DIFFERENTIAL EQUATIONS - Solved Exercises 1

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EXERCISES

Exercise 1

Find the general solution of the equations:

1a $y''-6y'+8y=0.$

1b $y''-6y'+9y=0.$

1c $y''-6y'+10y=0.$

Exercise 2

Solve the equation

2 $x^2y''+y=0.$

Exercise 3

Solve by two methods

3 $(4xy+y)dx+(2x^2+x)dy=0.$

Exercise 4

Solve by two methods

4 $\begin{cases} y_1'=3y_2\\ y_2'=3y_1.\end{cases}$

Exercise 5

Solve the system of equations

5 $\begin{cases} y_1'=y_1-2y_2\\ y_2'=2y_1+y_2.\end{cases}$

Exercise 6

Solve the system of equations

6 $\begin{cases} y_1'=3y_1+y_2\\ y_2'=3y_2\end{cases}$

SOME DEFINITIONS, THEOREMS and REMARKS

T1 Theorem (*Linear equations with constant coefficients*).

The space of all the real solutions of
$$a_0y^{(n)}+a_1y^{(n-1)}+\cdots+a_{n-1}y'+a_ny=0 \qquad (1)$$
where $a_0,\dots,a_n\in\mathbb{R}$, is a real vector space of dimension n.

R1 Remark.
In order to describe the space of all the real solutions of (1), it is suficient to find some particular solutions $y_1,\ y_2,\ \dots,\ y_n$ forming a basis in the space of solutions. The *general solution* can be written as
$$y(x)=C_1\,y_1(x)+C_2\,y_2(x)+\cdots+C_n\,y_n(x)$$
where $C_1,\ C_2,\ \dots,\ C_n$ are arbitrary real constants.

D1 Definition. The polynomial
$$P(\lambda)=a_0\lambda^n+a_1\lambda^{n-1}+\cdots+a_{n-1}\lambda+a_n$$
is called the *characteristic polynomial* of the equation (1).

T2 Theorem (Particular solutions).
$$\boxed{y(x)=\mathrm{e}^{\lambda x} \text{ is a solution of (1)}} \Leftrightarrow \boxed{P(\lambda)=0}$$

D2 Definition (*Complex exponential*).
$$\boxed{\mathrm{e}^{(\alpha+\beta\mathrm{i})x}=\mathrm{e}^{\alpha x}\cos\beta x+\mathrm{i}\,\mathrm{e}^{\alpha x}\sin\beta x}$$

T3 Theorem. General solution of $a_0y''+a_1y'+a_2y=0.$

$P(\lambda)=a_0\lambda^2+a_1\lambda+a_2$ has the roots $\lambda_{1,2}=\frac{-a_1\pm\sqrt{a_1^2-4a_0\,a_2}}{2a_0}.$

General solution:

- $\lambda_1\neq\lambda_2\in\mathbb{R} \;\Rightarrow\; y(x)=C_1\,\mathrm{e}^{\lambda_1x}+C_2\,\mathrm{e}^{\lambda_2x}.$
- $\lambda_1=\lambda_2=\lambda \;\Rightarrow\; y(x)=C_1\,\mathrm{e}^{\lambda x}+C_2\,x\,\mathrm{e}^{\lambda x}.$
- $\lambda_{1,2}=\alpha\pm\beta\mathrm{i} \;\Rightarrow\; y(x)=C_1\,\mathrm{e}^{\alpha x}\cos\beta x+C_2\,\mathrm{e}^{\alpha x}\sin\beta x.$

T4 Theorem (Euler's equation).
$$\boxed{a_0x^ny^{(n)}+\dots+a_{n-1}xy'+a_ny=0} \xrightarrow[x=\mathrm{e}^t]{\text{change}} \text{linear equation with constant coefficients}$$

T5 Theorem (*Primitives* of a continuous function)

$$\begin{array}{c}f:(a,b)\to\mathbb{R}\\ \text{continuous}\end{array} \Rightarrow \begin{array}{c}\text{Primitives of } f \text{ are: } F:(a,b)\to\mathbb{R},\\ \boxed{F(x)=\int_{x_0}^{x} f(t)\,dt+C}\\ x_0\in(a,b) \text{ is fixed}\end{array}$$

We have $F'(x)=f(x)$, that is
$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{x_0}^{x} f(t)\,dt\right)=f(x)}$$
for any $x\in(a,b)$.

T6 Theorem (*Separable equations*).
The solution $y(x)$ of $\boxed{y'=f(x)\,g(y)}$ is defined by
$$\boxed{\int_{y_0}^{y}\frac{1}{g(u)}du=\int_{x_0}^{x} f(t)\,dt+C}$$
$x_0,\ y_0$ constants, $g(y_0)\neq0.$

R2 Remark
$\boxed{y'=f(x,y)}$ can also be written as $\boxed{\frac{\mathrm{d}y}{\mathrm{d}x}=f(x,y)}$ or $\boxed{f(x,y)dx-dy=0.}$

D3 Definition (*Symmetric equations*).
$\begin{array}{l}x=\varphi(t)\\ y=\psi(t)\end{array}$ is a solution of $\boxed{P(x,y)dx+Q(x,y)dy=0}$ if
$$P(\varphi(t),\psi(t))\varphi'(t)+Q(\varphi(t),\psi(t))\psi'(t)=0, \text{ for any } t.$$

T7 Theorem (*Exact equations*).

$\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$ in a simply connected domain D $\Rightarrow$ The function $F:D\to\mathbb{R}$,
$$F(x,y)=\int_\gamma Pdx+Qdy,$$
where $\gamma:[a,b]\to D$ is an arbitrary path connecting a fixed point (x_0,y_0) with (x,y), defines a function satisfying the relation
$$P(x,y)dx+Q(x,y)dy=dF$$

R3 Remark.
In D, the equation $P(x,y)dx+Q(x,y)dy=0$ can be written as $dF=0$, and its solution is described implicitly by $F(x,y)=C.$

R4 Remark. By denoting
$$Y(x)=\begin{pmatrix}y_1(x)\\ y_2(x)\end{pmatrix},\quad A=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix},$$
$\boxed{\begin{cases}y_1'=a_{11}y_1+a_{12}y_2\\ y_2'=a_{21}y_1+a_{22}y_2\end{cases}}$ can be written as $\boxed{Y'=AY}$

T8 Theorem.

The space of all the real solutions of $Y'=AY$, where $a_{ij}\in\mathbb{R}$, is a real vector space of dimension 2.

D4 Definition. The polynomial
$$P(\lambda)=\begin{vmatrix}a_{11}-\lambda & a_{12}\\ a_{21} & a_{22}-\lambda\end{vmatrix}=\lambda^2-(a_{11}+a_{22})\lambda+a_{11}a_{22}-a_{12}a_{21}$$
is called the *characteristic polynomial* of $Y'=AY$.

T9 Theorem (Particular non-null solutions).
$$Y(x)=\begin{pmatrix}p\\ q\end{pmatrix}\mathrm{e}^{\lambda x}\neq\begin{pmatrix}0\\ 0\end{pmatrix} \text{ satisfies } Y'=AY \;\Leftrightarrow\; \begin{cases}P(\lambda)=0 \quad\text{and}\\ A\begin{pmatrix}p\\ q\end{pmatrix}=\lambda\begin{pmatrix}p\\ q\end{pmatrix}.\end{cases}$$

T10 Theorem. If λ_1 and λ_2 are the solutions of $P(\lambda)=0$, then:

- $\left.\begin{array}{l}\lambda_1,\ \lambda_2\in\mathbb{R}\\ A\begin{pmatrix}p_j\\ q_j\end{pmatrix}=\lambda_j\begin{pmatrix}p_j\\ q_j\end{pmatrix}\\ \text{linearly independent}\end{array}\right\} \Rightarrow Y(x)=C_1\begin{pmatrix}p_1\\ q_1\end{pmatrix}\mathrm{e}^{\lambda_1x}+C_2\begin{pmatrix}p_2\\ q_2\end{pmatrix}\mathrm{e}^{\lambda_2x}.$

- $\left.\begin{array}{l}\lambda_{1,2}=\alpha\pm\beta\mathrm{i}\notin\mathbb{R}\\ A\begin{pmatrix}p\\ q\end{pmatrix}=(\alpha+\beta\mathrm{i})\begin{pmatrix}p\\ q\end{pmatrix}\end{array}\right\} \Rightarrow Y(x)=C_1\,\mathfrak{Re}\left\{\begin{pmatrix}p\\ q\end{pmatrix}\mathrm{e}^{(\alpha+\beta\mathrm{i})x}\right\} +C_2\,\mathfrak{Im}\left\{\begin{pmatrix}p\\ q\end{pmatrix}\mathrm{e}^{(\alpha+\beta\mathrm{i})x}\right\}.$

- $\left.\begin{array}{l}\lambda_1=\lambda_2=\lambda\\ \dim\left\{\begin{pmatrix}p\\ q\end{pmatrix}\,\middle|\,A\begin{pmatrix}p\\ q\end{pmatrix}=\lambda\begin{pmatrix}p\\ q\end{pmatrix}\right\}=1\end{array}\right\} \Rightarrow$ General solution can be found of the form
$$Y(x)=\begin{pmatrix}c_1+c_2x\\ c_3+c_4x\end{pmatrix}\mathrm{e}^{\lambda x}$$
(only two of c_j are indep.).

SOLUTIONS

Exercise 1. We use T2 and T3.

$\boxed{1a}$ $\lambda^2-6\lambda+8=0 \quad\Rightarrow\quad \lambda_1=2,\ \lambda_2=4.$
$T3 \quad\Rightarrow\quad y(x)=C_1\,e^{2x}+C_2\,e^{4x}.$

$\boxed{1b}$ $\lambda^2-6\lambda+9=0 \quad\Rightarrow\quad \lambda_1=\lambda_2=3.$
$T3 \quad\Rightarrow\quad y(x)=C_1\,e^{3x}+C_2\,x\,e^{3x}.$

$\boxed{1c}$ $\lambda^2-6\lambda+10=0 \quad\Rightarrow\quad \lambda_{1,2}=3\pm i.$
$T3 \quad\Rightarrow\quad y(x)=C_1\,e^{3x}\cos x+C_2\,e^{3x}\sin x.$

Exercise 2 We use T4, T2 and T3.

$\boxed{2}$ We use the change of variables:

$$\begin{matrix} x\mapsto t, \\ y\mapsto z, \end{matrix}\quad \text{satisfying}\quad \begin{matrix} x=e^t, & y(x)=z(\ln x), \\ t=\ln x, & z(t)=y(e^t). \end{matrix}$$

Since $y'(x)=z'(\ln x)(\ln x)'=\frac{1}{x}\,z'(\ln x)$,

$$y''(x)=-\tfrac{1}{x^2}\,z'(\ln x)+\tfrac{1}{x^2}\,z''(\ln x),$$

in the new variables, the equation becomes

$$e^{2t}\left(-\tfrac{1}{e^{2t}}\,z'+\tfrac{1}{e^{2t}}\,z''\right)+z=0,$$

that is

$$z''-z'+z=0.$$

Because $\lambda^2-\lambda+1=0 \quad\Rightarrow\quad \lambda_{1,2}=\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$,

$T3 \quad\Rightarrow\quad z(t)=C_1\,e^{\frac{t}{2}}\cos\frac{t\sqrt{3}}{2}+C_2\,e^{\frac{t}{2}}\sin\frac{t\sqrt{3}}{2}$,

and consequently,

$$\begin{aligned} y(x)&=C_1\,e^{\frac{\ln x}{2}}\cos(\tfrac{\sqrt{3}}{2}\ln x)+C_2\,e^{\frac{\ln x}{2}}\sin(\tfrac{\sqrt{3}}{2}\ln x)\\ &=C_1\sqrt{x}\,\cos(\tfrac{\sqrt{3}}{2}\ln x)+C_2\sqrt{x}\,\sin(\tfrac{\sqrt{3}}{2}\ln x).\\ &=C_1\sqrt{x}\,\cos(\sqrt{3}\ln\sqrt{x})+C_2\sqrt{x}\,\sin(\sqrt{3}\ln\sqrt{x}).\end{aligned}$$

Exercise 3

$\boxed{3a}$ *Method 1.* The equation is exact in $D=\mathbb{R}^2$:

$$\frac{\partial(4xy+y)}{\partial y}=4x+1=\frac{\partial(2x^2+x)}{\partial x}.$$

By using T7 and the path

$$\gamma:[0,1]\to\mathbb{R}^2,\quad \gamma(t)=(xt,yt)$$

connecting $(0,0)$ with (x,y), we get

$$\begin{aligned} F(x,y)&=\int_\gamma (4xy+y)dx+(2x^2+x)dy\\ &=\int_0^1\left[(4xyt^2+yt)x+(2x^2t^2+xt)y\right]dt\\ &=\left(4x^2y\tfrac{t^3}{3}+xy\tfrac{t^2}{2}+2x^2y\tfrac{t^3}{3}+xy\tfrac{t^2}{2}\right)\Big|_{t=0}^{t=1}\\ &=2x^2y+xy.\end{aligned}$$

The equation can be written as (see R3)

$$d(2x^2y+xy)=0,$$

and its solution, described by

$$2x^2y+xy=C,$$

is

$$y:(a,b)\to\mathbb{R},\quad y(x)=\frac{C}{2x^2+x},$$

where (a,b) is an arbitrary interval not containing 0 or $-\frac{1}{2}$.

$\boxed{3b}$ *Method 2.* We use T6. The equation, written as

$$y'=-\frac{4x+1}{2x^2+x}y,$$

is a separable equation. Its solution is described by

$$\int\frac{dy}{y}=-\int\frac{4x+1}{2x^2+x}dx+\ln|C|,$$

where $\int\frac{dy}{y}=$ a primitive of $\quad y\mapsto\frac{1}{y}$,

$\int\frac{4x+1}{2x^2+x}dx=$ a primitive of $\quad x\mapsto\frac{4x+1}{2x^2+x}$,

that is, by the relation

$$\ln|y|=-\ln|2x^2+x|+\ln|C|,$$

For $x\notin\{0,-\frac{1}{2}\}$, this relation can be written as

$$|y(x)|=\left|\frac{C}{2x^2+x}\right|\quad\text{or}\quad y(x)=\frac{\pm C}{2x^2+x}.$$

Exercise 4

$\boxed{4a}$ *Method 1.* We have

$$\begin{cases} y_1'=3y_2\\ y_2'=3y_1.\end{cases}\Rightarrow\quad \begin{cases} y_1''=3y_2'\\ y_2'=3y_1.\end{cases}\Rightarrow\quad y_1''-9y_1=0.$$

$T3 \quad\Rightarrow\quad y_1(x)=C_1\,e^{3x}+C_2\,e^{-3x}.$

$y_1'=3y_2 \;\Rightarrow\; y_2(x)=\frac{1}{3}y_1'(x)=C_1\,e^{3x}-C_2\,e^{-3x}.$

$\boxed{4b}$ *Method 2.* The system can be written as $Y'=AY$, where.

$$Y(x)=\begin{pmatrix} y_1(x)\\ y_2(x)\end{pmatrix},\ A=\begin{pmatrix} 0 & 3\\ 3 & 0\end{pmatrix},$$

In this case,

$$\begin{vmatrix} 0-\lambda & 3\\ 3 & 0-\lambda\end{vmatrix}=0\;\Rightarrow\;\lambda_{1,2}=\pm 3\quad\text{and}$$

$$\begin{pmatrix} 0 & 3\\ 3 & 0\end{pmatrix}\begin{pmatrix} p\\ q\end{pmatrix}=3\begin{pmatrix} p\\ q\end{pmatrix}\quad\Rightarrow\quad\begin{pmatrix} p\\ q\end{pmatrix}=\begin{pmatrix} p\\ p\end{pmatrix},$$

$$\begin{pmatrix} 0 & 3\\ 3 & 0\end{pmatrix}\begin{pmatrix} p\\ q\end{pmatrix}=-3\begin{pmatrix} p\\ q\end{pmatrix}\quad\Rightarrow\quad\begin{pmatrix} p\\ q\end{pmatrix}=\begin{pmatrix} p\\ -p\end{pmatrix}.$$

and consequently (see T9 and T10)

$$\begin{pmatrix} y_1(x)\\ y_2(x)\end{pmatrix}=C_1\begin{pmatrix} 1\\ 1\end{pmatrix}e^{3x}+C_2\begin{pmatrix} 1\\ -1\end{pmatrix}e^{-3x},$$

that is

$$\begin{aligned} y_1(x)&=C_1\,e^{3x}+C_2\,e^{-3x},\\ y_2(x)&=C_1\,e^{3x}-C_2\,e^{-3x}.\end{aligned}$$

Exercise 5. The system can be written as $Y'=AY$, where.

$$Y(x)=\begin{pmatrix} y_1(x)\\ y_2(x)\end{pmatrix},\ A=\begin{pmatrix} 1 & -2\\ 2 & 1\end{pmatrix},$$

In this case,

$$\begin{vmatrix} 1-\lambda & -2\\ 2 & 1-\lambda\end{vmatrix}=0\;\Rightarrow\;\lambda_{1,2}=1\pm 2i\quad\text{and}$$

$$\begin{pmatrix} 1 & -2\\ 2 & 1\end{pmatrix}\begin{pmatrix} p\\ q\end{pmatrix}=(1+2i)\begin{pmatrix} p\\ q\end{pmatrix}\quad\Rightarrow\quad\begin{pmatrix} p\\ q\end{pmatrix}=\begin{pmatrix} p\\ -pi\end{pmatrix}.$$

By choosing the eigenvector $\begin{pmatrix} 1\\ -i\end{pmatrix}$, we get (see D2 and T10)

$$\begin{aligned}\begin{pmatrix} y_1(x)\\ y_2(x)\end{pmatrix}&=C_1\,\mathfrak{Re}\left\{\begin{pmatrix} 1\\ -i\end{pmatrix}e^{(1+2i)x}\right\}+C_2\,\mathfrak{Im}\left\{\begin{pmatrix} 1\\ -i\end{pmatrix}e^{(1+2i)x}\right\}\\ &=C_1\begin{pmatrix} e^x\cos 2x\\ e^x\sin 2x\end{pmatrix}+C_2\begin{pmatrix} e^x\sin 2x\\ -e^x\cos 2x\end{pmatrix}.\end{aligned}$$

that is

$$\begin{aligned} y_1(x)&=C_1\,e^x\cos 2x+C_2\,e^x\sin 2x,\\ y_2(x)&=C_1\,e^x\sin 2x-C_2\,e^x\cos 2x.\end{aligned}$$

Exercise 6. The system can be written as $Y'=AY$, where.

$$Y(x)=\begin{pmatrix} y_1(x)\\ y_2(x)\end{pmatrix},\ A=\begin{pmatrix} 3 & 1\\ 0 & 3\end{pmatrix},$$

We use T10. In this case,

$$\begin{vmatrix} 3-\lambda & 1\\ 0 & 3-\lambda\end{vmatrix}=0\;\Rightarrow\;\lambda_1=\lambda_2=3\quad\text{and}$$

$$\dim\left\{\begin{pmatrix} p\\ q\end{pmatrix}\;\middle|\;A\begin{pmatrix} p\\ q\end{pmatrix}=3\begin{pmatrix} p\\ q\end{pmatrix}\right\}=\dim\left\{\begin{pmatrix} p\\ 0\end{pmatrix}\;\middle|\;p\in\mathbb{R}\right\}=1.$$

$T10 \;\Rightarrow\;$ the general solution can be found of the form

$$Y(x)=\begin{pmatrix} c_1+c_2x\\ c_3+c_4x\end{pmatrix}e^{3x},\ \text{that is}\ \begin{matrix} y_1(x)=(c_1+c_2x)\,e^{3x},\\ y_2(x)=(c_3+c_4x)\,e^{3x}.\end{matrix}$$

By direct substitution into the system, we get $c_2=c_3,\ c_4=0$, and consequently

$$\begin{pmatrix} y_1(x)\\ y_2(x)\end{pmatrix}=\begin{pmatrix} c_1+c_2x\\ c_2\end{pmatrix}e^{3x}=c_1\begin{pmatrix} 1\\ 0\end{pmatrix}e^{3x}+c_2\begin{pmatrix} x\\ 1\end{pmatrix}e^{3x},$$

that is

$$\begin{aligned} y_1(x)&=C_1\,e^{3x}+C_2\,x\,e^{3x},\\ y_2(x)&=C_2\,e^{3x}.\end{aligned}$$