $\overrightarrow{u} = (\sqrt{2}/2, \sqrt{2}/2)$ for

$$f(x,y) = \Big(x^2 + x \cos y, \ln \sin \frac{x}{y}\Big).$$

13) Let

$$f(x,y) = \begin{cases} \dfrac{1 - \cos(x^3 + y^3)}{x^2 + y^2}, & (x,y) \neq (0,0), \\ 0, \quad (x,y) = (0,0). \end{cases}$$

Test this function for the existence of directional limits at the point $(0, 0)$.

Continuity

1) Test the following function for continuity:

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

2) Test the following function for continuity:

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

3) Decide upon the continuity of the following function:

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\dfrac{1}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

4) Test the following function for continuity:

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

5) Let

$$f(x,y) = \begin{cases} \dfrac{|x|^\alpha |y|^\beta}{(x^2+y^2)^\gamma}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

Prove that f is continuous $\Leftrightarrow \alpha + \beta > 2\gamma$.

6) Decide upon the continuity of the following function:

$$f(x,y) = \begin{cases} \dfrac{x^2}{y}, & y \neq 0, \\ 0, & y = 0. \end{cases}$$

7) Let

$$f(x,y) = \begin{cases} \dfrac{1-\cos(x^3+y^3)}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

Test this function for continuity and directional continuity at the point $(0,0)$.

8) Let

$$f(x,y)=\begin{cases}\dfrac{x^2y^2}{x^2y^2+(x^2-y)^2}, & (x,y)\neq(0,0),\\[2ex] 0, \quad (x,y)=(0,0).\end{cases}$$

Test this function for continuity and directional continuity at the point $(0,0)$.

Derivability

1) Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by

$$f(x,y)=\ln(1+x^2+y^2).$$

Compute, using the definition, the following partial derivatives of f: $\dfrac{\partial f}{\partial x}(1,1)$ and $\dfrac{\partial^2 f}{\partial x\partial y}(1,1)$.

2) Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by

$$f(x,y)=\sqrt{x^2+y^2}.$$

Prove that f is not differentiable at the point $(0,0)$.

3) Test the following function for differentiability:

$$f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & (x,y)\neq(0,0),\\[2ex] 0, \quad (x,y)=(0,0).\end{cases}$$

4) Test the following function for differentiability:

$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0),\\[2ex] 0, \quad (x,y)=(0,0).\end{cases}$$

5) Test the following function for differentiability:

$$f(x,y)=\begin{cases}(x^2+y^2)\sin\dfrac{1}{x^2+y^2}, & (x,y)\neq(0,0),\\[2ex] 0, \quad (x,y)=(0,0).\end{cases}$$

6) Compute the first and the second differential for the function $f:\mathbb{R}^2\to\mathbb{R},\quad f(x,y)=e^{x^2+y^2}$.

7) Compute the directional derivative of the function $f:\mathbb{R}^2\to\mathbb{R}^2,\quad f(x,y)=(x^2+y,x+y^2)$ at the point $a=(2,3)$, in the direction of the vector $\overrightarrow{u}=(1,1)$.

8) For the function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by:

$$f(x,y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2}, & (x,y) \neq (0,0), \\ \\ 0, \quad (x,y) = (0,0), \end{cases}$$

prove that

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0).$$

9) Compute the first and the second differential for the function $f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = \ln(1 + x^2 + y^2)$, at the point $(1,1)$.

10) Compute the first and the second differential for the function $F(x,y) = (x^2 + y^2)u(x+y)$, where $u \in C^2$.

11) Let $F(x,y) = xyu(x+y, xy)$, where $u \in C^2$. Compute $\mathrm{d}F$ and d^2F.

12) Let $F(x,y,z) = (x^2 + y^2 + z^2)u(x+y+z, xyz)$, where $u \in C^2$. Compute $\mathrm{d}F$ and d^2F.

13) Let $F(x,y) = u(x+y, xy, x^2+y^2)$, where $u \in C^2$. Compute $\mathrm{d}F$ and d^2F.

14) Show that the function

$$z(x,y) = f(x + \varphi(y)),$$

where $f, \varphi \in C^2$, satisfies the equation:

$$\frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial z}{\partial y}\frac{\partial^2 z}{\partial x^2}.$$

15) Show that the function

$$g(x,t) = \varphi(x - at) + \psi(x + at),$$

where $\varphi, \psi \in C^2, \; a \in \mathbb{R}$, satisfies the equation (the equation of the vibrating string):

$$\frac{\partial^2 g}{\partial t^2} = a^2 \frac{\partial^2 g}{\partial x^2}.$$

Differential Operators

1) Let $\overrightarrow{v} = (xyz, x+y+z, xy^2z^3)$. Compute div $\overrightarrow{v}$ and rot $\overrightarrow{v}$.

2) Let $\varphi(x,y,z) = xy^2z^3$. Compute $\mathrm{grad}\,\varphi$.

3) $\mathrm{div}\,(\varphi\overrightarrow{v}) = \varphi\,\mathrm{div}\,\overrightarrow{v} + (\mathrm{grad}\,\varphi)\cdot\overrightarrow{v}, \quad \forall \varphi, \overrightarrow{v} \in C^1.$

4) $\mathrm{div}\,(\overrightarrow{u} \times \overrightarrow{v}) = \overrightarrow{v}\cdot\mathrm{rot}\,\overrightarrow{u} - \overrightarrow{u}\cdot\mathrm{rot}\,\overrightarrow{v}, \quad \forall \overrightarrow{u}, \overrightarrow{v} \in C^1.$

5) $\operatorname{div}(\operatorname{grad}\varphi) = \triangle\varphi, \quad \forall \varphi \in C^2.$

6) $\operatorname{rot}(\operatorname{grad}\varphi) = 0, \quad \forall \varphi \in C^2.$

7) $\operatorname{rot}\,(\varphi \overrightarrow{v}) = \varphi \operatorname{rot}\overrightarrow{v} - \overrightarrow{v} \times \operatorname{grad}\varphi, \quad \forall \varphi, \overrightarrow{v} \in C^1.$

8) $\operatorname{rot}(\operatorname{rot}\overrightarrow{u}) = \operatorname{grad}\,(\operatorname{div}\overrightarrow{u}) - \triangle\overrightarrow{u}, \quad \forall \overrightarrow{u} \in C^2.$

9) $\operatorname{div}\,(\operatorname{rot}\overrightarrow{v}) = 0, \quad \forall \overrightarrow{v} \in C^2.$

Taylor's Formula

1) Write Taylor's formula of the order 2 about the point $(x_0, y_0) = (0, 0)$ for the function

$$f(x, y) = e^x \sin y.$$

2) Write Taylor's formula of the order 2 at the point $(x_0, y_0) = \left(0, \frac{\pi}{2}\right)$ for the function

$$f(x, y) = e^x \cos y.$$

3) Write Taylor's formula of the order 2 at the point $(x_0, y_0) = \left(0, \frac{\pi}{2}\right)$ for the function

$$f(x, y) = \cos x \cos y.$$

3) Write Taylor's formula of the order 2 at the point $(x_0, y_0) = (1, 1)$ for the function

$$f(x, y) = y^x.$$

Local Extrema

Find the local extrema of the following functions:

1) $f : \mathbb{R}^2 \to \mathbb{R},\ f(x, y) = x^3 + 3xy^2 - 15x - 12y.$

2) $f : \mathbb{R}^3 \to \mathbb{R},\ f(x, y, z) = x^2 + y^2 + z^2 + 4x + 2y - 8z.$

3) $f : \mathbb{R}^2 \to \mathbb{R},\ f(x, y) = x^2 - (y - 2)^2.$

4) $f : (0, \infty) \times (0, \infty) \to \mathbb{R},\ f(x, y) = xy + \dfrac{4}{x} + \dfrac{2}{y} - 3.$

Power Series

Compute the radius of convergence and the set of convergence for the following series:

1) $\displaystyle\sum_{n \geq 0} \frac{x^n}{n!}$; 2) $\displaystyle\sum_{n \geq 1} \frac{x^n}{n^n}$; 3) $\displaystyle\sum_{n \geq 0} n!\, x^n$; 4) $\displaystyle\sum_{n \geq 1} \frac{x^n}{n^2}$; 5) $\displaystyle\sum_{n \geq 1} \frac{x^n}{n 2^n}$;

6) $\displaystyle\sum_{n \geq 1} (-1)^{n-1} \frac{(x-5)^n}{n\, 3^n}$; 7) $\displaystyle\sum_{n \geq 1} \frac{(x-1)^{2n}}{n\, 9^n}$.