On the mathematical description of
the SINGLE-MODE GAUSSIAN STATES

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Abstract. This tutorial on the single-mode Gaussian states starts with a description of the mathematical fundamentals. In the second part, we present in a concise way the main definitions/theorems, and in the third part the proofs of the mathematical formulas.

Contents

MATHEMATICAL FOUNDATIONS

$\mathfrak{A}$ ■ TEMPERED DISTRIBUTIONS

1 Vector space of fast decreasing functions

$$\mathcal{S}(\mathbb{R})=\left\{\varphi:\mathbb{R}\longrightarrow\mathbb{C}\;\middle|\;\begin{array}{l}\varphi\in C^\infty(\mathbb{R}) \text{ satisfying the relation}\\ \lim_{q\to\pm\infty} q^k\,\varphi^{(m)}(q)=0,\quad \forall k,m\in\mathbb{N}\end{array}\right\}$$

2 $$\lim_{n\to\infty}\varphi_n=0 \;\overset{\text{def}}{\Leftrightarrow}\; \lim_{n\to\infty}\sup_{q\in\mathbb{R}}\left|q^k\,\varphi_n^{(m)}(q)\right|=0,\quad \forall k,m\in\mathbb{N}.$$

3 Vector space of *tempered distributions*

$$\mathcal{S}'(\mathbb{R})=\left\{\begin{array}{r}f:\mathcal{S}(\mathbb{R})\longrightarrow\mathbb{C}:\\ \varphi\mapsto\langle f,\varphi\rangle\end{array}\;\middle|\;\begin{array}{l}\langle f,\alpha\varphi+\beta\psi\rangle=\alpha\langle f,\varphi\rangle+\beta\langle f,\psi\rangle,\\ \lim\limits_{n\to\infty}\varphi_n=0\;\Rightarrow\;\lim\limits_{n\to\infty}\langle f,\varphi_n\rangle=0\end{array}\right\}$$

4 When it exists, the distribution corresponding to a usual function $f:\mathbb{R}\to\mathbb{C}$ is defined by the relation $\boxed{\langle f,\varphi\rangle\overset{\text{def}}{=}\int_{-\infty}^{\infty}\overline{f(q)}\,\varphi(q)\,dq}$

5 *Dirac distribution* $\boxed{\langle\delta_a,\varphi\rangle\overset{\text{def}}{=}\varphi(a)}$ and $\boxed{\langle\delta,\varphi\rangle\overset{\text{def}}{=}\varphi(0)}$

Definition.

6 *Multiplication of a distribution f* by a function $\eta(q)$ (when it exists) $\boxed{\langle\eta\,f,\varphi\rangle\overset{\text{def}}{=}\langle f,\eta\,\varphi\rangle}$

7 *Derivative of a distribution f* $\boxed{\langle f',\varphi\rangle\overset{\text{def}}{=}-\langle f,\varphi'\rangle}$

8 *Change of variable* $\boxed{\langle f(\alpha q+\beta),\varphi\rangle\overset{\text{def}}{=}\left\langle\frac{1}{|\alpha|}f,\varphi\left(\frac{q-\beta}{\alpha}\right)\right\rangle}$

$\mathfrak{B}$ ■ FOURIER TRANSFORM

1 **Definition** ($\hat{F}^{\pm1}[\varphi]$ exists only if the integral is convergent).

Fourier transform of $\varphi:\mathbb{R}\to\mathbb{C}$ is $\hat{F}[\varphi]:\mathbb{R}\to\mathbb{C}$, $\boxed{\hat{F}^{\pm1}[\varphi](p)\overset{\text{def}}{=}\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}e^{\mp\frac{i}{\hbar}pq}\varphi(q)\,dq}$

2 **Remarks.** $\boxed{\hat{F}^{-1}=\hat{F}^\dagger}$ $\boxed{\hat{F}^{-1}[\varphi]=\hat{F}[\check{\varphi}]}$ where $\check{\varphi}(q)\overset{\text{def}}{=}\varphi(-q)$

3 **Theorem.** $\boxed{(\hat{F}[\varphi])'=\hat{F}[-\frac{i}{\hbar}q\varphi(q)]}$ and $\boxed{\hat{F}[\varphi'](p)=\frac{i}{\hbar}p\,\hat{F}[\varphi](p)}$

4 **Theorem.** $\varphi\in\mathcal{S}(\mathbb{R})\;\Rightarrow\;\hat{F}^{\pm}[\varphi]\in\mathcal{S}(\mathbb{R})$.

5 **Definition.** *Fourier transform of* a distribution f $\boxed{\langle\hat{F}^{\pm1}[f],\varphi\rangle\overset{\text{def}}{=}\langle f,\hat{F}^{\pm1}[\varphi]\rangle}$

6 **Theorem.** $\boxed{(\hat{F}[f])'=\hat{F}[-\frac{i}{\hbar}q\,f]}$ and $\boxed{\hat{F}[f']=\frac{i}{\hbar}p\,\hat{F}[f]}$.

7 $$\left.\begin{array}{l}\langle\hat{F}[\delta_a],\varphi\rangle=\langle\delta_a,F[\varphi]\rangle\\ =\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}e^{-\frac{i}{\hbar}aq}\varphi(q)\,dq\end{array}\right\}\Rightarrow\boxed{\begin{array}{l}\hat{F}[\delta_a]=\frac{1}{\sqrt{2\pi\hbar}}\,e^{-\frac{i}{\hbar}aq}\\ \hat{F}[\delta]\;=\;\frac{1}{\sqrt{2\pi\hbar}}\end{array}}$$

8 $$\hat{F}[\delta]=\frac{1}{\sqrt{2\pi\hbar}}\;\Rightarrow\;\hat{F}^{-1}[1]=\hat{F}[1]=\sqrt{2\pi\hbar}\,\delta\;\Rightarrow\boxed{\hat{F}[1]=\sqrt{2\pi\hbar}\,\delta}$$

$\mathfrak{C}$ ■ DIRAC FUNCTION $\delta_a(q)$ ($a\in\mathbb{R}$ is a parameter)

1 Formally, that is, in a symbolical way

$$\left.\begin{array}{l}\langle f,\varphi\rangle=\int_{-\infty}^{\infty}f(q)\,\varphi(x)\,dq\\ \langle\delta_a,\varphi\rangle=\varphi(a)\end{array}\right\}\Rightarrow\boxed{\varphi(a)=\int_{-\infty}^{\infty}\delta_a(q)\,\varphi(q)\,dq}$$

where the right hand side is not a true integral, but in many cases it behaves similar to an integral, and

$$q\mapsto\delta_a(q)\overset{\text{def}}{=}\begin{cases}0 & \text{for } q\neq a\\ \infty & \text{for } q=a\end{cases}$$ is the *Dirac function.*

Theorem.

2 $$\left.\begin{array}{l}\langle\delta(q-a),\varphi\rangle=\langle\delta,\varphi(q+a)\rangle\\ \quad=\varphi(a)=\langle\delta_a,\varphi\rangle\end{array}\right\}\Rightarrow\boxed{\delta_a(q)=\delta(q-a)}$$

3 $$\left.\begin{array}{l}\langle\delta(\alpha q),\varphi\rangle=\left\langle\frac{1}{|\alpha|}\delta,\varphi\left(\frac{q}{\alpha}\right)\right\rangle\\ \quad=\frac{1}{|\alpha|}\varphi(0)=\left\langle\frac{1}{|\alpha|}\delta,\varphi\right\rangle\end{array}\right\}\Rightarrow\boxed{\begin{array}{l}\delta(\alpha q)=\frac{1}{|\alpha|}\delta(q)\\ \delta(-q)=\delta(q)\end{array}}$$

4 $$\delta_a(q)=\delta(q-a)=\delta(a-q)=\delta_q(a)\quad\Rightarrow\quad\boxed{\delta_a(q)=\delta_q(a)}$$

Theorem.

5 Formally, $\hat{F}[1]=\sqrt{2\pi\hbar}\,\delta$ can be written as $\boxed{\int_{-\infty}^{\infty}e^{-\frac{i}{\hbar}pq}\,dq=2\pi\hbar\,\delta(p)}$

Theorem. By denoting $\boxed{|a\rangle\equiv\delta_a}$, we formally get:

6 $$\langle a|\varphi\rangle=\langle\delta_a,\varphi\rangle=\varphi(a)\quad\Rightarrow\quad\boxed{\varphi(a)=\langle a|\varphi\rangle}$$

7 $$\left.\begin{array}{l}\varphi(a)=\int_{-\infty}^{\infty}\delta_a(q)\,\varphi(q)\,dq\\ \quad=\int_{-\infty}^{\infty}\varphi(q)\,\delta_q(a)\,dq\end{array}\right\}\Rightarrow\boxed{|\varphi\rangle=\int_{-\infty}^{\infty}\varphi(q)\,|q\rangle\,dq}$$

8 $$\langle a|b\rangle=\langle\delta_a,\delta_b\rangle=\delta_b(a)=\delta(a-b)\quad\Rightarrow\quad\boxed{\langle a|b\rangle=\delta(a-b)}$$

9 $$\left.\begin{array}{l}\mathbb{I}|\varphi\rangle\overset{\text{def}}{=}|\varphi\rangle=\int_{-\infty}^{\infty}\varphi(q)\,|q\rangle\,dq\\ \quad=\int_{-\infty}^{\infty}|q\rangle\langle q|\varphi\rangle\,dq\end{array}\right\}\Rightarrow\boxed{\mathbb{I}=\int_{-\infty}^{\infty}|q\rangle\langle q|\,dq}$$

10 $$\left.\begin{array}{l}\langle\eta(q)\delta_a,\varphi\rangle=\langle\delta_a,\eta(q)\varphi\rangle\\ \quad=\eta(a)\varphi(a)=\langle\eta(a)\delta_a,\varphi\rangle\end{array}\right\}\Rightarrow\boxed{\eta(q)|a\rangle=\eta(a)\,|a\rangle}$$

11 $$\left.\begin{array}{l}\langle q\delta_a,\varphi\rangle=\langle\delta_a,q\varphi\rangle\\ \quad=a\varphi(a)=\langle a\delta_a,\varphi\rangle\end{array}\right\}\Rightarrow\boxed{q|a\rangle=a\,|a\rangle}$$

$\mathfrak{D}$ ■ GAUSSIAN FUNCTIONS OF ONE VARIABLE

Definition. A *Gaussian function* is a function of the form

1 $$\boxed{g:\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R},\quad g(q)=e^{-\frac{1}{2}\kappa(q-q_0)^2}}$$

where $\kappa\in(0,\infty)$ and $q_0\in\mathbb{R}$.

Theorem.

2 $$\boxed{\int_{-\infty}^{\infty}e^{-\frac{1}{2}\kappa(q-q_0)^2}dq=\sqrt{\frac{2\pi}{\kappa}}}$$

3 $$\int_{-\infty}^{\infty}q\,e^{-\frac{1}{2}\kappa(q-q_0)^2}dq=q_0\sqrt{\frac{2\pi}{\kappa}}$$

4 $$\int_{-\infty}^{\infty}q^2\,e^{-\frac{1}{2}\kappa(q-q_0)^2}dq=\frac{1}{\kappa}\sqrt{\frac{2\pi}{\kappa}}$$

5 $$\boxed{\int_{-\infty}^{\infty}e^{\pm ipq}\,e^{-\frac{1}{2}\kappa q^2}\,dq=\frac{1}{\sqrt{\kappa}}e^{-\frac{1}{2}\frac{1}{\kappa}p^2}}$$

$\mathfrak{E}$ ■ GAUSSIAN FUNCTIONS OF TWO VARIABLES

Definition. A *Gaussian function* is a function $g:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$,

1 $$\boxed{\begin{array}{l}g(q,p)=e^{-\frac{1}{2}(q-q_0\;\;p-p_0)\begin{pmatrix}a&b\\b&c\end{pmatrix}\begin{pmatrix}q-q_0\\p-p_0\end{pmatrix}}\\ \quad=e^{-\frac{1}{2}[a(q-q_0)^2+2b(q-q_0)(p-p_0)+c(p-p_0)^2]}\end{array}}$$

where $\sigma=\begin{pmatrix}a&b\\b&c\end{pmatrix}$ has real entries, $\sigma>0$ and $(q_0,p_0)\in\mathbb{R}^2$.

Theorem.

2 $$\boxed{\int_{\mathbb{R}^2}e^{-\frac{1}{2}(q-q_0\;\;p-p_0)\,\sigma\begin{pmatrix}q-q_0\\p-p_0\end{pmatrix}}dqdp=\frac{2\pi}{\sqrt{\det\sigma}}}$$

3 $$\int_{\mathbb{R}^2}q\,e^{-\frac{1}{2}(q-q_0\;\;p-p_0)\,\sigma\begin{pmatrix}q-q_0\\p-p_0\end{pmatrix}}dqdp=q_0\frac{2\pi}{\sqrt{\det\sigma}}$$

4 $$\int_{\mathbb{R}^2}p\,e^{-\frac{1}{2}(q-q_0\;\;p-p_0)\,\sigma\begin{pmatrix}q-q_0\\p-p_0\end{pmatrix}}dqdp=p_0\frac{2\pi}{\sqrt{\det\sigma}}$$

5 $$\int_{\mathbb{R}^2}q^2\,e^{-\frac{1}{2}(q-q_0\;\;p-p_0)\,\sigma\begin{pmatrix}q-q_0\\p-p_0\end{pmatrix}}dqdp=\left(\frac{c}{\det\sigma}+q_0^2\right)\frac{2\pi}{\sqrt{\det\sigma}}$$

6 $$\int_{\mathbb{R}^2}p^2\,e^{-\frac{1}{2}(q-q_0\;\;p-p_0)\,\sigma\begin{pmatrix}q-q_0\\p-p_0\end{pmatrix}}dqdp=\left(\frac{a}{\det\sigma}+p_0^2\right)\frac{2\pi}{\sqrt{\det\sigma}}$$

7 $$\int_{\mathbb{R}^2}qp\,e^{-\frac{1}{2}(q-q_0\;\;p-p_0)\,\sigma\begin{pmatrix}q-q_0\\p-p_0\end{pmatrix}}dqdp=\left(\frac{-b}{\det\sigma}+q_0p_0\right)\frac{2\pi}{\sqrt{\det\sigma}}$$

Definition.

Fourier transform of $\varphi:\mathbb{R}^2\to\mathbb{C}$ is $\hat F[\varphi]:\mathbb{R}^2\to\mathbb{C}$

2 $$\boxed{\hat F^{\pm 1}[\varphi](p_1,p_2) \overset{\text{def}}{=} \frac{1}{2\pi\hbar}\int_{\mathbb{R}^2} e^{\mp\frac{i}{\hbar}(p_1q_1+p_2q_2)}\varphi(q_1,q_2)\,dq_1dq_2}$$

Theorem.

3 $$\boxed{\hat F\left[e^{-\frac12 (q_1\quad q_2)\,\sigma\begin{pmatrix}q_1\\ q_2\end{pmatrix}}\right](p_1,p_2) = \frac{1}{\sqrt{\det\sigma}}\; e^{-\frac12 (p_1\quad p_2)\,\sigma^{-1}\begin{pmatrix}p_1\\ p_2\end{pmatrix}}}$$

$\mathfrak{F}$ ■ HILBERT SPACE ℓ^2

1 $$\ell^2 \overset{\text{def}}{=} \left\{(\alpha_0,\alpha_1,\alpha_2,\ldots)\;\middle|\;\alpha_n\in\mathbb{C},\ \sum_{n=0}^{\infty}|\alpha_n|^2<\infty\right\}$$

2 $$\langle(\alpha_0,\alpha_1,\alpha_2,\ldots),(\beta_0,\beta_1,\beta_2,\ldots)\rangle \overset{\text{def}}{=} \sum_{n=0}^{\infty}\bar\alpha_n\,\beta_n$$

$\mathfrak{G}$ ■ HILBERT SPACE $L^2(\mathbb{R})$

1 $$L^2(\mathbb{R}) = \left\{\Psi:\mathbb{R}\to\mathbb{C},\ \Psi=\sum_{n=0}^{\infty}\alpha_n\Psi_n\;\middle|\;\alpha_n\in\mathbb{C},\ \sum_{n=0}^{\infty}|\alpha_n|^2<\infty\right\}$$

2 where $$\boxed{\Psi_n(q) \overset{\text{def}}{=} \frac{1}{\sqrt{2^n\,n!\,\sqrt{\pi}}}H_n(q)\,e^{-\frac{q^2}{2}}}$$

3 $$\boxed{\langle\Psi,\Phi\rangle \overset{\text{def}}{=} \int_{-\infty}^{\infty}\overline{\Psi(q)}\,\Phi(q)\,dq}$$

4 **Theorem**. $\boxed{\mathcal{S}(\mathbb{R})\subset L^2(\mathbb{R})\subset\mathcal{S}'(\mathbb{R})}$

Theorem.

5 $\boxed{\ell^2\to L^2(\mathbb{R}):(\alpha_0,\alpha_1,\alpha_2,\ldots)\mapsto\sum_{n=0}^{\infty}\alpha_n\Psi_n}$ is an isomorphism of Hilbert spaces

Theorem. *The position and momentum operators*

6 $\boxed{\begin{array}{c}\hat Q\Psi(q)\overset{\text{def}}{=}q\Psi(q)\\ \hat P\Psi\overset{\text{def}}{=}-i\hbar\frac{d}{dq}\Psi\end{array}}$ *satisfy* $\boxed{\begin{array}{c}\hat P=\hat F^\dagger\hat Q\hat F\\ [\hat Q,\hat P]=i\hbar\end{array}}$ $\boxed{\begin{array}{c}\hat Q|a\rangle=a\,|a\rangle\\ \hat Q=\int_{-\infty}^{\infty}q|q\rangle\langle q|\,dq\\ \hat P e^{\frac{i}{\hbar}aq}=a\,e^{\frac{i}{\hbar}aq}\end{array}}$

Theorem. *The annihilation and creation operators*

7 $\boxed{\begin{array}{c}\hat a\overset{\text{def}}{=}\frac{1}{\sqrt{2\hbar}}(\hat Q+i\hat P)\\ \hat a^\dagger\overset{\text{def}}{=}\frac{1}{\sqrt{2\hbar}}(\hat Q-i\hat P)\end{array}}$ *satisfy* $\boxed{\begin{array}{c}\hat a|\Psi_n\rangle=\sqrt{n}\,|\Psi_{n-1}\rangle\\ \hat a^\dagger|\Psi_n\rangle=\sqrt{n+1}\,|\Psi_{n+1}\rangle\\ \hat a^\dagger\hat a|\Psi_n\rangle=n\,|\Psi_n\rangle\\ [\hat a,\hat a^\dagger]=1\end{array}}$

$\mathfrak{H}$ ■ FOCK SPACE

1 $$\mathcal{F}\overset{\text{def}}{=}\left\{\sum_{n=0}^{\infty}\alpha_n|n\rangle\;\middle|\;\alpha_n\in\mathbb{C},\ \sum_{n=0}^{\infty}|\alpha_n|^2<\infty\right\}$$
where $\{|0\rangle,|1\rangle,|2\rangle,\ldots\}$ is a fixed orthonormal system.

Theorem. *The annihilation and creation operators*

2 $\boxed{\begin{array}{c}\hat a|n\rangle\overset{\text{def}}{=}\sqrt{n}\,|n-1\rangle\\ \hat a^\dagger|n\rangle\overset{\text{def}}{=}\sqrt{n+1}\,|n+1\rangle\end{array}}$ satisfy $\boxed{\begin{array}{c}\hat a^\dagger\hat a|n\rangle=n\,|n\rangle\\ [\hat a,\hat a^\dagger]=1\end{array}}$

Theorem. *The position and momentum operators*

3 $\boxed{\begin{array}{c}\hat Q\overset{\text{def}}{=}\sqrt{\frac{\hbar}{2}}(\hat a+\hat a^\dagger)\\ \hat P\overset{\text{def}}{=}-i\sqrt{\frac{\hbar}{2}}(\hat a-\hat a^\dagger)\end{array}}$ *satisfy* $\boxed{\begin{array}{c}[\hat Q,\hat P]=i\hbar\\ \hat a=\frac{1}{\sqrt{2\hbar}}(\hat Q+i\hat P)\\ \hat a^\dagger=\frac{1}{\sqrt{2\hbar}}(\hat Q-i\hat P)\end{array}}$

Definition. *Displacement operators*

!4 $$\boxed{\begin{aligned}D(\lambda)&\overset{\text{def}}{=}e^{\lambda\hat a^\dagger-\bar\lambda\hat a}\\ &=e^{-\frac{|\lambda|^2}{2}}e^{\lambda\hat a^\dagger}e^{-\bar\lambda\hat a}\\ &=e^{\frac{i}{2\hbar}pq}e^{-\frac{i}{\hbar}q\hat P}e^{\frac{i}{\hbar}p\hat Q}\\ &=e^{\frac{i}{\hbar}(p\hat Q-q\hat P)}=D(q,p)\end{aligned}} \qquad \begin{array}{c}\lambda\in\mathbb{C}\\ \lambda=\frac{1}{\sqrt{2\hbar}}(q+p\,i)\\ \frac{1}{\pi}d^2\lambda=\frac{1}{2\pi\hbar}dq\,dp\\ (q,p)\in\mathbb{R}^2\end{array}$$

5 *Coherent states* $$\boxed{\begin{aligned}|\lambda\rangle&\overset{\text{def}}{=}D(\lambda)|0\rangle\\ &\equiv D(q,p)|0\rangle\overset{\text{def}}{=}|q,p\rangle\end{aligned}} \qquad \begin{array}{c}\lambda\in\mathbb{C}\\ (q,p)\in\mathbb{R}^2\end{array}$$

Theorem.

6 $\boxed{|\lambda\rangle=e^{-\frac12|\lambda|^2}\sum_{n=1}^{\infty}\frac{\lambda^n}{\sqrt{n!}}\,|n\rangle}$ $\boxed{\hat a|\lambda\rangle=\lambda|\lambda\rangle}$

7 $\boxed{\frac{1}{\pi}\int_{\mathbb{C}}|\lambda\rangle\langle\lambda|\,d^2\lambda=\mathbb{I}}$ $\boxed{\frac{1}{2\pi\hbar}\int_{\mathbb{R}^2}|q,p\rangle\langle q,p|\,dqdp=\mathbb{I}}$

$\mathfrak{I}$ ■ LINEAR OPERATORS

Definition. When the series/integrals are convergent:

1 $\boxed{[A,B]\overset{\text{def}}{=}AB-BA}$

2 $\boxed{\operatorname{tr}A\overset{\text{def}}{=}\sum_{n=0}^{\infty}\langle n|A|n\rangle}$

3 $\boxed{\langle A,B\rangle\overset{\text{def}}{=}\operatorname{tr}(A^\dagger B)}$

4 $\boxed{e^A\overset{\text{def}}{=}\sum_{n=0}^{\infty}\frac{1}{n!}A^n=\mathbb{I}+\frac{1}{1!}A+\frac{1}{2!}A^2+\frac{1}{3!}A^3+\cdots}$

Theorem.

5 $\boxed{[A,BC]=B[A,C]+[A,B]C}$

6 $\boxed{\operatorname{tr}A=\int_{-\infty}^{\infty}\langle q|A|q\rangle\,dq}$

7 $\boxed{\operatorname{tr}(AB)=\operatorname{tr}(BA)}$

8 $\boxed{e^{a\frac{d}{dq}}\Psi(q)=\Psi(q)+\frac{a}{1!}\Psi'(q)+\frac{a^2}{2!}\Psi''(q)+\cdots=\Psi(q+a)}$

9 $\boxed{\left.\begin{array}{c}[A,[A,B]]=0\\ [B,[A,B]]=0\end{array}\right\}\Rightarrow e^{A+B}=e^{-\frac12[A,B]}e^A\,e^B}$

10 $\boxed{e^{\hat A}\hat B e^{-\hat A}=\hat B+\frac{1}{1!}[\hat A,\hat B]+\frac{1}{2!}[\hat A,[\hat A,\hat B]]+\ldots}$

SINGLE-MODE GAUSSIAN STATES

A POSITION AND MOMENTUM OPERATORS

■ **Hilbert space**

1 $\boxed{L^2(\mathbb{R}) \equiv \mathcal{F} = \overline{\mathrm{span}\{|0\rangle, |1\rangle, |2\rangle, ...\}}}$ $\quad \langle n|m\rangle = \delta_{nm}$, $\quad \sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbb{I}$

■ **Theorem.** *The annihilation and creation operators*

2 $\boxed{\begin{array}{c} \hat{a}|n\rangle \stackrel{\text{def}}{=} \sqrt{n}\,|n-1\rangle \\ \hat{a}^\dagger|n\rangle \stackrel{\text{def}}{=} \sqrt{n+1}\,|n+1\rangle \end{array}}$ satisfy $\boxed{\begin{array}{c} \hat{a}^\dagger\hat{a}|n\rangle = n\,|n\rangle \\ [\hat{a}, \hat{a}^\dagger] = 1 \end{array}}$

■ **Theorem.** *The position and momentum operators*

3 $\boxed{\begin{array}{l} \hat{Q} \stackrel{\text{def}}{=} \sqrt{\frac{\hbar}{2}}(\hat{a}+\hat{a}^\dagger) \\ \hat{P} \stackrel{\text{def}}{=} -\mathrm{i}\sqrt{\frac{\hbar}{2}}(\hat{a}-\hat{a}^\dagger) \end{array}}$ *satisfy* $\boxed{\begin{array}{l} [\hat{Q}, \hat{P}] = \mathrm{i}\hbar \\ \hat{a} = \frac{1}{\sqrt{2\hbar}}(\hat{Q}+\mathrm{i}\hat{P}) \\ \hat{a}^\dagger = \frac{1}{\sqrt{2\hbar}}(\hat{Q}-\mathrm{i}\hat{P}) \end{array}}$

■ **Commutation relations**

4 $\boxed{[\hat{Q}, \hat{P}] = \mathrm{i}\hbar}$

5 $\hat{R} \equiv \begin{pmatrix} \hat{R}_1 \\ \hat{R}_2 \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} \hat{Q} \\ \hat{P} \end{pmatrix}$ $\quad \boxed{[\hat{R}_k, \hat{R}_\ell] = \mathrm{i}\,\Omega_{k\ell}}$ $\quad \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

6 **Theorem.** $\boxed{\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})}$

Theorem. By extending $\hat{Q}$ and $\hat{P}$ up to $\mathcal{S}'(\mathbb{R})$, formally :

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$\boxed{\begin{array}{c} \hat{Q}\psi \stackrel{\text{def}}{=} q\psi \\ |a\rangle \stackrel{\text{def}}{=} \delta_a \end{array}} \Rightarrow \boxed{\begin{array}{l} \hat{Q}|a\rangle = a\,|a\rangle \\ \hat{Q} = \int_{-\infty}^{\infty} q|q\rangle\langle q|\,dq \\ \mathrm{e}^{\frac{\mathrm{i}}{\hbar}a\hat{Q}}\psi(q) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}aq}\psi(q) \\ \langle a|Q|b\rangle = a\,\delta(b-a) \end{array}}$

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$\boxed{\begin{array}{c} \hat{P}\psi \stackrel{\text{def}}{=} -\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}q}\psi \\ |\tilde{a}\rangle \stackrel{\text{def}}{=} \hat{F}^\dagger|a\rangle \end{array}} \Rightarrow \boxed{\begin{array}{l} \hat{P} = \hat{F}^\dagger\hat{Q}\hat{F} \\ \langle q|\tilde{a}\rangle = \frac{1}{\sqrt{2\pi\hbar}}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}aq} \\ \hat{P}|\tilde{a}\rangle = a\,|\tilde{a}\rangle \\ \mathbb{I} = \int_{-\infty}^{\infty} |\tilde{p}\rangle\langle\tilde{p}|\,dp \\ \hat{P} = \int_{-\infty}^{\infty} p|\tilde{p}\rangle\langle\tilde{p}|\,dp \\ \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}a\hat{P}}\psi(q) = \psi(q-a) \end{array}}$

B DISPLACEMENT OPERATORS

■ *Displacement operators*

1 $\boxed{\begin{array}{rl} D(\lambda) & \stackrel{\text{def}}{=} \mathrm{e}^{\lambda\hat{a}^\dagger - \bar{\lambda}\hat{a}} \\ & = \mathrm{e}^{-\frac{|\lambda|^2}{2}}\,\mathrm{e}^{\lambda\hat{a}^\dagger}\,\mathrm{e}^{-\bar{\lambda}\hat{a}} \\ & = \mathrm{e}^{\frac{\mathrm{i}}{2\hbar}pq}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}q\hat{P}}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p\hat{Q}} \\ & = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(p\hat{Q}-q\hat{P})} \\ & = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\Lambda^T\Omega\hat{R}} = D(\Lambda) \end{array}}$ $\quad \lambda = \frac{1}{\sqrt{2\hbar}}(q+p\,\mathrm{i})$, $\quad \frac{1}{\pi}d^2\lambda = \frac{1}{2\pi\hbar}dq\,dp$, $\quad \Lambda \equiv \begin{pmatrix} q \\ p \end{pmatrix}, X \equiv \begin{pmatrix} x \\ y \end{pmatrix}$

■ **Theorem.**

2 $\boxed{D(q,p)\psi(a) = \mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}pq}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pa}\,\psi(a-q)}$

3 $\boxed{\langle a|D(q,p)|b\rangle = \mathrm{e}^{\frac{\mathrm{i}}{2\hbar}p(a+b)}\delta(a-b-q)}$

4 $\boxed{D(\Lambda)D(X) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\Lambda^T\Omega X}D(\Lambda+X)}$

5 $\boxed{D^\dagger(\Lambda) = D(-\Lambda)}$ $\quad \Lambda \equiv \begin{pmatrix} q \\ p \end{pmatrix}, X \equiv \begin{pmatrix} x \\ y \end{pmatrix}$

6 $\boxed{\mathrm{tr}\,D(\Lambda) = 2\pi\hbar\,\delta(\Lambda)}$

7 $\boxed{\langle D(q,p), D(x,y)\rangle = 2\pi\hbar\,\delta(q-x)\,\delta(p-y)}$

8 $\boxed{\hat{F}^\dagger D(q,p)\hat{F} = D(-p,q)}$

C PARITY OPERATORS

■ *Parity operators*

1 $\boxed{\Pi(\Lambda) \stackrel{\text{def}}{=} D(\Lambda)\Pi D^\dagger(\Lambda)}$ where $\boxed{\Pi \stackrel{\text{def}}{=} (-1)^{\hat{a}^\dagger\hat{a}}}$

■ **Theorem.**

2 $\boxed{\Pi\psi(q) = \psi(-q)}$ $\quad \boxed{\hat{F}^2 = \Pi = (\hat{F}^\dagger)^2}$

3 $\boxed{\Pi(q,p)\psi(a) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}2p(q-a)}\,\psi(2q-a)}$

4 $\boxed{\langle a|\Pi(q,p)|b\rangle = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}p(a-b)}\delta(2q-a-b)}$

5 $\boxed{\langle 2\Pi(q,p), 2\Pi(x,y)\rangle = 2\pi\hbar\,\delta(q-x)\,\delta(p-y)}$

6 $\boxed{\hat{F}^\dagger\Pi\hat{F} = \Pi}$ $\quad \boxed{\hat{F}^\dagger\Pi(q,p)\hat{F} = \Pi(-p,q)}$

D — WEYL TRANSFORM, CHARACTERISTIC AND WIGNER FUNCTION

■ **Definition.**

1 *Density operator* = an operator $\hat\varrho$ such that $\begin{cases}\hat\varrho\geq 0\\ \operatorname{tr}\hat\varrho=1\end{cases}$

■ **Definition.**

2 *Characteristic function of an operator* $\hat O$: $\chi[\hat O](\Lambda)\overset{\text{def}}{=}\operatorname{tr}(\hat O\, D(\Lambda))$

3 *Weyl transform of an operator* $\hat O$: $O^{\mathrm{w}}(\Lambda)\overset{\text{def}}{=}2\operatorname{tr}(\hat O\,\Pi(\Lambda))$ $\quad\Lambda\equiv\begin{pmatrix}q\\p\end{pmatrix}$

4 *Wigner function of a state* $\hat\varrho$: $W[\hat\varrho](\Lambda)\overset{\text{def}}{=}\frac{1}{2\pi\hbar}\varrho^{\mathrm{w}}(\Lambda)=\frac{1}{\pi\hbar}\operatorname{tr}(\hat\varrho\,\Pi(\Lambda))$

■ **Theorem.**

5 $\hat O=\frac{1}{2\pi\hbar}\int_{\mathbb{R}^2}\chi[\hat O](\Lambda)\;D(-\Lambda)\,d^2\Lambda$ $\quad d^2\Lambda=dq\,dp$, $\; d^2X=dx\,dy$

6 $\hat\varrho=2\int_{\mathbb{R}^2}W[\hat\varrho](\Lambda)\,\Pi(\Lambda)\,d^2\Lambda$ $\quad W[\hat\varrho]=\frac{1}{2\pi\hbar}\varrho^{\mathrm{w}}$

7 $\operatorname{tr}(\hat O_1\hat O_2)=\frac{1}{2\pi\hbar}\int_{\mathbb{R}^2}\chi[\hat O_1](\Lambda)\;\chi[\hat O_2](-\Lambda)\;d^2\Lambda$

8 $\operatorname{tr}(\hat O_1\hat O_2)=\frac{1}{2\pi\hbar}\int_{\mathbb{R}^2}O_1^{\mathrm{w}}(\Lambda)\;O_2^{\mathrm{w}}(\Lambda)\;d^2\Lambda$

9 $\operatorname{tr}(\hat\varrho\hat O)=\int_{\mathbb{R}^2}W[\hat\varrho](\Lambda)\;O^{\mathrm{w}}(\Lambda)\;d^2\Lambda$

10 $O^{\mathrm{w}}(q,p)=\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}px}\langle q+\tfrac{x}{2}|\hat O|q-\tfrac{x}{2}\rangle\,dx$

11 $W[\hat\varrho](q,p)=\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}px}\langle q+\tfrac{x}{2}|\hat\varrho|q-\tfrac{x}{2}\rangle\,dx$

12 $W[\hat F\hat\varrho\hat F^{\dagger}](q,p)=W[\hat\varrho](-p,q)$

13 $\Pi(\Lambda)=\frac{1}{4\pi\hbar}\int_{\mathbb{R}^2}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\Lambda^T\Omega X}\,D(X)\,d^2X$ $\quad X\equiv\begin{pmatrix}x\\y\end{pmatrix}$

14 $W[\hat\varrho](\Lambda)\equiv W[\hat\varrho](q,p)=\frac{1}{(2\pi\hbar)^2}\int_{\mathbb{R}^2}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\Lambda^T\Omega X}\;\chi[\hat\varrho](X)\,d^2X$

15 $\chi[\hat\varrho](X)\equiv\chi[\hat\varrho](x,y)=\int_{\mathbb{R}^2}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\Lambda^T\Omega X}\;W[\hat\varrho](\Lambda)\,d^2\Lambda$

■ **Theorem.** If $\hat\varrho=|\psi\rangle\langle\psi|$ is a pure state, then:

16 $W[\hat\varrho](q,p)=\frac{1}{\pi\hbar}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}2px}\;\overline{\psi(q-x)}\;\psi(q+x)\,dx$

17 $\quad=\frac{1}{\pi\hbar}\int_{-\infty}^{\infty}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}2qy}\;\overline{F[\psi](p-y)}\;F[\psi](p+y)\,dy$

18 $\int_{-\infty}^{\infty}W[\hat\varrho](\Lambda)\,dp=|\psi(q)|^2$ $\quad \int_{-\infty}^{\infty}W[\hat\varrho](\Lambda)\,dq=|\hat F[\psi](p)|^2$

SINGLE-MODE GAUSSIAN STATES

■ **Definition.**

1 For two linear operators $\hat A$ and $\hat B$: $\;[\hat A,\hat B]\overset{\text{def}}{=}\hat A\hat B-\hat B\hat A$, $\;\{\hat A,\hat B\}\overset{\text{def}}{=}\hat A\hat B+\hat B\hat A$

2 *Mean value of an observable* $\hat O$ in state $\hat\varrho$: $\langle\hat O\rangle\equiv\langle\hat O\rangle_{\varrho}\overset{\text{def}}{=}\operatorname{tr}(\varrho\,\hat O)$

3 *First-moments vector* in state $\hat\varrho$: $\langle\hat R\rangle\overset{\text{def}}{=}\operatorname{tr}(\varrho\,\hat R)\equiv\begin{pmatrix}\langle\hat R_1\rangle\\ \langle\hat R_2\rangle\end{pmatrix}\equiv\begin{pmatrix}\langle\hat Q\rangle\\ \langle\hat P\rangle\end{pmatrix}$

4 *Covariance matrix* in state $\hat\varrho$: $\begin{pmatrix}\langle\hat Q^2\rangle-\langle\hat Q\rangle^2 & \frac12\{\hat Q-\langle\hat Q\rangle,\hat P-\langle\hat P\rangle\}\\ \frac12\{\hat Q-\langle\hat Q\rangle,\hat P-\langle\hat P\rangle\} & \langle\hat P^2\rangle-\langle\hat P\rangle^2\end{pmatrix}$

■ **Theorem.**

5 $\langle\hat O\rangle_{\varrho}=\operatorname{tr}(\hat\varrho\hat O)=\int_{\mathbb{R}^2}W[\hat\varrho](\Lambda)\;O^{\mathrm{w}}(\Lambda)\;d^2\Lambda$

6 $(\hat Q^n)^w(\Lambda)=q^n$, $\;(\hat P^n)^w(\Lambda)=p^n$ $\quad\langle\hat Q^n\rangle_{\varrho}=\int_{\mathbb{R}^2}q^nW[\hat\varrho](\Lambda)\,d^2\Lambda$, $\;\langle\hat P^n\rangle_{\varrho}=\int_{\mathbb{R}^2}p^nW[\hat\varrho](\Lambda)\,d^2\Lambda$

■ **<u>Theorem.</u>**

7 *If* $\sigma=\begin{pmatrix}\sigma_{11}&\sigma_{12}\\ \sigma_{12}&\sigma_{22}\end{pmatrix}$ *satisfy* $\sigma_{ij}\in\mathbb{R}$, $\sigma>0$ *and* $\mathbf{r}=\begin{pmatrix}r_1\\r_2\end{pmatrix}\in\mathbb{R}^2$,

then the Gaussian function

8 $W[\varrho](X)=\frac{1}{\pi\sqrt{\det\sigma}}\mathrm{e}^{-\frac12(X-\mathbf{r})^T\sigma^{-1}(X-\mathbf{r})}$

9 *is the Wigner function of a quantum state* $\hat\varrho$ $\;\Leftrightarrow\;$ $\sigma+\frac{\mathrm{i}}{2}\Omega\geq 0$.

10 *In this case* $\begin{cases}\mathbf{r}=\textit{first-moments vector in state }\hat\varrho\\ \sigma=\textit{covariance matrix in state }\hat\varrho.\end{cases}$

■ **<u>Definition</u>** The quantum states described in the previous theorem are called <u>*Gaussian states*</u>. The state ϱ from theorem, well determined by σ and $\mathbf{r}$, is sometimes denoted by $\varrho(\sigma,\mathbf{r})$.

■ **<u>Theorem.</u>**

The characteristic function of $\varrho(\sigma,\mathbf{r})$ *is the*

11 *Gaussian* function $\chi[\varrho](\Lambda)=\mathrm{e}^{-\frac12\Lambda^T\Omega\sigma\Omega^T\Lambda-\mathrm{i}\Lambda^T\Omega\mathbf{r}}$

■ **Definition.**

12 *Purity* of a state ϱ is $\mu(\varrho)\overset{\text{def}}{=}\operatorname{tr}(\varrho^2)$

■ **Theorem.**

13 The purity of the Gaussian state $\varrho(\sigma,\mathbf{r})$ is $\mu(\varrho(\sigma,\mathbf{r}))=\frac{1}{2\sqrt{\det(\sigma)}}$

■ **Theorem.**

14 *Coherent state* $|q,p\rangle=D(q,p)|0\rangle$ is a Gaussian state with $\sigma=\begin{pmatrix}\frac12&0\\0&\frac12\end{pmatrix}$, $\mathbf{r}=\begin{pmatrix}q\\p\end{pmatrix}$.

■ **Theorem.** *Thermal state* (where $\beta=\hbar\omega/(k_BT)$)

15 $\nu_{th}(N)=\frac{\mathrm{e}^{-\beta\hat a^\dagger\hat a}}{\operatorname{tr}\left(\mathrm{e}^{-\beta\hat a^\dagger\hat a}\right)}=\frac{N^{\hat a^\dagger\hat a}}{(1+N)^{\hat a^\dagger\hat a+1}}=\frac{1}{1+N}\sum_{m=0}^{\infty}\left(\frac{N}{1+N}\right)^m|m\rangle\langle m|$ is a Gaussian state with $\sigma=\begin{pmatrix}\frac{1+2N}{2}&0\\0&\frac{1+2N}{2}\end{pmatrix}$

GAUSSIAN TRANSFORMS

■ **Theorem.**
For $\alpha = \frac{1}{\sqrt{2}}(a+b\mathrm{i}) \in \mathbb{C}$ *and an operator* O, *we have:*

1 $\boxed{\hat{D}^\dagger(\alpha)\begin{pmatrix}\hat{a}\\ \hat{a}^\dagger\end{pmatrix}\hat{D}(\alpha) = \begin{pmatrix}\hat{a}\\ \hat{a}^\dagger\end{pmatrix} + \begin{pmatrix}\alpha\\ \bar{\alpha}\end{pmatrix}}$

2 $\boxed{\hat{D}^\dagger(\alpha)\begin{pmatrix}\hat{Q}\\ \hat{P}\end{pmatrix}\hat{D}(\alpha) = \begin{pmatrix}\hat{Q}\\ \hat{P}\end{pmatrix} + \begin{pmatrix}a\\ b\end{pmatrix}}$

that is $\boxed{\hat{D}^\dagger(\alpha)\hat{\mathbf{R}}\hat{D}(\alpha) = \hat{\mathbf{R}} + \mathbf{d}}$ $\quad \mathbf{d} = \begin{pmatrix}a\\ b\end{pmatrix}$

3 $\boxed{\hat{D}^\dagger(\alpha)D(\Lambda)\hat{D}(\alpha) = \mathrm{e}^{-\mathrm{i}\Lambda^T\Omega\mathbf{d}}\, D(\Lambda)}$ $\quad \Lambda \equiv \begin{pmatrix}q\\ p\end{pmatrix}$

4 $\boxed{\chi[\hat{D}(\alpha)\hat{O}\hat{D}^\dagger(\alpha)](\Lambda) = \mathrm{e}^{-\mathrm{i}\Lambda^T\Omega\mathbf{d}}\,\chi[\hat{O}](\Lambda)}$

5 $\boxed{W[\hat{D}(\alpha)\hat{O}\hat{D}^\dagger(\alpha)](X) = ...}$ $\quad X = \begin{pmatrix}x\\ y\end{pmatrix}$

■ **Theorem.**
For a state ϱ *and a transform* $\boxed{\hat{U} = \mathrm{e}^{-\mathrm{i}\hat{H}}}$ with

$\boxed{\hat{H} = \mathcal{A}\hat{a}^\dagger\hat{a} + \mathcal{B}(\hat{a}^\dagger)^2 + \bar{\mathcal{A}}\hat{a}\hat{a}^\dagger + \bar{\mathcal{B}}\hat{a}^2}$ $\quad \mathcal{A}, \mathcal{B} \in \mathbb{C}$

we have:

6 $\boxed{U^\dagger\begin{pmatrix}\hat{a}\\ \hat{a}^\dagger\end{pmatrix}\hat{U} = \mathrm{e}^{\mathrm{i}\begin{pmatrix}\mathcal{A} & \mathcal{B}\\ -\bar{\mathcal{B}} & -\bar{\mathcal{A}}\end{pmatrix}}\begin{pmatrix}\hat{a}\\ \hat{a}^\dagger\end{pmatrix} + \begin{pmatrix}\lambda\\ \bar{\lambda}\end{pmatrix}}$

7 $\boxed{U^\dagger\begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix}\hat{U} = S\begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix} + \begin{pmatrix}a\\ b\end{pmatrix}}$ where $S = ...$

8 $\boxed{U^\dagger D(\Lambda)\hat{U} = ...}$

9 $\boxed{\chi[\hat{U}\hat{O}\hat{U}^\dagger] = ...}$

10′ $\boxed{W[\hat{U}\hat{O}\hat{U}^\dagger] = ...}$

■ **Theorem.** *For a Gaussian state* $\varrho \equiv \varrho(\sigma, d)$ *with*

$\boxed{W[\varrho](X) = \frac{1}{\pi\sqrt{\det\sigma}}\mathrm{e}^{-\frac{1}{2}(X-\langle\hat{R}\rangle)^T\sigma^{-1}(X-\langle\hat{R}\rangle)}}$

and a transform $\boxed{\hat{U} = \mathrm{e}^{-\mathrm{i}\hat{H}}}$ *with*

$\boxed{\hat{H} = \mathrm{i}(\lambda\hat{a}^\dagger + \mathcal{A}\hat{a}^\dagger\hat{a} + \mathcal{B}(\hat{a}^\dagger)^2) + \mathrm{h.c.}}$ $\quad \lambda, \mathcal{A}, \mathcal{B} \in \mathbb{C}$

we have:

$\boxed{W[\hat{U}\varrho\hat{U}^\dagger](X) = ...}$

that is, $\hat{U}\varrho\hat{U}^\dagger$ is also a Gaussian state, namely

$\boxed{\hat{U}\varrho(\sigma, d)\hat{U}^\dagger = \varrho(S\sigma S^T + d, Sd)}$

■ **Definition.**
Symplectic group $\boxed{\mathrm{Sp}(2,\mathbb{R}) \overset{\mathrm{def}}{=} \{\ F \in \mathrm{GL}(2,\mathbb{R})\ |\ \ F\Omega F^T = \Omega\ \}}$

■ **Theorem.** Under the transformation $R \mapsto R' = FR$,

- $\boxed{\dot{R}_k = \Omega_{k\ell}\frac{\partial H}{\partial R_\ell}}$ ramain invariant $\iff$ $F \in \mathrm{Sp}(2,\mathbb{R})$.
- $\boxed{[\hat{R}_k, \hat{R}_\ell] = \mathrm{i}\,\Omega_{k\ell}}$ ramain invariant $\iff$ $F \in \mathrm{Sp}(2,\mathbb{R})$.

■ **Theorem** (Williamson).
σ real, $\sigma^T = \sigma$, $\sigma > 0$ $\Rightarrow$ there exist $F \in \mathrm{Sp}(2,\mathbb{R})$, $d \geq \frac{1}{2}$ such that $\boxed{\sigma = F\begin{pmatrix}d & 0\\ 0 & d\end{pmatrix}F^T}$

■ **Theorem** *There exists a one-to-one correspondence*

$\boxed{\hat{U}_s \longleftrightarrow S}$ *such that* $\boxed{U_s^\dagger\begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix}\hat{U}_s = S^{-1}\begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix}}$

between the transforms of the form

$\boxed{\hat{U}_s = \mathrm{e}^{-\frac{\mathrm{i}}{2}\hat{H}}}$ *with* $\boxed{\hat{H} = \alpha\hat{a}^\dagger\hat{a} + \beta(\hat{a}^\dagger)^2 + \bar{\beta}\hat{a}^2 + \alpha\hat{a}\hat{a}^\dagger}$ $\quad \alpha \in \mathbb{R}$, $\beta \in \mathbb{C}$

and the elements S *of Sp(2,*$\mathbb{R}$*).*

■ **Theorem** (*Phase shift*).

$\boxed{U(\theta) = \mathrm{e}^{-\mathrm{i}\theta\hat{a}^\dagger\hat{a}}} \longleftrightarrow$ symplectic matrix $\boxed{\mathcal{R}_\theta = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}}$

Proof.
$$U^\dagger(\theta)\hat{a}U(\theta) = \mathrm{e}^{-\mathrm{i}\theta}\hat{a} \Rightarrow U^\dagger(\theta)\begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix}U(\theta) = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix}$$

■ **Theorem** $\left(\begin{matrix}\textit{Single-mode}\\ \text{squeezing}\end{matrix}\right)$

$\xi = r\mathrm{e}^{\mathrm{i}\psi}$ $\boxed{S(\xi) = \mathrm{e}^{\frac{1}{2}[\xi(\hat{a}^\dagger)^2 - \bar{\xi}\hat{a}^2}} \leftrightarrow$ symplectic matrix $\boxed{\begin{matrix}\Sigma_\xi = \cosh r\ \mathbb{I}_2 + \mathcal{R}_\xi,\\ \mathcal{R}_\xi = \sinh r\begin{pmatrix}\cos\psi & \sin\psi\\ \sin\psi & -\cos\psi\end{pmatrix}\end{matrix}}$

Proof.
$$\mathcal{S}^\dagger(\xi)\begin{pmatrix}\hat{a}\\ \hat{b}\end{pmatrix}\mathcal{S}(\xi) = \begin{pmatrix}\cosh r & \mathrm{e}^{\mathrm{i}\psi}\sinh r\\ \mathrm{e}^{-\mathrm{i}\psi}\sinh r & \cosh r\end{pmatrix}\begin{pmatrix}\hat{a}\\ \hat{b}\end{pmatrix}.$$

PROOFS

Mathematical foundations

$\boxed{\mathfrak{B}3}$

$$\begin{aligned}(\hat F[\varphi])'(p) &= \tfrac{1}{\sqrt{2\pi\hbar}}\tfrac{\mathrm d}{\mathrm dp}\int_{-\infty}^{\infty}\mathrm e^{-\frac{\mathrm i}{\hbar}pq}\varphi(q)\,dq\\ &= \tfrac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\left(-\tfrac{\mathrm i}{\hbar}p\right)\mathrm e^{-\frac{\mathrm i}{\hbar}pq}\varphi(q)\,dq\\ &= \hat F[-\tfrac{\mathrm i}{\hbar}q\varphi(q)](p)\\ \hat F[\varphi'](p) &= \tfrac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\mathrm e^{-\frac{\mathrm i}{\hbar}pq}\varphi'(q)\,dq\\ &= -\tfrac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\left(\mathrm e^{-\frac{\mathrm i}{\hbar}pq}\right)'\varphi(q)\,dq\\ &= \tfrac{\mathrm i}{\hbar}p\,\hat F[\varphi](p).\end{aligned}$$

$\boxed{\mathfrak{B}6}$

$$\begin{aligned}\langle(\hat F[f])',\varphi\rangle &= -\langle\hat F[f],\varphi'\rangle = -\langle f,\hat F[\varphi']\rangle\\ &= -\langle f,\tfrac{\mathrm i}{\hbar}p\,\hat F[\varphi]\rangle = \langle -\tfrac{\mathrm i}{\hbar}p\,f,\hat F[\varphi]\rangle = \langle\hat F[-\tfrac{\mathrm i}{\hbar}q\,f],\varphi\rangle\\ \langle\hat F[f'],\varphi\rangle &= \langle f',\hat F[\varphi]\rangle = -\langle f,(\hat F[\varphi])'\rangle\\ &= -\langle f,\hat F[-\tfrac{\mathrm i}{\hbar}q\varphi(q)]\rangle = \langle\hat F[f],\tfrac{\mathrm i}{\hbar}q\varphi(q)\rangle = \langle\tfrac{\mathrm i}{\hbar}p\,\hat F[f],\varphi\rangle.\end{aligned}$$

$\boxed{\mathfrak{C}2}$

$$\begin{aligned}&\int_{\mathbb R^2}\mathrm e^{-\frac12(q-q_0\ \ p-p_0)\,\sigma\begin{pmatrix}q-q_0\\ p-p_0\end{pmatrix}}dqdp\\ &= \int_{\mathbb R^2}\mathrm e^{-\frac12[a(q-q_0)^2+2b(q-q_0)(p-p_0)+c(p-p_0)^2]}\,dqdp\\ &\qquad\Downarrow\ q-q_0\mapsto q,\quad p-p_0\mapsto p\\ &= \int_{\mathbb R^2}\mathrm e^{-\frac12[aq^2+2bqp+cp^2]}\,dqdp\\ &= \int_{\mathbb R^2}\mathrm e^{-\frac a2\left(q+\frac ba p\right)^2}\mathrm e^{-\frac12\left(c-\frac{b^2}{a}\right)p^2}\,dqdp\\ &\qquad\Downarrow\ q+\tfrac ba p\mapsto q,\quad p\mapsto p\\ &= \int_{-\infty}^{\infty}\mathrm e^{-\frac a2 q^2}\,dq\int_{-\infty}^{\infty}\mathrm e^{-\frac{\det\sigma}{2a}p^2}\,dp\\ &\qquad\Downarrow\ \mathfrak{D}2\\ &= \sqrt{\tfrac{2\pi}{a}}\sqrt{\tfrac{2a\pi}{\det\sigma}} = \tfrac{2\pi}{\sqrt{\det\sigma}}.\end{aligned}$$

$\boxed{\mathfrak{C}3}$

$$\begin{aligned}&\int_{\mathbb R^2}q\ \mathrm e^{-\frac12(q-q_0\ \ p-p_0)\,\sigma\begin{pmatrix}q-q_0\\ p-p_0\end{pmatrix}}dqdp\\ &= \int_{\mathbb R^2}q\ \mathrm e^{-\frac12[a(q-q_0)^2+2b(q-q_0)(p-p_0)+c(p-p_0)^2]}\,dqdp\\ &\qquad\Downarrow\ q-q_0\mapsto q,\quad p-p_0\mapsto p\\ &= \int_{\mathbb R^2}(q+q_0)\,\mathrm e^{-\frac12[aq^2+2bqp+cp^2]}\,dqdp\\ &= \int_{\mathbb R^2}(q+q_0)\,\mathrm e^{-\frac a2\left(q+\frac ba p\right)^2}\mathrm e^{-\frac12\left(c-\frac{b^2}{a}\right)p^2}\,dqdp\\ &\qquad\Downarrow\ q+\tfrac ba p\mapsto q,\quad p\mapsto p\\ &= \int_{\mathbb R^2}(q-\tfrac ba p+q_0)\,\mathrm e^{-\frac a2 q^2}\ \mathrm e^{-\frac{\det\sigma}{2a}p^2}\,dqdp\\ &= \int_{-\infty}^{\infty}q\,\mathrm e^{-\frac a2 q^2}\,dq\int_{-\infty}^{\infty}\mathrm e^{-\frac{\det\sigma}{2a}p^2}\,dp\\ &\qquad -\tfrac ba\int_{-\infty}^{\infty}\mathrm e^{-\frac a2 q^2}\,dq\int_{-\infty}^{\infty}p\,\mathrm e^{-\frac{\det\sigma}{2a}p^2}\,dp\\ &\qquad +q_0\int_{-\infty}^{\infty}\mathrm e^{-\frac a2 q^2}\,dq\int_{-\infty}^{\infty}\mathrm e^{-\frac{\det\sigma}{2a}p^2}\,dp\\ &\qquad\Downarrow\ \mathfrak{D}2\\ &= q_0\sqrt{\tfrac{2\pi}{a}}\sqrt{\tfrac{2a\pi}{\det\sigma}} = q_0\tfrac{2\pi}{\sqrt{\det\sigma}}.\end{aligned}$$

$\boxed{\mathfrak{H}4}$

$$\left.\begin{aligned}\hat Q &\overset{\mathrm{def}}{=}\sqrt{\tfrac{\hbar}{2}}(\hat a+\hat a^\dagger)\\ \hat P &\overset{\mathrm{def}}{=}-\mathrm i\sqrt{\tfrac{\hbar}{2}}(\hat a-\hat a^\dagger)\\ \lambda &= \tfrac{1}{\sqrt{2\hbar}}(q+p\,\mathrm i)\end{aligned}\right\}\Rightarrow\quad \begin{aligned}D(\lambda)&\overset{\mathrm{def}}{=}\mathrm e^{\lambda\hat a^\dagger-\bar\lambda\hat a}\\ &=\mathrm e^{\frac{\mathrm i}{\hbar}(p\hat Q-q\hat P)}\overset{\mathrm{def}}{=}D(q,p).\end{aligned}$$

By using $\boxed{\tilde{\mathfrak{J}}9}$, we get:

$$\begin{aligned}\mathrm e^{\lambda\hat a^\dagger-\bar\lambda\hat a} &= \mathrm e^{-\frac12[\lambda\hat a^\dagger,-\bar\lambda\hat a]}\mathrm e^{\lambda\hat a^\dagger}\mathrm e^{-\bar\lambda\hat a}\\ &= \mathrm e^{\frac{|\lambda|^2}{2}[\hat a^\dagger,\hat a]}\mathrm e^{\lambda\hat a^\dagger}\mathrm e^{-\bar\lambda\hat a}\\ &= \mathrm e^{-\frac{|\lambda|^2}{2}}\mathrm e^{\lambda\hat a^\dagger}\mathrm e^{-\bar\lambda\hat a}\\ \mathrm e^{\frac{\mathrm i}{\hbar}(p\hat Q-q\hat P)} &= \mathrm e^{-\frac12[-\frac{\mathrm i}{\hbar}q\hat P,\frac{\mathrm i}{\hbar}p\hat Q]}\mathrm e^{-\frac{\mathrm i}{\hbar}q\hat P}\mathrm e^{\frac{\mathrm i}{\hbar}p\hat Q}\\ &= \mathrm e^{-\frac{1}{2\hbar^2}pq[\hat P,\hat Q]}\mathrm e^{-\frac{\mathrm i}{\hbar}q\hat P}\mathrm e^{\frac{\mathrm i}{\hbar}p\hat Q}\\ &= \mathrm e^{\frac{\mathrm i}{2\hbar}pq}\mathrm e^{-\frac{\mathrm i}{\hbar}q\hat P}\mathrm e^{\frac{\mathrm i}{\hbar}p\hat Q}.\end{aligned}$$

$\boxed{\mathfrak{I}6}$

$$\begin{aligned}\mathrm{tr}\,A &= \sum_{n=0}^{\infty}\langle n|A|n\rangle = \sum_{n=0}^{\infty}\langle n|\mathbb I A\mathbb I|n\rangle\\ &= \sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle n|q\rangle\langle q|A|a\rangle\langle a|n\rangle dqda\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{n=0}^{\infty}\langle a|n\rangle\langle n|q\rangle\langle q|A|a\rangle dqda\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle a|q\rangle\langle q|A|a\rangle dqda\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(a-q)\,\langle q|A|a\rangle dqda\\ &= \int_{-\infty}^{\infty}\langle q|A|q\rangle\,dq.\end{aligned}$$

$\boxed{\mathfrak{I}7}$

$$\begin{aligned}\mathrm{tr}(AB) &= \sum_{n=0}^{\infty}\langle n|AB|n\rangle\\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\langle n|A|m\rangle\langle m|B|n\rangle\\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle m|B|n\rangle\langle n|A|m\rangle = \mathrm{tr}(BA).\end{aligned}$$

$\boxed{\mathfrak{I}8}$ In view of Taylor's formula:

$$\begin{aligned}\mathrm e^{a\frac{\mathrm d}{\mathrm dq}}\Psi(q) &= \left(\mathbb I+\tfrac{a}{1!}\tfrac{\mathrm d}{\mathrm dq}+\tfrac{a^2}{2!}\tfrac{\mathrm d^2}{\mathrm dq^2}+\cdots\right)\Psi(q)\\ &= \Psi(q)+\tfrac{a}{1!}\Psi'(q)+\tfrac{a^2}{2!}\Psi''(q)+\cdots\\ &= \Psi(q+a).\end{aligned}$$

$\boxed{\mathfrak{I}9}$

The derivative of the operator $\hat F(t)=\mathrm e^{t\hat A}\,\mathrm e^{t\hat B}\,\mathrm e^{-t(\hat A+\hat B)}$ depending on the real variable t satisfies the relation $\hat F'(t)=\mathrm e^{t\hat A}\,[\hat A,\mathrm e^{t\hat B}]\,\mathrm e^{-t(\hat A+\hat B)}$ which can be written as $\hat F'(t)=t[\hat A,\hat B]\hat F(t)$. By starting from this equation written as $(\ln F)'=t[\hat A,\hat B]$ we get the relation $\int_0^1(\ln F)'dt=\int_0^1 t[\hat A,\hat B]dt$ leading to $F(1)=\mathrm e^{\frac12[\hat A,\hat B]}$, that is $\mathrm e^{\hat A}\,\mathrm e^{\hat B}\,\mathrm e^{-(\hat A+\hat B)}=\mathrm e^{\frac12[\hat A,\hat B]}$.

A — Position and momentum operators

$\boxed{A7'}$

$$\hat{Q}|a\rangle = \hat{Q}\delta_a = q\delta_a = a\delta_a = a\,|a\rangle$$

$\boxed{A8'}$

$$\begin{aligned}\hat{Q} &= \hat{Q}\mathbb{I} = \hat{Q}\int_{-\infty}^{\infty}|q\rangle\langle q|\,dq\\ &= \int_{-\infty}^{\infty}\hat{Q}|q\rangle\langle q|\,dq = \int_{-\infty}^{\infty}q|q\rangle\langle q|\,dq.\end{aligned}$$

$\boxed{A9'}$

$$\begin{aligned}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}a\hat{Q}}\psi(q) &= \left(\mathbb{I} + \tfrac{1}{1!}\tfrac{\mathrm{i}}{\hbar}a\hat{Q} + \tfrac{1}{2!}\left(\tfrac{\mathrm{i}}{\hbar}a\hat{Q}\right)^2 + \cdots\right)\psi(q)\\ &= \psi(q) + \tfrac{1}{1!}\tfrac{\mathrm{i}}{\hbar}aq\psi(q) + \tfrac{1}{2!}\left(\tfrac{\mathrm{i}}{\hbar}aq\right)^2\psi(q) + \cdots\\ &= \mathrm{e}^{\frac{\mathrm{i}}{\hbar}aq}\psi(q).\end{aligned}$$

$\boxed{A11'}$ By using $\boxed{\mathfrak{B}6}$, we get:

$$\begin{aligned}\hat{F}^\dagger\hat{Q}\hat{F}[\psi](q) &= \tfrac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\,pq}\hat{Q}(\hat{F}[\psi])(p)\,dp\\ &= \tfrac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\,pq}p\hat{F}[\psi](p)\,dp\\ &= \tfrac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\,pq}\tfrac{\hbar}{\mathrm{i}}\hat{F}[\psi'](p)\,dp\\ &= -\mathrm{i}\hbar\hat{F}^\dagger[\hat{F}[\psi']](q) = -\mathrm{i}\hbar\psi'(q)\\ &= -\mathrm{i}\hbar\tfrac{\mathrm{d}}{\mathrm{d}q}\psi(q)\end{aligned}$$

$\boxed{A12'}$

$$\begin{aligned}\langle q|\tilde{a}\rangle &= \langle q|\hat{F}^\dagger|a\rangle = \hat{F}^\dagger[\delta_a](q)\\ &= \tfrac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq}\delta_a(p)\,dp = \tfrac{1}{\sqrt{2\pi\hbar}}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}aq}.\end{aligned}$$

$\boxed{A13'}$

$$\begin{aligned}\hat{P}|\tilde{a}\rangle &= F^\dagger\hat{Q}\hat{F}F^\dagger|a\rangle = F^\dagger\hat{Q}|a\rangle\\ &= F^\dagger a|a\rangle = a\,F^\dagger|a\rangle = a\,|\tilde{a}\rangle.\end{aligned}$$

$\boxed{A14'}$

$$\begin{aligned}\mathbb{I} &= \hat{F}^\dagger\hat{F} = \hat{F}^\dagger\mathbb{I}\hat{F} = \hat{F}^\dagger\mathbb{I}\hat{F}\\ &= \int_{-\infty}^{\infty}\hat{F}^\dagger|p\rangle\langle p|\hat{F}\,dp = \int_{-\infty}^{\infty}|\tilde{p}\rangle\langle\tilde{p}|\,dp.\end{aligned}$$

$\boxed{A15'}$

$$\hat{P} = \hat{P}\mathbb{I} = \int_{-\infty}^{\infty}\hat{P}|\tilde{p}\rangle\langle\tilde{p}|\,dp = \int_{-\infty}^{\infty}p|\tilde{p}\rangle\langle\tilde{p}|\,dp.$$

$\boxed{A16'}$

$$\begin{aligned}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}a\hat{P}}\psi(q) &= \mathrm{e}^{-a\frac{\mathrm{d}}{\mathrm{d}q}}\psi(q)\\ &= \left(\mathbb{I} - \tfrac{a}{1!}\tfrac{\mathrm{d}}{\mathrm{d}q} + \tfrac{a^2}{2!}\tfrac{\mathrm{d}^2}{\mathrm{d}q^2} - \cdots\right)\psi(q)\\ &= \psi(q) - \tfrac{a}{1!}\psi'(q) + \tfrac{a^2}{2!}\psi''(q) - \cdots\\ &= \psi(q-a).\end{aligned}$$

$\boxed{\textbf{B}}$ $\boxed{\textbf{Displacement operators}}$

$\boxed{B1'}$

$$\left.\begin{aligned}\hat{Q} &\stackrel{\text{def}}{=} \sqrt{\tfrac{\hbar}{2}}(\hat{a}+\hat{a}^\dagger)\\ \hat{P} &\stackrel{\text{def}}{=} -\mathrm{i}\sqrt{\tfrac{\hbar}{2}}(\hat{a}-\hat{a}^\dagger)\\ \lambda &= \tfrac{1}{\sqrt{2\hbar}}(q+p\,\mathrm{i})\end{aligned}\right\} \Rightarrow \quad \begin{aligned}D(\lambda) &\stackrel{\text{def}}{=} \mathrm{e}^{\lambda\hat{a}^\dagger - \bar{\lambda}\hat{a}}\\ &= \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(p\hat{Q}-q\hat{P})}\\ &= \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\Lambda^T\Omega\hat{R}} \stackrel{\text{def}}{=} D(\Lambda).\end{aligned}$$

By using $\boxed{\mathfrak{I}9}$, we get:

$$\begin{aligned}\mathrm{e}^{\lambda\hat{a}^\dagger-\bar{\lambda}\hat{a}} &= \mathrm{e}^{-\frac{1}{2}[\lambda\hat{a}^\dagger,-\bar{\lambda}\hat{a}]}\mathrm{e}^{\lambda\hat{a}^\dagger}\mathrm{e}^{-\bar{\lambda}\hat{a}}\\ &= \mathrm{e}^{\frac{|\lambda|^2}{2}[\hat{a}^\dagger,\hat{a}]}\mathrm{e}^{\lambda\hat{a}^\dagger}\mathrm{e}^{-\bar{\lambda}\hat{a}}\\ &= \mathrm{e}^{-\frac{|\lambda|^2}{2}}\mathrm{e}^{\lambda\hat{a}^\dagger}\mathrm{e}^{-\bar{\lambda}\hat{a}}\\ \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(p\hat{Q}-q\hat{P})} &= \mathrm{e}^{-\frac{1}{2}[-\frac{\mathrm{i}}{\hbar}q\hat{P},\frac{\mathrm{i}}{\hbar}p\hat{Q}]}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}q\hat{P}}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p\hat{Q}}\\ &= \mathrm{e}^{-\frac{1}{2\hbar^2}pq[\hat{P},\hat{Q}]}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}q\hat{P}}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p\hat{Q}}\\ &= \mathrm{e}^{\frac{\mathrm{i}}{2\hbar}pq}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}q\hat{P}}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p\hat{Q}}.\end{aligned}$$

$\boxed{B2'}$

$$\begin{aligned}D(q,p)\psi(a) &\equiv \langle a|D(q,p)|\psi\rangle\\ &= \langle a|\mathrm{e}^{\frac{\mathrm{i}}{2\hbar}pq}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}q\hat{P}}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p\hat{Q}}|\psi\rangle\\ &= \mathrm{e}^{\frac{\mathrm{i}}{2\hbar}pq}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}q\hat{P}}(\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p\hat{Q}}\psi)(a)\\ &= \mathrm{e}^{\frac{\mathrm{i}}{2\hbar}pq}(\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p\hat{Q}}\psi)(a-q)\\ &= \mathrm{e}^{\frac{\mathrm{i}}{2\hbar}pq}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p(a-q)}\psi(a-q)\\ &= \mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}pq}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pa}\,\psi(a-q)\end{aligned}$$

$\boxed{B3'}$

$$\begin{aligned}\langle a|D(q,p)|b\rangle &= \mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}pq}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pa}\,\delta_b(a-q)\\ &= \mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}pq}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pa}\,\delta(a-b-q)\\ &= \mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}p(a-b)}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pa}\,\delta(a-b-q)\\ &= \mathrm{e}^{\frac{\mathrm{i}}{2\hbar}p(a+b)}\delta(a-b-q)\end{aligned}$$

$\boxed{B4'}$

$$\begin{aligned}D(\Lambda)D(X)\psi(a) &\equiv D(q,p)D(x,y)\psi(a)\\ &= \mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}pq}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pa}(D(x,y)\psi)(a-q)\\ &= \mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}pq}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pa}\mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}xy}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}y(a-q)}\,\psi(a-q-x)\\ &= \mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}(qy-px)}\,\mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}(p+y)(q+x)}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(p+y)a}\psi(a-q-x)\\ &= \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\Lambda^T\Omega X}\mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}(p+y)(q+x)}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(p+y)a}\psi(a-q-x)\\ &= \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\Lambda^T\Omega X}D(\Lambda+X)\psi(a).\end{aligned}$$

$\boxed{B5'}$

$$\begin{aligned}D^\dagger(q,p) &= \left(\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(p\hat{Q}-q\hat{P})}\right)^\dagger\\ &= \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(p\hat{Q}-q\hat{P})} = D(-q,-p) = D(-\Lambda)\end{aligned}$$

$\boxed{B6'}$

$$\begin{aligned}\operatorname{tr} D(\Lambda) &= \int_{-\infty}^{\infty}\langle a|D(q,p)|a\rangle\,da = \int_{-\infty}^{\infty}\mathrm{e}^{\frac{\mathrm{i}}{2\hbar}p(a+a)}\delta(a-a-q)\,da\\ &= \delta(q)\int_{-\infty}^{\infty}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pa}\,da = 2\pi\hbar\,\delta(q)\delta(p) = 2\pi\hbar\,\delta(\Lambda)\end{aligned}$$

$\boxed{B7'}$

$$\begin{aligned}\langle D(q,p), D(x,y)\rangle &= \operatorname{tr}(D^\dagger(q,p)D(x,y))\\ &= \operatorname{tr}(D(-q,-p)D(x,y))\\ &= \mathrm{e}^{\frac{\mathrm{i}}{2\hbar}(qy-px)}\operatorname{tr} D(x-q,y-p)\\ &= 2\pi\hbar\,\mathrm{e}^{\frac{\mathrm{i}}{2\hbar}(qy-px)}\delta(q-x)\,\delta(p-y)\\ &= 2\pi\hbar\,\delta(q-x)\,\delta(p-y)\end{aligned}$$

$\boxed{B8'}$

$$\left.\begin{aligned}&\hat{P} = \hat{F}^\dagger\hat{Q}\hat{F} \;\Rightarrow\; \hat{F}^\dagger\hat{P}\hat{F} = (\hat{F}^\dagger)^2\hat{Q}\hat{F}^2 = \Pi\hat{Q}\Pi\\ &\Pi\hat{Q}\Pi\psi(q) = \hat{Q}\Pi\psi(-q) = -q\,\Pi\psi(-q) = -q\,\psi(q)\end{aligned}\right\}$$

$$\begin{aligned}\Rightarrow \hat{F}^\dagger D(q,p)\hat{F} &= \hat{F}^\dagger\left(\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(p\hat{Q}-q\hat{P})}\right)\hat{F}\\ &= \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(phatF^\dagger Q\hat{F}-qhatF^\dagger P\hat{F})}\\ &= \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(p\hat{P}+q\hat{Q})} = D(-p,q).\end{aligned}$$

$\boxed{\textbf{C}}$ $\boxed{\textbf{Parity operators}}$

$\boxed{C2'}$

$$\begin{aligned}
\Pi\psi(q) &= \langle q|\Pi|\psi\rangle = \langle q|\Pi|\sum_{n=0}^{\infty}|n\rangle\langle n|\psi\rangle = \langle q|\sum_{n=0}^{\infty}\Pi|n\rangle\langle n|\psi\rangle\\
&= \langle q|\sum_{n=0}^{\infty}(-1)^n|n\rangle\langle n|\psi\rangle = \sum_{n=0}^{\infty}(-1)^n\langle q|n\rangle\langle n|\psi\rangle\\
&= \sum_{n=0}^{\infty}(-1)^n\Psi_n(q)\langle\Psi_n|\psi\rangle = \sum_{n=0}^{\infty}\Psi_n(-q)\langle\Psi_n|\psi\rangle\\
&= \sum_{n=0}^{\infty}\langle -q|\Psi_n\rangle\langle\Psi_n|\psi\rangle = \langle -q|\psi\rangle = \psi(-q).
\end{aligned}$$

$$\begin{aligned}
\hat{F}^2[\psi](p) &= \hat{F}[\hat{F}[\psi]](p) = \tfrac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}pq}\hat{F}[\psi](q)\,dq\\
&= \tfrac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}pq}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}qx}\hat{\psi}(x)\,dxdq\\
&= \tfrac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}q(p+x)}\,\hat{\psi}(x)\,dqdx\\
&= \int_{-\infty}^{\infty}\delta(p+x)\,\hat{\psi}(x)\,dx = \psi(-p) = \Pi\psi(p).
\end{aligned}$$

$$(\hat{F}^\dagger)^2 = \hat{F}^\dagger\hat{F}^\dagger = (\hat{F}\hat{F})^\dagger = (\hat{F}^2)^\dagger = \Pi^\dagger = \Pi.$$

$\boxed{C3'}$

$$\begin{aligned}
\Pi(q,p)\psi(a) &= D(q,p)\Pi D(-q,-p)\psi(a)\\
&= \mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}pq}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pa}\,\Pi D(-q,-p)\psi(a-q)\\
&= \mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}pq}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pa}\,D(-q,-p)\psi(q-a)\\
&= \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}pq}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pa}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}p(q-a)}\,\psi(q-a+q)\\
&= \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}2pq}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}2pa}\,\psi(2q-a)\\
&= \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}2p(q-a)}\,\psi(2q-a).
\end{aligned}$$

$\boxed{C4'}$

$$\begin{aligned}
\langle a|\Pi(q,p)|b\rangle &= \Pi(q,p)\delta_b(a)\\
&= \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}2p(q-a)}\,\delta_b(2q-a)\\
&= \mathrm{e}^{\frac{\mathrm{i}}{\hbar}p(2a-2q)}\,\delta(2q-a-b)\\
&= \mathrm{e}^{\frac{\mathrm{i}}{\hbar}p(2a-a-b)}\,\delta(2q-a-b)\\
&= \mathrm{e}^{\frac{\mathrm{i}}{\hbar}p(a-b)}\delta(2q-a-b).
\end{aligned}$$

$\boxed{C5'}$

$$\begin{aligned}
\langle 2\Pi(q,p), 2\Pi(x,y)\rangle &= 4\,\mathrm{tr}(\Pi^\dagger(q,p)\Pi(x,y)) = 4\,\mathrm{tr}(\Pi(q,p)\Pi(x,y))\\
&= 4\,\mathrm{tr}(D(q,p)\Pi D(-q,-p)D(x,y)\Pi D(-x,-y))\\
&= 4\,\mathrm{tr}(D(-x,-y)D(q,p)\Pi D(-q,-p)D(x,y)\Pi)\\
&= 4\,\mathrm{tr}(\Pi(q-x,p-y)\Pi)\\
&= 4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle a|\Pi(q-x,p-y)|b\rangle\langle b|\Pi|a\rangle\,dbda\\
&= 4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(p-y)(a-b)}\delta(2q-2x-a-b)\delta(a+b)\,dbda\\
&= 4\int_{-\infty}^{\infty}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(p-y)(a+a)}\delta(2q-2x)\,da\\
&= 4\int_{-\infty}^{\infty}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}2a(p-y)}\delta(2q-2x)\,da\\
&= 2\int_{-\infty}^{\infty}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}2a(p-y)}\delta(q-x)\,da\\
&= 2\delta(q-x)\int_{-\infty}^{\infty}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}2a(p-y)}\,da\\
&= 2\delta(q-x)\,2\pi\hbar\delta(2p-2y)\\
&= 2\pi\hbar\,\delta(q-x)\,\delta(p-2).
\end{aligned}$$

$\boxed{C6'}$ By using $\boxed{\mathfrak{C}5}$, $\boxed{B5'}$, $\boxed{C1'}$, $\boxed{C4'}$, we get

$$\begin{aligned}
\hat{F}^\dagger\Pi\hat{F}\psi(p) &= \tfrac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq}\Pi\hat{F}\psi(q)\,dq\\
&= \tfrac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq}\hat{F}[\psi](-q)\,dq\\
&= \tfrac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}qx}\psi(x)\,dqdx\\
&= \tfrac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}q(p+x)}\psi(x)\,dqdx\\
&= \int_{-\infty}^{\infty}\delta(p+x)\psi(x)\,dx = \psi(-p) = \Pi\psi(p).
\end{aligned}$$

$$\begin{aligned}
\hat{F}^\dagger\Pi(q,p)\hat{F} &= \hat{F}^\dagger D(q,p)\Pi D(-q,-p)\hat{F}\\
&= \hat{F}^\dagger D(q,p)\hat{F}\hat{F}^\dagger\Pi\hat{F}\hat{F}^\dagger D(-q,-p)\hat{F}\\
&= D(-p,q)\Pi D(p,-q) = \Pi(-p,q).
\end{aligned}$$

D | Weyl transform, characteristic and Wigner function

$\boxed{D5'}$

$$
\begin{aligned}
&\left\langle a \left| \frac{1}{2\pi\hbar} \int\limits_{\mathbb{R}^2} \chi[\hat{O}](\Lambda)\ D(-\Lambda)\, d^2\Lambda \right| b \right\rangle \\
&= \left\langle a \left| \frac{1}{2\pi\hbar} \int\limits_{\mathbb{R}^2} \mathrm{tr}(\hat{O} D(q,p))\, D(-q,-p)\, dqdp \right| b \right\rangle \\
&= \frac{1}{2\pi\hbar} \int\limits_{\mathbb{R}^2} \mathrm{tr}(\hat{O} D(q,p))\ \langle a\, | D(-q,-p) |\, b \rangle\ dqdp \\
&= \frac{1}{2\pi\hbar} \int\limits_{\mathbb{R}^4} \langle x|\hat{O}|y\rangle \langle y|D(q,p))|x\rangle\ \langle a\, | D(-q,-p) |\, b \rangle\ dxdydqdp \\
&= \frac{1}{2\pi\hbar} \int\limits_{\mathbb{R}^4} \langle x|\hat{O}|y\rangle \mathrm{e}^{\frac{\mathrm{i}}{2\hbar} p(x+y)} \delta(y-x-q)\, \mathrm{e}^{-\frac{\mathrm{i}}{2\hbar} p(a+b)} \delta(a-b+q)\, dxdydqdp \\
&= \frac{1}{2\pi\hbar} \int\limits_{\mathbb{R}^3} \langle x|\hat{O}|y\rangle \mathrm{e}^{\frac{\mathrm{i}}{2\hbar} p(x+y-a-b)} \delta(y-x-b+a)\, dpdxdy \\
&= \int\limits_{\mathbb{R}^2} \langle x|\hat{O}|y\rangle \delta(x+y-a-b) \delta(y-x-b+a)\, dxdy \\
&= \int\limits_{\mathbb{R}^2} \langle x|\hat{O}|y\rangle \delta(x-a) \delta(y-b)\, dxdy \\
&= \langle a|\hat{O}|b\rangle
\end{aligned}
$$

$\boxed{D6'}$

$$
\begin{aligned}
&\left\langle a \left| 2 \int\limits_{\mathbb{R}^2} W[\hat{\varrho}](\Lambda)\ \Pi(\Lambda)\, d^2\Lambda \right| b \right\rangle \\
&= \left\langle a \left| \frac{2}{\pi\hbar} \int\limits_{\mathbb{R}^2} \mathrm{tr}(\hat{\varrho}\Pi(q,p))\, \Pi(q,p)\, dqdp \right| b \right\rangle \\
&= \frac{2}{\pi\hbar} \int\limits_{\mathbb{R}^2} \mathrm{tr}(\hat{\varrho}\Pi(q,p))\, \langle a|\Pi(q,p)|b\rangle\, dqdp \\
&= \frac{2}{\pi\hbar} \int\limits_{\mathbb{R}^4} \langle x|\hat{\varrho}|y\rangle \langle y|\Pi(q,p)|x\rangle\ \langle a\, | \Pi(q,p) |\, b\rangle\ dxdydqdp \\
&= \frac{2}{\pi\hbar} \int\limits_{\mathbb{R}^4} \langle x|\hat{\varrho}|y\rangle \mathrm{e}^{\frac{\mathrm{i}}{\hbar} p(y-x)} \delta(2q-x-y)\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar} p(a-b)} \delta(2q-a-b)\, dxdydqdp \\
&= \frac{2}{\pi\hbar} \int\limits_{\mathbb{R}^4} \langle x|\hat{\varrho}|y\rangle \mathrm{e}^{\frac{\mathrm{i}}{\hbar} p(y-x+a-b)} \delta(2q-x-y)\, \delta(2q-a-b)\, dydxdqdp \\
&= \frac{2}{\pi\hbar} \int\limits_{\mathbb{R}^3} \langle x|\hat{\varrho}|2q-x\rangle \mathrm{e}^{\frac{\mathrm{i}}{\hbar} p(2q-2x+a-b)}\ \delta(2q-a-b)\, dxdqdp \\
&= \frac{1}{\pi\hbar} \int\limits_{\mathbb{R}^3} \langle x|\hat{\varrho}|2q-x\rangle \mathrm{e}^{\frac{\mathrm{i}}{\hbar} p(2q-2x+a-b)}\ \delta(q-\tfrac{a}{2}-\tfrac{b}{2})\, dqdxdp \\
&= \frac{1}{\pi\hbar} \int\limits_{\mathbb{R}^2} \langle x|\hat{\varrho}|a+b-x\rangle \mathrm{e}^{\frac{\mathrm{i}}{\hbar} p(a+b-2x+a-b)}\, dxdp \\
&= \frac{1}{\pi\hbar} \int\limits_{\mathbb{R}^2} \langle x|\hat{\varrho}|a+b-x\rangle \mathrm{e}^{\frac{\mathrm{i}}{\hbar} p(2a-2x)}\, dpdx \\
&= 2 \int\limits_{-\infty}^{\infty} \langle x|\hat{\varrho}|a+b-x\rangle \delta(2a-2x)\, dx \\
&= \int\limits_{-\infty}^{\infty} \langle x|\hat{\varrho}|a+b-x\rangle \delta(a-x)\, dx \\
&= \langle a|\hat{\varrho}|b\rangle
\end{aligned}
$$

$\boxed{D7'}$

$$
\begin{aligned}
&\frac{1}{2\pi\hbar} \int\limits_{\mathbb{R}^2} \chi[\hat{O}_1](\Lambda)\ \chi[\hat{O}_2](-\Lambda)\ d^2\Lambda \\
&= \frac{1}{2\pi\hbar} \int\limits_{\mathbb{R}^2} \mathrm{tr}(\hat{O}_1 D(q,p))\ \mathrm{tr}(\hat{O}_2 D(-q,-p))\ dqdp \\
&= \frac{1}{2\pi\hbar} \int\limits_{\mathbb{R}^6} \langle a|\hat{O}_1|b\rangle \langle b|D(q,p)|a\rangle \langle x|\hat{O}_2|y\rangle \langle y|D(-q,-p)|x\rangle\, dadbdxdydqdp \\
&= \frac{1}{2\pi\hbar} \int\limits_{\mathbb{R}^6} \langle a|\hat{O}_1|b\rangle \langle x|\hat{O}_2|y\rangle \mathrm{e}^{\frac{\mathrm{i}}{2\hbar} p(a+b-x-y)} \delta(b-a-q)\, \delta(y-x+q) \\
&\qquad\qquad\qquad dqdadbdxdydp \\
&= \frac{1}{2\pi\hbar} \int\limits_{\mathbb{R}^5} \langle a|\hat{O}_1|b\rangle \langle x|\hat{O}_2|y\rangle \mathrm{e}^{\frac{\mathrm{i}}{2\hbar} p(a+b-x-y)} \delta(b-a-x+y)\, dpdadbdxdy \\
&= \int\limits_{\mathbb{R}^4} \langle a|\hat{O}_1|b\rangle \langle x|\hat{O}_2|y\rangle \delta(\tfrac{1}{2}(a+b-x-y)) \delta(b-a-x+y)\, dydadbdx \\
&= \int\limits_{\mathbb{R}^3} \langle a|\hat{O}_1|b\rangle \langle x|\hat{O}_2|-b+a+x\rangle \delta(\tfrac{1}{2}(a+b-x+b-a-x))\, dadbdx \\
&= \int\limits_{\mathbb{R}^3} \langle a|\hat{O}_1|b\rangle \langle x|\hat{O}_2|-b+a+x\rangle \delta(b-x)\, dxdadb \\
&= \int\limits_{\mathbb{R}^3} \langle a|\hat{O}_1|b\rangle \langle b|\hat{O}_2|a\rangle\, dadb \\
&= \mathrm{tr}(\hat{O}_1 \hat{O}_2)
\end{aligned}
$$

$\boxed{D8'}$

$$
\begin{aligned}
&\frac{1}{2\pi\hbar} \int\limits_{\mathbb{R}^2} O_1^{\mathrm{w}}(\Lambda)\ O_2^{\mathrm{w}}(\Lambda)\ d^2\Lambda \\
&= \frac{2}{\pi\hbar} \int\limits_{\mathbb{R}^2} \mathrm{tr}(\hat{O}_1 \Pi(q,p))\, \mathrm{tr}(\hat{O}_2 \Pi(q,p))\, dqdp \\
&= \frac{2}{\pi\hbar} \int\limits_{\mathbb{R}^6} \langle a|\hat{O}_1|b\rangle \langle b|\Pi(q,p))|a\rangle\ \langle x|\hat{O}_2|y\rangle \langle y|\Pi(q,p))|x\rangle\, dadbdxdydqdp \\
&= \frac{2}{\pi\hbar} \int\limits_{\mathbb{R}^6} \langle a|\hat{O}_1|b\rangle \langle x|\hat{O}_2|y\rangle \mathrm{e}^{\frac{\mathrm{i}}{\hbar} p(b-a+y-x)} \delta(2q-a-b)\, \delta(2q-x-y) \\
&\qquad\qquad\qquad dydqdadbdxdp \\
&= \frac{2}{\pi\hbar} \int\limits_{\mathbb{R}^5} \langle a|\hat{O}_1|b\rangle \langle x|\hat{O}_2|2q-x\rangle \mathrm{e}^{\frac{\mathrm{i}}{\hbar} p(b-a+2q-x-x)} \delta(2q-a-b) \\
&\qquad\qquad\qquad dqdadbdxdp \\
&= \frac{1}{\pi\hbar} \int\limits_{\mathbb{R}^5} \langle a|\hat{O}_1|b\rangle \langle x|\hat{O}_2|2q-x\rangle \mathrm{e}^{\frac{\mathrm{i}}{\hbar} p(b-a+2q-2x)} \delta(q-\tfrac{a}{2}-\tfrac{b}{2})\, dqdadbdxdp \\
&= \frac{1}{\pi\hbar} \int\limits_{\mathbb{R}^4} \langle a|\hat{O}_1|b\rangle \langle x|\hat{O}_2|a+b-x\rangle \mathrm{e}^{\frac{\mathrm{i}}{\hbar} p(b-a+a+b-2x)}\, dpdadbdx \\
&= 2 \int\limits_{\mathbb{R}^3} \langle a|\hat{O}_1|b\rangle \langle x|\hat{O}_2|a+b-x\rangle \delta(2b-2x)\, dxdadb \\
&= \int\limits_{\mathbb{R}^3} \langle a|\hat{O}_1|b\rangle \langle x|\hat{O}_2|a+b-x\rangle \delta(b-x)\, dxdadb \\
&= \int\limits_{\mathbb{R}^3} \langle a|\hat{O}_1|b\rangle \langle b|\hat{O}_2|a\rangle\, dadb \\
&= \mathrm{tr}(\hat{O}_1 \hat{O}_2)
\end{aligned}
$$

$\boxed{D9'}$

$$
\begin{aligned}
\mathrm{tr}(\hat{\varrho}\hat{O}) &= \frac{1}{(2\pi\hbar)^2} \int\limits_{\mathbb{R}^2} \varrho^{\mathrm{w}}(\Lambda)\ O^{\mathrm{w}}(\Lambda)\ d^2\Lambda \\
&= \int\limits_{\mathbb{R}^2} W[\hat{\varrho}](\Lambda)\ O^{\mathrm{w}}(\Lambda)\ d^2\Lambda.
\end{aligned}
$$

$\boxed{D10'}$

$$
\begin{aligned}
O^{\mathrm{w}}(q,p) &= 2\mathrm{tr}(\hat{O}\Pi(q,p)) = \int\limits_{\mathbb{R}^2} \langle a|\hat{O}|b\rangle \langle b|\hat{\Pi}|a\rangle\, dadb \\
&= \int\limits_{\mathbb{R}^2} \langle a|\hat{O}|b\rangle \mathrm{e}^{\frac{\mathrm{i}}{\hbar} p(b-a)} \delta(2q-a-b)\, dadb \\
&= \int\limits_{\mathbb{R}^2} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} p(2q-a-a)} \langle a|\hat{O}|2q-a\rangle\, da \\
&= \int\limits_{\mathbb{R}^2} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} 2p(q-a)} \langle a|\hat{O}|2q-a\rangle\, da \\
&\qquad \Downarrow\ a = q + \tfrac{x}{2} \\
&= \int\limits_{-\infty}^{\infty} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} px} \langle q+\tfrac{x}{2}|\hat{O}|q-\tfrac{x}{2}\rangle\, dx.
\end{aligned}
$$

$\boxed{D11'}$

$$
\begin{aligned}
W[\hat{\varrho}](q,p) &= \frac{1}{2\pi\hbar} \varrho^{w}(q,p) \\
&= \frac{1}{2\pi\hbar} \int\limits_{-\infty}^{\infty} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} px} \langle q+\tfrac{x}{2}|\hat{\varrho}|q-\tfrac{x}{2}\rangle\, dx.
\end{aligned}
$$

$\boxed{D12'}$

$$
\begin{aligned}
W[\hat{F}\hat{\varrho}\hat{F}^{\dagger}](q,p) &= \frac{1}{\pi\hbar} \mathrm{tr}(\hat{F}\hat{\varrho}\hat{F}^{\dagger}\Pi(q,p)) \\
&= \frac{1}{\pi\hbar} \mathrm{tr}(\hat{\varrho}\hat{F}^{\dagger}\Pi(q,p)\hat{F}) \\
&= \frac{1}{\pi\hbar} \mathrm{tr}(\hat{\varrho}\Pi(-p,q)) \\
&= W[\hat{\varrho}](-p,q).
\end{aligned}
$$

$\boxed{D13'}$

$$
\begin{aligned}
&\left\langle a \left| \frac{1}{4\pi\hbar} \int\limits_{\mathbb{R}^2} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} \Lambda^T \Omega X}\, D(X)\, d^2X \right| b \right\rangle \\
&= \frac{1}{4\pi\hbar} \int\limits_{\mathbb{R}^2} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} \Lambda^T \Omega X}\ \langle a\, | D(X) |\, b\rangle\ d^2X \\
&= \frac{1}{4\pi\hbar} \int\limits_{\mathbb{R}^2} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} (qy-px)}\, \mathrm{e}^{\frac{\mathrm{i}}{2\hbar} y(a+b)} \delta(a-b-x)\, dxdy \\
&= \frac{1}{4\pi\hbar} \int\limits_{\mathbb{R}^2} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} (qy-pa+pb)}\, \mathrm{e}^{\frac{\mathrm{i}}{2\hbar} y(a+b)}\, dy \\
&= \frac{1}{4\pi\hbar} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} p(a+b)} \int\limits_{\mathbb{R}^2} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} y(q-\frac{a}{2}-\frac{b}{2})}\, dy \\
&= \tfrac{1}{2} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} p(a+b)}\, \delta(q-\tfrac{a}{2}-\tfrac{b}{2}) \\
&= \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} p(a+b)}\, \delta(2q-a-b) \\
&= \langle a|\Pi(q,p)|b\rangle \equiv \langle a|\Pi(\Lambda)|b\rangle.
\end{aligned}
$$

$\boxed{D14'}$

$$
\begin{aligned}
&\frac{1}{(2\pi\hbar)^2}\int_{\mathbb{R}^2} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\Lambda^T\Omega X}\,\chi[\hat{\varrho}](X)\,d^2X\\
&=\frac{1}{(2\pi\hbar)^2}\int_{\mathbb{R}^2} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\Lambda^T\Omega X}\,\mathrm{tr}(\hat{\varrho}D(x,y))\,d^2X\\
&=\frac{1}{\pi\hbar}\,\mathrm{tr}\left(\hat{\varrho}\,\frac{1}{4\pi\hbar}\int_{\mathbb{R}^2} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\Lambda^T\Omega X}\,D(X)\,d^2X\right)\\
&=\frac{1}{\pi\hbar}\,\mathrm{tr}(\varrho\,\Pi(X))\\
&=W[\hat{\varrho}](\Lambda).
\end{aligned}
$$

$\boxed{D15'}$

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\Lambda^T\Omega X}\,W[\hat{\varrho}](\Lambda)\,d^2\Lambda\\
&=\frac{1}{(2\pi\hbar)^2}\int_{\mathbb{R}^4} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(qy-px)}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(qb-pa)}\,\chi[\hat{\varrho}](a,b)\,dadbdqdp\\
&=\frac{1}{(2\pi\hbar)^2}\int_{\mathbb{R}^4} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}q(y-b)}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}p(x-a)}\,\chi[\hat{\varrho}](a,b)\,dqdpdadb\\
&=\int_{\mathbb{R}^2} \delta(y-b)\,\delta(x-a)\,\chi[\hat{\varrho}](a,b)\,dadb\\
&=\chi[\hat{\varrho}](x,y)\\
&=\chi[\hat{\varrho}](X).
\end{aligned}
$$

$\boxed{D16'}$

$$
\begin{aligned}
W[\hat{\varrho}](q,p)&=\frac{1}{\pi\hbar}\,\mathrm{tr}(\hat{\varrho}\,\Pi(q,p))\\
&=\frac{1}{\pi\hbar}\int_{\mathbb{R}^2}\langle a|\hat{\varrho}|b\rangle\langle b|\Pi(q,p))|a\rangle\,dbda\\
&=\frac{1}{\pi\hbar}\int_{\mathbb{R}^2}\langle a|\psi\rangle\langle\psi|b\rangle \mathrm{e}^{\frac{\mathrm{i}}{\hbar}p(b-a)}\,\delta(2q-a-b)\,dbda\\
&=\frac{1}{\pi\hbar}\int_{\mathbb{R}^2}\langle a|\psi\rangle\langle\psi|2q-a\rangle \mathrm{e}^{\frac{\mathrm{i}}{\hbar}p(2q-a-a)}\,da\\
&=\frac{1}{\pi\hbar}\int_{\mathbb{R}^2}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}2p(q-a)}\,\langle a|\psi\rangle\langle\psi|2q-a\rangle\,da\\
&\qquad\Downarrow\ a=q+x\\
&=\frac{1}{\pi\hbar}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}2px}\,\overline{\psi(q-x)}\,\psi(q+x)\,dx.
\end{aligned}
$$

$\boxed{D17'}$

$$
\begin{aligned}
&\frac{1}{\pi\hbar}\int_{-\infty}^{\infty}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}2qy}\,\overline{F[\psi](p-y)}\,F[\psi](p+y)\,dy\\
&=\frac{1}{2(\pi\hbar)^2}\int_{\mathbb{R}^3}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}2qy}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(p-y)a}\,\overline{\psi(a)}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(p+y)b}\psi(b)\,dadbdy\\
&=\frac{1}{2(\pi\hbar)^2}\int_{\mathbb{R}^3}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}y(2q-a-b)}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p(a-b)}\,\overline{\psi(a)}\,\psi(b)\,dydadb\\
&=\frac{1}{\pi\hbar}\int_{\mathbb{R}^2}\delta(2q-a-b)\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p(a-b)}\,\overline{\psi(a)}\,\psi(b)\,dadb\\
&=\frac{1}{\pi\hbar}\int_{-\infty}^{\infty}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p(a-2q+a)}\,\overline{\psi(a)}\,\psi(2q-a)\,da\\
&=\frac{1}{\pi\hbar}\int_{-\infty}^{\infty}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}2p(a-q)}\,\overline{\psi(a)}\,\psi(2q-a)\,da\\
&\qquad\Downarrow\ a=q-x\\
&=\frac{1}{\pi\hbar}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}2px}\,\overline{\psi(q-x)}\,\psi(q+x)\,da\\
&\qquad\Downarrow\ \boxed{D16'}\\
&=W[\hat{\varrho}](q,p).
\end{aligned}
$$

$\boxed{D18'}$

$$
\begin{aligned}
\int_{-\infty}^{\infty}W[\hat{\varrho}](q,p)\,dp&=\frac{1}{\pi\hbar}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}2px}\,\overline{\psi(q-x)}\,\psi(q+x)\,dpdx\\
&\qquad\Downarrow\ \boxed{\mathfrak{c}5}\\
&=2\int_{-\infty}^{\infty}\delta(-2x)\,\overline{\psi(q-x)}\,\psi(q+x)\,dx\\
&\qquad\Downarrow\ \boxed{\mathfrak{c}3}\\
&=\int_{-\infty}^{\infty}\delta(x)\,\overline{\psi(q-x)}\,\psi(q+x)\,dx\\
&=\overline{\psi(q)}\,\psi(q)=|\psi(q)|^2\\
\int_{-\infty}^{\infty}W[\hat{\varrho}](\Lambda)\,dq&\overset{\boxed{D17'}}{=}\frac{1}{\pi\hbar}\int_{-\infty}^{\infty}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}2qy}\,\overline{F[\psi](p-y)}\,F[\psi](p+y)\,dy\\
&\qquad\Downarrow\ \boxed{\mathfrak{c}5}\\
&=2\int_{-\infty}^{\infty}\delta(-2x)\,\overline{F[\psi](p-y)}\,\psi(q+x)\,dx\\
&\qquad\Downarrow\ \boxed{\mathfrak{c}3}\\
&=\int_{-\infty}^{\infty}\delta(x)\,\overline{F[\psi](p-y)}\,\psi(q+x)\,dx\\
&=\overline{F[\psi](p)}\,F[\psi](p)=|F[\psi](p)|^2.
\end{aligned}
$$

$\boxed{\textbf{E}}$ $\boxed{\textbf{Gaussian states of continuous variable}}$

$\boxed{1}$ ■ $\boxed{2}$ ■

$\boxed{\textbf{F}}$ $\boxed{\textbf{Gaussian transforms}}$

$\boxed{F1'}$

$$\left.\begin{array}{r}\mathrm{e}^{\hat{A}}\hat{B}\mathrm{e}^{-\hat{A}}=\hat{B}+\frac{1}{1!}[\hat{A},\hat{B}]+\frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]]+\ldots\\ \hat{A}=-\alpha\hat{a}^{+}+\bar{\alpha}\hat{a}\ \Rightarrow\ \begin{array}{l}[\hat{A},\hat{a}]=\alpha,\ [\hat{A},[\hat{A},\hat{a}]]=0,\ \ldots\\ [\hat{A},\hat{a}^{\dagger}]=\bar{\alpha},\ [\hat{A},[\hat{A},\hat{a}^{\dagger}]]=0,\ \ldots\end{array}\end{array}\right\}$$

$$\Rightarrow\left\{\begin{array}{l}D^{\dagger}(\alpha)\hat{a}D(\alpha)=\mathrm{e}^{\hat{A}}\hat{a}\mathrm{e}^{-\hat{A}}=\hat{a}+\alpha\\ D^{\dagger}(\alpha)\hat{a}^{\dagger}D(\alpha)=\mathrm{e}^{\hat{A}}\hat{a}^{\dagger}\mathrm{e}^{-\hat{A}}=\hat{a}^{\dagger}+\bar{\alpha}.\end{array}\right.$$

$$\Rightarrow\hat{D}^{\dagger}(\alpha)\begin{pmatrix}\hat{a}\\ \hat{a}^{\dagger}\end{pmatrix}\hat{D}(\alpha)=\begin{pmatrix}\hat{D}^{\dagger}(\alpha)\hat{a}\hat{D}(\alpha)\\ \hat{D}^{\dagger}(\alpha)\hat{a}^{\dagger}\hat{D}(\alpha)\end{pmatrix}$$

$$=\begin{pmatrix}\hat{a}+\alpha\\ \hat{a}^{\dagger}+\bar{\alpha}\end{pmatrix}=\begin{pmatrix}\hat{a}\\ \hat{a}^{\dagger}\end{pmatrix}+\begin{pmatrix}\alpha\\ \bar{\alpha}\end{pmatrix}.$$

$\boxed{F2'}$ ■ $\hat{q}=\frac{1}{\sqrt{2}}(\hat{a}+\hat{a}^{\dagger}),\quad \hat{p}=\frac{1}{\mathrm{i}\sqrt{2}}(\hat{a}-\hat{a}^{\dagger})\ \Rightarrow\ldots$

$\boxed{F3'}$ ■ $\hat{D}^{\dagger}(\alpha)\mathrm{e}^{-\mathrm{i}\Lambda^{T}\Omega\hat{\mathbf{R}}}\hat{D}(\alpha)=\mathrm{e}^{-\mathrm{i}\Lambda^{T}\Omega\hat{D}^{\dagger}(\alpha)\hat{\mathbf{R}}\hat{D}(\alpha)}=\ldots$

$\boxed{F4'}$ ■ $\chi[\hat{D}(\alpha)\hat{O}\hat{D}^{\dagger}(\alpha)](\Lambda)=\mathrm{tr}(\hat{D}(\alpha)\hat{O}\hat{D}^{\dagger}(\alpha)\,D(\Lambda))$

$$=\mathrm{tr}(\hat{O}\,\hat{D}^{\dagger}(\alpha)\,D(\Lambda)\hat{D}(\alpha))=\ldots$$

$\boxed{F5'}$ ■ $W[\hat{D}(\alpha)\hat{O}\hat{D}^{\dagger}(\alpha)](X)=\frac{1}{2\pi^2}\int_{\mathbb{R}^2}d^2\Lambda\ \mathrm{e}^{-\mathrm{i}\Lambda^{T}\Omega X}\,\chi[\hat{O}](\Lambda)$

$\boxed{F6'}$ By using $\boxed{\mathfrak{C}5}$, $\boxed{B5'}$, $\boxed{C1'}$, $\boxed{C4'}$, we get

$$\begin{aligned}(\hat{Q}^n)^w(q,p)&=2\mathrm{tr}(\hat{Q}^n\Pi(q,p))\\ &=2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle x|\hat{Q}^n|y\rangle\langle y|\Pi(q,p)|x\rangle\,dxdy\\ &=2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}y^n\,\delta(y-x)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(y-x)}\delta(2q-x-y)\,dxdy\\ &=2\int_{-\infty}^{\infty}x^n\,\delta(2q-2x)\,dx=\int_{-\infty}^{\infty}x^n\,\delta(q-x)\,dx=q^n.\end{aligned}$$

$$\begin{aligned}(\hat{P}^n)^w(\Lambda)&=2\mathrm{tr}(\hat{P}^n\Pi(q,p))\\ &=2\mathrm{tr}((\hat{F}^{\dagger}\hat{Q}\hat{F})^n\Pi(q,p))\\ &=2\mathrm{tr}(\hat{F}^{\dagger}\hat{Q}^n\hat{F}\Pi(q,p))\\ &=2\mathrm{tr}(\hat{Q}^n\hat{F}\Pi(q,p)\hat{F}^{\dagger})\\ &=2\mathrm{tr}(\hat{Q}^n\Pi(p,-q))\\ &=(\hat{Q}^n)^w(p,-q)=p^n.\end{aligned}$$

$\boxed{E1'}$ ■ Since $\mathrm{e}^{\hat{A}}\hat{B}\mathrm{e}^{-\hat{A}}=\hat{B}+\frac{1}{1!}[\hat{A},\hat{B}]+\frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]]+\ldots$

$$\hat{A}=-\alpha\hat{a}^{+}+\bar{\alpha}\hat{a}\ \Rightarrow\ \begin{array}{l}[\hat{A},\hat{a}]=\alpha,\ [\hat{A},[\hat{A},\hat{a}]]=0,\ \ldots\\ [\hat{A},\hat{a}^{\dagger}]=\bar{\alpha},\ [\hat{A},[\hat{A},\hat{a}^{\dagger}]]=0,\ \ldots\end{array}\ \Rightarrow\ldots$$

DEFINITIONS USED IN THE CASE OF GAUSSIAN STATES

- Definitions &Relations from:
  - A. Ferraro *et al.*, Bibliopolis, Napoli, 2005.
  - Stefano Olivares, *Quantum optics in phase space*, Eur. Phys. J. Special Topics 203, 3-24 (2012).

$$\boxed{\hbar=1}\quad \boxed{\begin{array}{c}\hat q=\frac{1}{\sqrt2}(\hat a+\hat a^\dagger)\\ \hat p=-\frac{\mathrm i}{\sqrt2}(\hat a-\hat a^\dagger)\\ [\hat q,\hat p]=\mathrm i\end{array}}\quad \boxed{\begin{array}{c}\hat a=\frac{1}{\sqrt2}(\hat q+\mathrm i\hat p)\\ \hat a^\dagger=\frac{1}{\sqrt2}(\hat q-\mathrm i\hat p)\\ [\hat a,\hat a^\dagger]=1\end{array}}$$

$$\boxed{\chi[\varrho](\Lambda)=\mathrm e^{-\frac12\Lambda^T\Omega\sigma\Omega^T\Lambda-\mathrm i\Lambda^T\Omega\langle\hat R\rangle}}\qquad \boxed{\sigma_{k\ell}\stackrel{\text{def}}{=}\tfrac12\langle\hat R_k\,\hat R_\ell+\hat R_\ell\,\hat R_k\rangle-\langle\hat R_k\rangle\langle\hat R_\ell\rangle}$$

$$\boxed{W[\varrho](X)=\frac{1}{\pi\sqrt{\det\sigma}}\mathrm e^{-\frac12(X-\langle\hat R\rangle)^T\sigma^{-1}(X-\langle\hat R\rangle)}}\qquad \mathrm{tr}\,\rho^2=\frac{1}{2\sqrt{\det\sigma}},\quad \sigma+\frac{\mathrm i}{2}\begin{pmatrix}0&1\\-1&0\end{pmatrix}\ge0.$$

- Definitions &Relations used in:
  - G. Adesso *et al.*, Open Syst. Inf. Dyn. 21, 1440001 (2014),
  - X.-B. Wang *et al.*, Phys. Rep. 448 (2007), 1-111.

$$\boxed{\hbar=1}\quad \boxed{\begin{array}{c}\hat q=\frac{1}{\sqrt2}(\hat a+\hat a^\dagger)\\ \hat p=-\frac{\mathrm i}{\sqrt2}(\hat a-\hat a^\dagger)\\ [\hat q,\hat p]=\mathrm i\end{array}}\quad \boxed{\begin{array}{c}\hat a=\frac{1}{\sqrt2}(\hat q+\mathrm i\hat p)\\ \hat a^\dagger=\frac{1}{\sqrt2}(\hat q-\mathrm i\hat p)\\ [\hat a,\hat a^\dagger]=1\end{array}}$$

$$\boxed{\chi[\varrho](\Lambda)=\mathrm e^{-\frac14\Lambda^T\Omega\sigma\Omega^T\Lambda-\mathrm i\Lambda^T\Omega\langle\hat R\rangle}}\qquad \boxed{\sigma_{k\ell}\stackrel{\text{def}}{=}\langle\hat R_k\,\hat R_\ell+\hat R_\ell\,\hat R_k\rangle-2\,\langle\hat R_k\rangle\langle\hat R_\ell\rangle}$$

$$\boxed{W[\varrho](X)=\frac{1}{\pi\sqrt{\det\sigma}}\mathrm e^{-(X-\langle\hat R\rangle)^T\sigma^{-1}(X-\langle\hat R\rangle)}}\qquad \mathrm{tr}\,\rho^2=\frac{1}{\sqrt{\det\sigma}},\quad \sigma+\mathrm i\begin{pmatrix}0&1\\-1&0\end{pmatrix}\ge0.$$

- Definitions &Relations used in:
  - A. Ferraro *et al.*, Bibliopolis, Napoli, 2005.

$$\boxed{\hbar=\tfrac12}\quad \boxed{\begin{array}{c}\hat q=\frac12(\hat a+\hat a^\dagger)\\ \hat p=-\frac{\mathrm i}{2}(\hat a-\hat a^\dagger)\\ [\hat q,\hat p]=\frac{\mathrm i}{2}\end{array}}\quad \boxed{\begin{array}{c}\hat a=\hat q+\mathrm i\hat p\\ \hat a^\dagger=\hat q-\mathrm i\hat p\\ [\hat a,\hat a^\dagger]=1\end{array}}$$

$$\boxed{\chi[\varrho](\Lambda)=\mathrm e^{-\frac12\Lambda^T\Omega\sigma\Omega^T\Lambda-\mathrm i\Lambda^T\Omega\langle\hat R\rangle}}\qquad \boxed{\sigma_{k\ell}\stackrel{\text{def}}{=}\tfrac14\langle\hat R_k\,\hat R_\ell+\hat R_\ell\,\hat R_k\rangle-\langle\hat R_k\rangle\langle\hat R_\ell\rangle}$$

$$\boxed{W[\varrho](X)=\frac{1}{2\pi\sqrt{\det\sigma}}\mathrm e^{-\frac12(X-\langle\hat R\rangle)^T\sigma^{-1}(X-\langle\hat R\rangle)}}\qquad \mathrm{tr}\,\rho^2=\frac{1}{3\sqrt{\det\sigma}},\quad \sigma+\frac{\mathrm i}{4}\begin{pmatrix}0&1\\-1&0\end{pmatrix}\ge0.$$

- Definitions &Relations used in:
  - C. Weedbrook *et al.* , Rev. Mod. Phys. **84**, 621 (2012),
  - C. Navarrete-Benlloch, IOPscience, 2015.

$$\boxed{\hbar=2}\quad \boxed{\begin{array}{c}\hat q=\hat a+\hat a^\dagger\\ \hat p=-\mathrm i(\hat a-\hat a^\dagger)\\ [\hat q,\hat p]=\mathrm i\end{array}}\quad \boxed{\begin{array}{c}\hat a=\frac12(\hat q+\mathrm i\hat p)\\ \hat a^\dagger=\frac12(\hat q-\mathrm i\hat p)\\ [\hat a,\hat a^\dagger]=1\end{array}}$$

$$\boxed{\chi[\varrho](\Lambda)=\mathrm e^{-\frac12\Lambda^T\Omega\sigma\Omega^T\Lambda-\mathrm i\Lambda^T\Omega\langle\hat R\rangle}}\qquad \boxed{\sigma_{k\ell}\stackrel{\text{def}}{=}\tfrac12\langle\hat R_k\,\hat R_\ell+\hat R_\ell\,\hat R_k\rangle-\langle\hat R_k\rangle\langle\hat R_\ell\rangle}$$

$$\boxed{W[\varrho](X)=\frac{1}{2\pi\sqrt{\det\sigma}}\mathrm e^{-\frac12(X-\langle\hat R\rangle)^T\sigma^{-1}(X-\langle\hat R\rangle)}}\qquad \mathrm{tr}\,\rho^2=\text{........},\quad \sigma+\mathrm i\begin{pmatrix}0&1\\-1&0\end{pmatrix}\ge0.$$