ELEMENTS OF LINEAR ALGEBRA AND SOME APPLICATIONS

(Real and complex finite-dimensional vector spaces)

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Abstract. This is a tutorial[1] on Linear Algebra and some of its applications to the description of quantum systems.

A VECTOR SPACES (also called *LINEAR SPACES*)

■ $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$
natural numbers, integer numbers, rational numbers, real numbers, complex numbers

■ **Definition.** Let either $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$ ($\mathbb{K}$=field of scalars).

Vector space over $\mathbb{K}$ ($\mathbb{K}$-vector space) = $\mathcal{V}$ (space of vectors) + $\mathcal{V}\times\mathcal{V}\to\mathcal{V}:(x,y)\mapsto x+y$ (vector addition), $\mathbb{K}\times\mathcal{V}\to\mathcal{V}:(\alpha,x)\mapsto \alpha x$ (scalar multiplication)

satisfying the axioms:

1
1. $(x+y)+z=x+(y+z), \quad \forall x,y,z\in\mathcal{V}$
2. there exists $0\in\mathcal{V}$ such that $x+0=x, \quad \forall x\in\mathcal{V}$
3. $\forall\, x\in\mathcal{V}$ there exists $-x\in\mathcal{V}$ such that $x+(-x)=0$
4. $x+y=y+x, \quad \forall x,y\in\mathcal{V}$
5. $\alpha(x+y)=\alpha x+\alpha y, \quad \forall\alpha\in\mathbb{K}, \ \forall x,y\in\mathcal{V}$
6. $(\alpha+\beta)x=\alpha x+\beta x, \quad \forall\alpha,\beta\in\mathbb{K}, \ \forall x\in\mathcal{V}$
7. $\alpha(\beta x)=(\alpha\beta)x, \quad \forall\alpha,\beta\in\mathbb{K}, \ \forall x\in\mathcal{V}$
8. $1x=x, \quad \forall x\in\mathcal{V}$.

■ A *linear combination* of $v_1,v_2,...,v_n$ is any vector of the form $\alpha_1v_1+\alpha_2v_2+...+\alpha_nv_n$ with $\alpha_1,\alpha_2,\dots,\alpha_n\in\mathbb{K}$

■ **Theorem.**

2, 3, 4, 5 $\boxed{\mathcal{V} \text{ is a } \mathbb{K}\text{-vector space}}$, $x-y\overset{\text{def}}{=}x+(-y)$ $\Rightarrow$
$\alpha x=0 \ \Leftrightarrow \ \begin{cases}\alpha=0 & \text{or}\\ x=0\end{cases}$
$\alpha(-x)=(-\alpha)x=-\alpha x$
$\alpha(x-y)=\alpha x-\alpha y$
$(\alpha-\beta)x=\alpha x-\beta x$

■ **Theorem.** Let $\mathcal{V}$ be a $\mathbb{K}$-vector space and let $\mathcal{W}\subseteq\mathcal{V}$.

6 $\left.\begin{matrix}x,y\in\mathcal{W}\\ \alpha\in\mathbb{K}\end{matrix}\right\}\Rightarrow\left\{\begin{matrix}x+y\in\mathcal{W}\\ \alpha x\in\mathcal{W}\end{matrix}\right. \quad\Leftrightarrow\quad \left.\begin{matrix}x,y\in\mathcal{W}\\ \alpha,\beta\in\mathbb{K}\end{matrix}\right\}\Rightarrow \alpha x+\beta y\in\mathcal{W}$

■ **Definition.**

7 $\mathcal{W}$ is called a *subspace* of $\mathcal{V}$ if $\left.\begin{matrix}x,y\in\mathcal{W}\\ \alpha,\beta\in\mathbb{K}\end{matrix}\right\}\Rightarrow \alpha x+\beta y\in\mathcal{W}$.

■ **Theorem.**

8 Any subspace of $\mathcal{V}$ is itself a vector space.

9 Null space $\{0\}$ and $\mathcal{V}$ are subspaces of $\mathcal{V}$.

10 $\boxed{\mathrm{span}\{v_1,v_2,...,v_n\}\overset{\text{def}}{=}\{\alpha_1v_1+\alpha_2v_2+...+\alpha_nv_n \mid \alpha_k\in\mathbb{K}\}}$ is a subspace of $\mathcal{V}$ (subspace spanned by $\{v_1,v_2,...,v_n\}$).

■ **Definition.**

11 $\mathcal{S}=\{v_1,v_2,...,v_n\}$ is a *spanning set* of $\mathcal{V}$ ($\mathcal{S}$ *spans* $\mathcal{V}$) if $\boxed{\mathcal{V}=\mathrm{span}\{v_1,v_2,...,v_n\}}$

$\mathcal{V}$ is a *finite-dimensional* vector space if there exsits a finite set $\{v_1,...,v_n\}$ such that $\mathcal{V}=\mathrm{span}\{v_1,v_2,...,v_n\}$.

■ **Theorem.**

12 $\mathcal{S}=\{v_1,v_2,...,v_n\}$ spans $\mathcal{V}$, v_k is a linear combination of the other vectors $\Rightarrow$ $\mathcal{S}\backslash\{v_k\}$ spans $\mathcal{V}$.

13 None of $v_1,v_2,...,v_n$ is a linear combination of the other vectors $\Leftrightarrow$ $\alpha_1v_1+\alpha_2v_2+...+\alpha_nv_n=0 \Downarrow \alpha_1=\alpha_2=...=\alpha_n=0$

■ **Definition.**

14 $\{v_1,v_2,...,v_n\}$ is a *linearly independent* set of vectors if $\alpha_1v_1+\alpha_2v_2+...+\alpha_nv_n=0 \Downarrow \alpha_1=\alpha_2=...=\alpha_n=0$

■ **Theorem.**

15 $\mathcal{S}=\{v_1,v_2,...,v_n\}$ linearly independent $\Rightarrow$ $0\not\in\mathcal{S}$.

16 • $0\not\in\mathcal{S}=\{v_1,v_2,...,v_n\}$ and • v_k is not a linear combination of $v_1,v_2,...,v_{k-1}$, for any $2\le k\le n$ $\Leftrightarrow$ $\mathcal{S}$ is linearly independent.

■ **Definition.**

17 $\mathfrak{B}=\{v_1,v_2,...,v_n\}$ is a *basis* of $\mathcal{V}$ if $\mathfrak{B}$ spans $\mathcal{V}$ and $\mathfrak{B}$ linearly indep.

■ **Theorem.**

18 $\mathfrak{B}=\{v_1,v_2,...,v_n\}$ basis of $\mathcal{V}$ $\Rightarrow$
- any $x\in\mathcal{V}$ can be written as $\boxed{x=x_1v_1+x_2v_2+...+x_nv_n}$
- the coefficients $x_1,x_2,\dots,x_n$ of x are uniquely determined (*coordinates of* x *in basis* $\mathfrak{B}$).

19 *From any spanning set* $\mathcal{S}$ *of* $\mathcal{V}$, *we can extract a basis.*

20 *Any finite-dimensional vector space* $\mathcal{V}\neq\{0\}$ *admits a basis.*

21 • $\mathcal{M}=\{v_1,v_2,...,v_n\}$ linearly indep. • $\mathcal{S}=\{w_1,w_2,...,w_k\}$ spans $\mathcal{V}$ $\Rightarrow$ • $n\le k$; • $\mathcal{M}$ can be extended up to a basis of $\mathcal{V}$ by adding vectors from $\mathcal{S}$.

22 *Any two bases of a vector space have the same number of vectors.*

■ **Definition.**

23 *Dimension* of a vector space over $\mathbb{K}$ is $\dim_{\mathbb{K}}\mathcal{V}\overset{\text{def}}{=}\begin{cases}\text{number of vectors of a basis} & \text{if } \mathcal{V}\neq\{0\};\\ 0 & \text{if } \mathcal{V}=\{0\}.\end{cases}$

■ **Theorem.**

24 $\mathcal{W}\subseteq\mathcal{V}$ is a subspace $\Rightarrow$ $\dim\mathcal{W}\le\dim\mathcal{V}$.

25 $\mathcal{W}\subseteq\mathcal{V}$ and $\dim\mathcal{W}=\dim\mathcal{V}$ $\Rightarrow$ $\mathcal{W}=\mathcal{V}$.

■ **Theorem.**

26 $\mathcal{W}_1,\mathcal{W}_2\subseteq\mathcal{V}$ subspaces $\not\Rightarrow$ $\mathcal{W}_1\cup\mathcal{W}_2$ is a subspace.

27 $\mathcal{W}_1,\mathcal{W}_2\subseteq\mathcal{V}$ subspaces $\Rightarrow$ $\mathcal{W}_1\cap\mathcal{W}_2$ is a subspace.

28 $\mathrm{span}\{v_1,v_2,...,v_n\}$ is the intersection of all the subspaces containing $\{v_1,v_2,...,v_n\}$.

■ **Theorem.**

29 $\mathcal{W}_1,\mathcal{W}_2\subseteq\mathcal{V}$ subspaces $\Rightarrow$
- $\mathcal{W}_1+\mathcal{W}_2=\left\{w_1+w_2 \;\middle|\; \begin{matrix}w_1\in\mathcal{W}_1\\ w_2\in\mathcal{W}_2\end{matrix}\right\}$ is a subspace of $\mathcal{V}$;
- $\dim(\mathcal{W}_1+\mathcal{W}_2)=\dim\mathcal{W}_1+\dim\mathcal{W}_2-\dim(\mathcal{W}_1\cap\mathcal{W}_2)$.

30 The representation of any $x\in\mathcal{W}_1+\mathcal{W}_2$ as $x=w_1+w_2$ with $w_k\in\mathcal{W}_k$ is unique $\Leftrightarrow$ $\mathcal{W}_1\cap\mathcal{W}_2=\{0\}$.

■ **Definition.**

31 The *sum of the subspaces* $\mathcal{W}_1,\mathcal{W}_2\subseteq\mathcal{V}$ is $\mathcal{W}_1+\mathcal{W}_2\overset{\text{def}}{=}\left\{w_1+w_2 \;\middle|\; \begin{matrix}w_1\in\mathcal{W}_1\\ w_2\in\mathcal{W}_2\end{matrix}\right\}$

32 $\mathcal{W}_1+\mathcal{W}_2$ is called a *direct sum* and denoted $\mathcal{W}_1\oplus\mathcal{W}_2$ if $\mathcal{W}_1\cap\mathcal{W}_2=\{0\}$.

33 $\mathcal{W}_2$ is a *complement* of $\mathcal{W}_1$ in $\mathcal{V}$ if $\mathcal{V}=\mathcal{W}_1\oplus\mathcal{W}_2$.

■ **Theorem.**

34 $\mathcal{V}_1,\mathcal{V}_2$ vector spaces over the same field $\mathbb{K}$ $\Rightarrow$
- $\mathcal{V}=\mathcal{V}_1\times\mathcal{V}_2$ with $(x_1,x_2)+(y_1,y_2)=(x_1+y_1,x_2+y_2)$, $\lambda(x_1,x_2)=(\lambda x_1,\lambda x_2)$ is a vector space.
- $\dim(\mathcal{V}_1\times\mathcal{V}_2)=\dim\mathcal{V}_1+\dim\mathcal{V}_2$.

[1] Version 16 Oct. 2021 (for future updates see https://unibuc.ro/user/nicolae.cotfas/)

■ **Theorem.**

35 By *restriction of scalars* from $\mathbb{C}$ to $\mathbb{R}$, a complex vector space $\mathcal{V}$ becomes a real vector space $\boxed{\dim_{\mathbb{R}}\mathcal{V}=2\dim_{\mathbb{C}}\mathcal{V}}$

36 By *extension of scalars* from $\mathbb{R}$ to $\mathbb{C}$, a complex vector space $\mathcal{V}^{\mathbb{C}}$ is obtained from a real space $\mathcal{V}$

- $\mathcal{V}^{\mathbb{C}}=\mathcal{V}\times\mathcal{V}$ with
  $(x,y)+(x',y')=(x+x',y+y')$
  $(\alpha+\beta\mathrm{i})(x,y)=(\alpha x-\beta y,\alpha y+\beta x)$
- $\dim_{\mathbb{C}}\mathcal{V}^{\mathbb{C}}=\dim_{\mathbb{R}}\mathcal{V}$

37 $\mathcal{W}\subseteq\mathcal{V}$ subspace $\Rightarrow$

- $x\sim y$ if $x-y\in\mathcal{W}$ is an equivalence relation.
- The set of equivalence classes $\mathcal{V}/\mathcal{W}=\{\,\hat{x}=x+\mathcal{W}\mid x\in\mathcal{V}\,\}$ with $\hat{x}+\hat{y}=\widehat{x+y}$, $\lambda\hat{x}=\widehat{\lambda x}$, is a vector space (*factor space*).
- $\dim\mathcal{V}/\mathcal{W}=\dim\mathcal{V}-\dim\mathcal{W}$.

■ **Remark**. The vector space

38 $$\mathbb{K}^n=\{(x_1,x_2,...,x_n)\mid\ x_1,x_2,...,x_n\in\mathbb{K}\,\}$$

can as well be regarded as the space of column matrices

39 $$\mathbb{K}^n=\left\{\begin{pmatrix}x^1\\x^2\\\vdots\\x^n\end{pmatrix}\ \middle|\ x^1,x^2,...,x^n\in\mathbb{K}\right\},$$

where we have used superscripts instead of subscripts.

■ **Notation** (*Einstein summation convention*).

Representation of a vector with respect to an ordered basis

40 $$x=x^1v_1+x^2v_2+\cdots+x^nv_n=\sum_{k=1}^{n}x^kv_k$$

can be written as

41 $\boxed{x=x^kv_k}$ where $\boxed{x^kv_k\overset{\text{def}}{=}x^1v_1+x^2v_2+\cdots+x^nv_n}$

The numbers $x^1,x^2,...,x^n$ are the *coordinates* of x.

■ **Definition.** Given two bases of the $\mathbb{K}-$vector space $\mathcal{V}$

42 $\mathfrak{B}=\{v_1,v_2,...,v_n\}$ (the "old" basis),
$\mathfrak{B}'=\{v_1',v_2',...,v_n'\}$ (the "new" basis),

the vectors v_j' can be expressed in terms of v_k

43 $$\begin{cases}v_1'=\alpha_1^1v_1+\alpha_1^2v_2+...+\alpha_1^nv_n\\ v_2'=\alpha_2^1v_1+\alpha_2^2v_2+...+\alpha_2^nv_n\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ v_n'=\alpha_n^1v_1+\alpha_n^2v_2+...+\alpha_n^nv_n\end{cases}\quad\text{that is}\quad\boxed{v_j'=\alpha_j^kv_k}$$

The matrix

44 $$\mathbf{S}=\begin{pmatrix}\alpha_1^1&\alpha_2^1&\cdots&\alpha_n^1\\\alpha_1^2&\alpha_2^2&\cdots&\alpha_n^2\\\vdots&\vdots&\ddots&\vdots\\\alpha_1^n&\alpha_2^n&\cdots&\alpha_n^n\end{pmatrix}$$

is the *change of basis matrix* from $\mathfrak{B}$ to $\mathfrak{B}'$.

■ **Theorem.** The inverse matrix

45 $$\mathbf{S}^{-1}=\begin{pmatrix}\beta_1^1&\beta_2^1&\cdots&\beta_n^1\\\beta_1^2&\beta_2^2&\cdots&\beta_n^2\\\vdots&\vdots&\ddots&\vdots\\\beta_1^n&\beta_2^n&\cdots&\beta_n^n\end{pmatrix}\ \text{satisfying}\ \boxed{\beta_k^m\alpha_j^k=\delta_j^m}\ \boxed{\alpha_m^j\beta_k^m=\delta_k^j}\ \boxed{v_k=\beta_k^jv_j'}$$

is the *change of basis matrix* from $\mathfrak{B}'$ to $\mathfrak{B}$.

■ **Theorem** (New coordinates x'^j in terms of the old ones).

46 $$\boxed{\begin{aligned}v_j'&=\alpha_j^kv_k\\ x=x^kv_k&=x'^jv_j'\\ v_k&=\beta_k^jv_j'\end{aligned}}\Rightarrow\ \boxed{x'^j=\beta_k^jx^k}\quad x^k=\alpha_j^kx'^j\qquad\boxed{\begin{pmatrix}x'^1\\x'^2\\\vdots\\x'^n\end{pmatrix}=\mathbf{S}^{-1}\begin{pmatrix}x^1\\x^2\\\vdots\\x^n\end{pmatrix}}$$

B LINEAR MAPS AND MATRICES

■ **Definition.** Let $\mathcal{V},\mathcal{W}$ be vector spaces over the same field $\mathbb{K}$.

1 A *linear map* from $\mathcal{V}$ to $\mathcal{W}$ (linear mapping, linear operator) is a map $A:\mathcal{V}\to\mathcal{W}$ satisfying the relation $\boxed{A(\alpha x+\beta y)=\alpha Ax+\beta Ay}$ for any $x,y\in\mathcal{V}$ and $\alpha,\beta\in\mathbb{K}$.

2 *Notation.* $\mathcal{L}(\mathcal{V},\mathcal{W})=$ set of all the linear maps $A:\mathcal{V}\to\mathcal{W}$.
$\mathrm{End}(\mathcal{V})\equiv\mathcal{L}(\mathcal{V})=$ set of all the linear maps $A:\mathcal{V}\to\mathcal{V}$.

■ **Theorem.**

$A:\mathcal{V}\to\mathcal{W}$ linear map $\Rightarrow$

3 • *Kernel* $\boxed{\mathrm{Ker}\,A=\{x\in\mathcal{V}\mid Ax=0\}}$ is a subspace of $\mathcal{V}$.

4 • *Image* (range) $\boxed{\mathrm{Im}\,A=\{Ax\mid x\in\mathcal{V}\}}$ is a subspace of $\mathcal{W}$.

5 • $\dim\mathcal{V}=\dim\mathrm{Ker}\,A+\dim\mathrm{Im}\,A$

■ **Theorem.**

6 Linear map $A:\mathcal{V}\to\mathcal{W}$ is injective $\Leftrightarrow$ $\mathrm{Ker}\,A=\{0\}$.

7 Linear map $A:\mathcal{V}\to\mathcal{W}$ is surjective $\Leftrightarrow$ $\mathrm{Im}\,A=\mathcal{W}$.

8 Linear map $A:\mathcal{V}\to\mathcal{W}$ is bijective $\Rightarrow A^{-1}:\mathcal{W}\to\mathcal{V}$ is linear.

■ **Definition.**

- An *isomorphism* of vector spaces is a bijective linear map.
- Two vector spaces are *isomorphic spaces* if an isomorphism exists between them.

■ **Theorem.**

9 $\boxed{\text{Vector spaces } \mathcal{V},\mathcal{W} \text{ over the same field are isomorphic}}\Leftrightarrow\boxed{\dim\mathcal{V}=\dim\mathcal{W}}$

10 $\boxed{\mathfrak{B}=\{v_1,v_2,...,v_n\}$ is an ordered basis in the $\mathbb{K}$-vector space $\mathcal{V}} \Rightarrow \boxed{\mathcal{V}\longrightarrow\mathbb{K}^n:\ x=x^kv_k\mapsto(x^1,x^2,...,x^n)$ is an isomorphism$}$

■ **Definition.**

11 $\mathfrak{B}_v=\{v_1,v_2,...,v_n\}$ basis in $\mathcal{V}$
$\mathfrak{B}_w=\{w_1,...,w_m\}$ basis in $\mathcal{W}$
$A:\mathcal{V}\to\mathcal{W}$ linear map
$\boxed{Av_k=a_k^jw_j}$

$$\mathbf{A}=\begin{pmatrix}a_1^1&a_2^1&\cdots&a_n^1\\a_1^2&a_2^2&\cdots&a_n^2\\\vdots&\vdots&\ddots&\vdots\\a_1^m&a_2^m&\cdots&a_n^m\end{pmatrix}$$

is the *matrix of* A with respect to $\mathfrak{B}_v$ and $\mathfrak{B}_w$.

■ **Remark.** For $x=x^kv_k$, $y=y^jw_j$, in matrix representation

12 $\boxed{y=Ax}$ $y^j=a_k^jx^k$ becomes $$\begin{pmatrix}y^1\\y^2\\\vdots\\y^m\end{pmatrix}=\begin{pmatrix}a_1^1&a_2^1&\cdots&a_n^1\\a_1^2&a_2^2&\cdots&a_n^2\\\vdots&\vdots&\ddots&\vdots\\a_1^m&a_2^m&\cdots&a_n^m\end{pmatrix}\begin{pmatrix}x^1\\x^2\\\vdots\\x^n\end{pmatrix}.$$

■ **Theorem.** Let $\mathfrak{B}=\{v_1,...,v_n\}$, $\mathfrak{B}'=\{v_1',...,v_n'\}$ be bases in $\mathcal{V}$.

13 $A:\mathcal{V}\to\mathcal{V}$ linear map
$Av_k=a_k^jv_j$
$Av_k'=a'^j_kv_j'$
$v_k'=\alpha_k^jv_j$

$$\mathbf{A}=\begin{pmatrix}a_1^1&\cdots&a_n^1\\\vdots&\ddots&\vdots\\a_1^n&\cdots&a_n^n\end{pmatrix},\quad\mathbf{A}'=\begin{pmatrix}a'^1_1&\cdots&a'^1_n\\\vdots&\ddots&\vdots\\a'^n_1&\cdots&a'^n_n\end{pmatrix},$$

$$\mathbf{S}=\begin{pmatrix}\alpha_1^1&\cdots&\alpha_n^1\\\vdots&\ddots&\vdots\\\alpha_1^n&\cdots&\alpha_n^n\end{pmatrix}\quad\Rightarrow\quad\boxed{\mathbf{A}'=\mathbf{S}^{-1}\mathbf{A}\mathbf{S}}.$$

■ **Theorem.** $\mathfrak{B}_u=\{u_1,...,u_n\}$, $\mathfrak{B}_v=\{v_1,...,v_m\}$, $\mathfrak{B}_w=\{w_1,...,w_k\}$.

14 $\mathcal{U}\overset{A}{\to}\mathcal{V}\overset{B}{\to}\mathcal{W}$ linear maps
$Au_k=a_k^jv_j$
$Bv_k=b_k^jw_j$
$C=BA$
$Cu_k=c_k^jw_j$

$$\mathbf{A}=\begin{pmatrix}a_1^1&\cdots&a_n^1\\\vdots&\ddots&\vdots\\a_1^m&\cdots&a_n^m\end{pmatrix},\quad\mathbf{B}=\begin{pmatrix}b_1^1&\cdots&b_m^1\\\vdots&\ddots&\vdots\\b_1^k&\cdots&b_m^k\end{pmatrix},$$

$$\mathbf{C}=\begin{pmatrix}c_1^1&\cdots&\alpha_n^1\\\vdots&\ddots&\vdots\\c_1^k&\cdots&\alpha_n^k\end{pmatrix}\quad\Rightarrow\quad\boxed{\mathbf{C}=\mathbf{B}\mathbf{A}}.$$

■ **Definition.** Let $\mathcal{V}$ be a vector space over $\mathbb{K}$.

15 λ is an *eigenvalue* of the linear operator $A:\mathcal{V}\to\mathcal{V}$ if
- $\lambda\in\mathbb{K}$ and
- there exists $v\in\mathcal{V}$, $v\neq 0$ (*eigenvector* corresp. to λ) such that $\boxed{Av=\lambda v}$.

■ **Theorem.**

16 λ *eigenvalue* of linear operator $A:\mathcal{V}\to\mathcal{V}$ $\Rightarrow$ $\boxed{\mathcal{V}_\lambda=\{\,x\in\mathcal{V}\mid Ax=\lambda x\,\}}$ is a subspace of $\mathcal{V}$ (*eigenspace* corresp. to λ).

$\mathfrak{B}=\{v_1,v_2,\dots,v_n\}$ basis in $\mathcal{V}$. $A:\mathcal{V}\to\mathcal{V}$ linear operator. $Av_k=a^j_k v_j$ $\Rightarrow$

17 • *Characteristic polynomial* of A $\boxed{P(\lambda)=\det(\mathbf{A}-\lambda\mathbf{I})}$,

$$P(\lambda)=\begin{vmatrix} a^1_1-\lambda & a^1_2 & \cdots & a^1_n\\ a^2_1 & a^2_2-\lambda & \cdots & a^2_n\\ \vdots & \vdots & \ddots & \vdots\\ a^n_1 & a^n_2 & \cdots & a^n_n-\lambda\end{vmatrix}$$

18 • *Trace* of A

$$\boxed{\operatorname{tr}A=a^1_1+a^2_2+\cdots+a^n_n}$$

19 • *Determinant* of A

$$\boxed{\det A=\begin{vmatrix} a^1_1 & \cdots & a^1_n\\ \vdots & \ddots & \vdots\\ a^n_1 & \cdots & a^n_n\end{vmatrix}}$$

defined by using the matrix of A with respect to $\mathfrak{B}$, do not depend on the basis $\mathfrak{B}$ we chose.

■ **Theorem.** Let $\mathcal{V}$ be a $\mathbb{K}$-vector space.

20 • $A,B:\mathcal{V}\to\mathcal{V}$ linear operators $\Rightarrow$ $\boxed{\operatorname{tr}(AB)=\operatorname{tr}(BA)}$

21 • $A,B,C:\mathcal{V}\to\mathcal{V}$ linear operators $\Rightarrow$ $\boxed{\operatorname{tr}(ABC)=\operatorname{tr}(BCA)}$

22 • λ is an eigenvalue of the linear operator $A:\mathcal{V}\to\mathcal{V}$ $\Leftrightarrow$ • $\lambda\in\mathbb{K}$ and • $P(\lambda)=0$, that is $\det(\mathbf{A}-\lambda\mathbf{I})=0$.

■ **Definition.**

23 An *invariant subspace* of a linear operator $A:\mathcal{V}\to\mathcal{V}$ (A-invariant subspace) is a subspace $\mathcal{W}\subset\mathcal{V}$ such that $A(\mathcal{W})\subset\mathcal{W}$.

■ **Theorem.**

24 λ is an eigenvalue of the linear operator $A:\mathcal{V}\to\mathcal{V}$ $\Rightarrow$ $\mathcal{V}_\lambda$ is A-invariant.

25 Matrix of a linear operator $A:\mathcal{V}\to\mathcal{V}$ relative to a basis $\mathfrak{B}=\{v_1,v_2,\dots,v_n\}$ is a diagonal matrix $\Leftrightarrow$ All the vectors $v_1,v_2,\dots,v_n$ are eigenvectors of A

26 $v_1,v_2,\dots,v_n$ are eigenvectors of a linear operator $A:\mathcal{V}\to\mathcal{V}$ corresponding to distinct eigenvalues $\Rightarrow$ $\{v_1,v_2,\dots,v_n\}$ is a linearly independent set.

■ **Definition.** Let $\mathcal{V}$ be a vector space over $\mathbb{K}$.

27 Linear operator $A:\mathcal{V}\to\mathcal{V}$ is *diagonalizable* if there exists an ordered basis of $\mathcal{V}$ consisting of eigenvectors of A.

28 *Algebraic multiplicity* of an eigenvalue λ of A is the number of times λ appears as a root of the characteristic polynomial.

29 *Geometric multiplicity* of an eigenvalue λ of A is the dimension of the corresponding eigenspace $\mathcal{V}_\lambda$.

■ **Theorem.**

30 Linear operator $A:\mathcal{V}\to\mathcal{V}$ is *diagonalizable* $\Leftrightarrow$ All the roots of the characteristic polynomial belong to $\mathbb{K}$, and for each of them, the algebraic and geometric multiplicities are equal.

C TENSORS

■ **Definition.** Let $\mathcal{V}$ be a vector space over the field $\mathbb{K}$.

1 *Linear functional* (or *linear form*) is a linear map $\varphi:\mathcal{V}\to\mathbb{K}$.

■ **Theorem.**

2 $\boxed{\begin{array}{r}\mathcal{V}^*=\{\,\varphi:\mathcal{V}\to\mathbb{K}\mid \text{linear form}\,\}\\ (\varphi+\psi)(x)=\varphi(x)+\psi(x)\\ (\lambda\varphi)(x)=\lambda\varphi(x)\end{array}}$ is a vector space (*dual space* of $\mathcal{V}$) and $\dim\mathcal{V}^*=\dim\mathcal{V}$.

■ **Definition.**

3 *Dual basis* of $\mathfrak{B}=\{v_1,v_2,\dots,v_n\}$ is $\boxed{\begin{array}{c}\mathfrak{B}^*=\{v^1,v^2,\dots,v^n\},\\ v^j:\mathcal{V}\to\mathbb{K},\quad v^j(v_k)=\delta^j_k,\\ \text{that is}\quad v^j(x^k v_k)=x^j.\end{array}}$

■ **Theorem.**

4 $\mathfrak{B}=\{v_1,v_2,\dots,v_n\}$ basis in $\mathcal{V}$ $\Rightarrow$ $\boxed{\text{Matrix of } \varphi\in\mathcal{V}^* \text{ relative to } \mathfrak{B} \text{ is } (\,\varphi(v_1)\ \ \varphi(v_2)\ \ \dots\ \ \varphi(v_n)\,).}$

5 Identification of $\mathcal{V}$ with its *bidual* $\mathcal{V}^{**}$ $\boxed{\mathcal{V}\to(\mathcal{V}^*)^*: x\mapsto \boxed{\begin{array}{c}\Psi_x:\mathcal{V}^*\to\mathbb{K}\\ \Psi_x(\varphi)=\varphi(x)\end{array}} \text{ is isomorphism}}$

■ **Theorem.**

6 $\mathfrak{B}=\{v_1,v_2,\dots,v_n\}$, $\quad v'_j=\alpha^k_j v_k$
$\mathfrak{B}'=\{v'_1,v'_2,\dots,v'_n\}$, $\quad v_k=\beta^j_k v'_j$
$\mathfrak{B}^*=\{v^1,v^2,\dots,v^n\}$ dual of $\mathfrak{B}$
$\mathfrak{B}'^*=\{v'^1,v'^2,\dots,v'^n\}$ dual of $\mathfrak{B}'$

7 $x=x^i v_i=x'^j v'_j$
$\varphi=\varphi_m v^m=\varphi'_k v'^k$

8 $A:\mathcal{V}\to\mathcal{V}$ linear operator
$Av_m=a^i_m v_i$, $\quad Av'_k=a'^j_k v'_j$

$\Rightarrow$

$v'^j=\beta^j_i v^i$

$\boxed{x'^j=\beta^j_i x^i}$

$\boxed{\varphi'_k=\alpha^m_k\varphi_m}$

$\boxed{a'^j_k=\beta^j_i\alpha^m_k a^i_m}$

■ **Definition.**

A (p,q)*-tensor* (p contravariant and q covariant indices) is a mathematical object T described relative to a basis $\mathfrak{B}$ by its coordinates $T^{j_1j_2\dots j_p}_{k_1k_2\dots k_q}$ which, under a basis change $\{v_1,\dots,v_n\}\mapsto\{v'_1,\dots,v'_n\}$, $\begin{array}{l}v'_j=\alpha^k_j v_k\\ v_k=\beta^j_k v'_j\end{array}$ transform as

9 $$\boxed{T'^{j_1j_2\dots j_p}_{k_1k_2\dots k_q}=\beta^{j_1}_{i_1}\beta^{j_2}_{i_2}\dots\beta^{j_p}_{i_p}\alpha^{m_1}_{k_1}\alpha^{m_2}_{k_2}\dots\alpha^{m_q}_{k_q}T^{i_1i_2\dots i_p}_{m_1m_2\dots m_q}}$$

■ **Theorem.**
- Elements of $\mathcal{V}$ are tensors of type $(1,0)$.
- Elements of $\mathcal{V}^*$ are tensors of type $(0,1)$.
- Any linear operator $A:\mathcal{V}\to\mathcal{V}$ is a tensor of type (1,1).
- Any multilinear map

10 $$T:\underbrace{\mathcal{V}^*\times\mathcal{V}^*\times\cdots\times\mathcal{V}^*}_{p\text{ times}}\times\underbrace{\mathcal{V}\times\mathcal{V}\times\cdots\times\mathcal{V}}_{q\text{ times}}\longrightarrow\mathbb{K}$$

is a tensor of type (p,q) with

11 $$T^{j_1j_2\dots j_p}_{k_1k_2\dots k_q}=T(v^{j_1},\dots,v^{j_p},v_{k_1},\dots,v_{k_q})$$

and any tensor corresponds to such a map.

■ **Theorem.** The following relations define tensors.

- *Tensor addition*

12 $$\boxed{(T+S)^{j_1j_2\dots j_p}_{k_1k_2\dots k_q}=T^{j_1j_2\dots j_p}_{k_1k_2\dots k_q}+S^{j_1j_2\dots j_p}_{k_1k_2\dots k_q}}$$

- *Scalar multiplication*

13 $$\boxed{(\lambda T)^{j_1j_2\dots j_p}_{k_1k_2\dots k_q}=\lambda\,T^{j_1j_2\dots j_p}_{k_1k_2\dots k_q}}$$

- *Tensor product*

14 $$\boxed{(T\otimes S)^{j_1\dots j_p i_1\dots i_r}_{k_1\dots k_q m_1\dots m_s}=T^{j_1\dots j_p}_{k_1\dots k_q}\,S^{i_1\dots i_r}_{m_1\dots m_s}}$$

- *Tensor* (r,s) *contraction*

15 $$\boxed{S^{j_1\dots j_{r-1}j_{r+1}\dots j_p}_{k_1\dots k_{s-1}k_{s+1}\dots k_q}=\sum_{m=1}^{n}T^{j_1\dots j_{r-1}mj_{r+1}\dots j_p}_{k_1\dots k_{s-1}mk_{s+1}\dots k_q}}$$

D FINITE-DIMENSIONAL HILBERT SPACES

■ **Definition.** Let either $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$ ($\mathbb{K}$=field of scalars).

1 $\dfrac{\textit{Inner product space}}{\text{over the field } \mathbb{K}} = \boxed{\begin{array}{c}\mathcal{H}\\ \text{(vector}\\ \text{space)}\end{array}} + $ *inner product* $\boxed{\mathcal{H}\times\mathcal{H}\rightarrow\mathbb{K}:(x,y)\mapsto\langle x,y\rangle}$ satisfying the relations:
$\langle x,\alpha y+\beta z\rangle=\alpha\langle x,y\rangle+\beta\langle x,z\rangle$,
$\langle x,y\rangle=\overline{\langle y,x\rangle}$, for any x,y,z,
$\langle x,x\rangle>0$, for $x\neq 0$.

■ **Remark.** One can prove that:

2 Any finite-dimensional inner product space is a Hilbert space.

3 $\boxed{\begin{array}{c}\mathcal{H}\times\mathcal{H}\rightarrow\mathbb{K}:\\(x,y)\mapsto\langle x,y\rangle\\ \text{inner product}\end{array}} \Rightarrow$ $\boxed{\mathcal{H}\rightarrow\mathbb{R}:x\mapsto||x||=\sqrt{\langle x,x\rangle}}$ is a norm.

4 $\boxed{\mathcal{H}\times\mathcal{H}\rightarrow\mathbb{R}:(x,y)\mapsto||x-y||}$ is a distance.

■ **Theorem** (Cauchy-Schwarz).

5 $\boxed{\begin{array}{c}\mathcal{H}\times\mathcal{H}\rightarrow\mathbb{K}:\\(x,y)\mapsto\langle x,y\rangle\\ \text{inner product}\end{array}} \Rightarrow$
- $\boxed{|\langle x,y\rangle|\leq||x||\,||y||}$ for any $x,y\in\mathcal{H}$,
- For non-null vectors: $|\langle x,y\rangle|=||x||\,||y|| \;\Leftrightarrow$ there exists $\lambda\in\mathbb{K}$ such that $x=\lambda y$.

■ **Definition.**

6 $x\in\mathcal{H}$ is a *unit vector* if $||x||=\sqrt{\langle x,x\rangle}=1$.

7 $x,y\in\mathcal{H}$ are *orthogonal vectors* $\boxed{x\perp y \quad\Leftrightarrow\quad \langle x,y\rangle=0.}$

8 $\{v_1,v_2,...,v_n\}$ is an *orthogonal set* (list, system) if $\langle v_j,v_k\rangle=0$ for $j\neq k$.

9 $\{v_1,...,v_n\}$ is an *orthonormal set* (list, system) if $\langle v_j,v_k\rangle=\delta_{jk}=\begin{cases}1 \text{ for } j=k,\\ 0 \text{ for } j\neq k.\end{cases}$

10 A basis $\{e_1,e_2,...,e_n\}$ is an *orthonormal basis* if $\langle e_j,e_k\rangle=\delta_{jk}$.

■ **Theorem.**

11 $\langle x,y\rangle=0 \quad\Rightarrow\quad ||x+y||^2=||x||^2+||y||^2 \quad (\textit{Pythagoras}).$

12 $\left.\begin{array}{r}\{v_1,v_2,...,v_n\} \text{ orthogonal}\\ 0\notin\{v_1,v_2,...,v_n\}\end{array}\right\}\Rightarrow\{v_1,v_2,...,v_n\}$ linearly indep.

13 *Orthogonal projection* of x on $w\neq 0$ is $P_w x=\frac{\langle w,x\rangle}{\langle w,w\rangle}w$.

14 *Orthogonal projection* on a unit vector u is $P_u x=\langle u,x\rangle u$.

■ **Theorem** (Gram-Schmidt).

15 $\{v_1,...,v_n\}$ linearly indep. $\Rightarrow$
- $\{w_1,w_2,...,w_n\}$, where
  $w_1=v_1$,
  $w_2=v_2-\frac{\langle w_1,v_2\rangle}{\langle w_1,w_1\rangle}w_1$,
  ....
  $w_n=v_n-\frac{\langle w_1,v_n\rangle}{\langle w_1,w_1\rangle}w_1-\cdots-\frac{\langle w_{n-1},v_n\rangle}{\langle w_{n-1},w_{n-1}\rangle}w_{n-1}$,
  is an orthogonal set such that
- span $\{w_1,w_2,...,w_k\}$=span $\{v_1,v_2,...,v_k\}$, for any $k\in\{1,2,...,n\}$.

16 $\{v_1,...,v_n\}$ linearly indep. $\Rightarrow$
- $\{u_1,u_2,...,u_n\}$, where
  $u_1=\frac{v_1}{||v_1||}$,
  $u_2=\frac{v_2-\langle u_1,v_2\rangle u_1}{||v_2-\langle u_1,v_2\rangle u_1||}$,
  ....
  $u_n=\frac{v_n-\langle u_1,v_n\rangle u_1-\langle u_2,v_n\rangle u_2-\cdots-\langle u_{n-1},v_n\rangle u_{n-1}}{||v_n-\langle u_1,v_n\rangle u_1-\langle u_2,v_n\rangle u_2-\cdots-\langle u_{n-1},v_n\rangle u_{n-1}||}$,
  is an orthonormal set such that
- span $\{u_1,u_2,...,u_k\}$=span $\{v_1,v_2,...,v_k\}$, for any $k\in\{1,2,...,n\}$.

■ **Theorem.**

17 $\boxed{\begin{array}{c}\{e_1,e_2...,e_n\}\\ \text{orthonormal}\\ \text{basis in } \mathcal{H}\end{array}} \Rightarrow$ For any $x,y\in\mathcal{H}$ we have:
- $x=\sum_{k=1}^{n}\langle e_k,x\rangle e_k$,
- $\langle x,y\rangle=\sum_{k=1}^{n}\langle x,e_k\rangle\langle e_k,y\rangle$,
- $||x||=\sqrt{\sum_{k=1}^{n}|\langle e_k,x\rangle|^2}$.

■ **Theorem.**

18 $\left.\begin{array}{c}\mathcal{M}\subseteq\mathcal{H}\\ \text{subset}\end{array}\right\}\Rightarrow$ $\boxed{\mathcal{M}^{\perp}=\{\,x\in\mathcal{H}\mid\langle x,y\rangle=0,\ \forall y\in\mathcal{M}\,\}}$ is a vector subspace of $\mathcal{H}$ and $\boxed{\dim\mathcal{M}^{\perp}=\dim\mathcal{H}-\dim(\text{span}\,\mathcal{M}).}$

19 $\left.\begin{array}{c}\mathcal{K}\subseteq\mathcal{H}\\ \text{subspace}\end{array}\right\}\Rightarrow\boxed{\mathcal{H}=\mathcal{K}\oplus\mathcal{K}^{\perp}}$

■ **Theorem.**

20 $\left.\begin{array}{c}\langle x,v\rangle=\langle y,v\rangle\\ \text{for any } v\in\mathcal{H}\end{array}\right\}\Rightarrow x=y.$

21 For any $\varphi\in\mathcal{H}^*$ $\boxed{\text{there exists } a\in\mathcal{H} \text{ uniquely determined and such that } \varphi(x)=\langle a,x\rangle, \text{ for all } x\in\mathcal{H}}$

22 The dual of $\mathcal{H}$ is $\boxed{\mathcal{H}^*=\{\,\mathcal{H}\rightarrow\mathbb{K}:x\mapsto\langle a,x\rangle\mid a\in\mathcal{H}\,\}}$

■ **Notation** (*Dirac's "bra-ket" notation*).

23 $\boxed{\begin{array}{c}\varphi(x)=\langle a,x\rangle\\ \text{for any } x\in\mathcal{H}\end{array}}$ suggests us the notations $\boxed{\begin{array}{c}\langle a| \text{ for } \varphi\\ |x\rangle \text{ for } x\end{array}}$

24 Thus, $\boxed{\begin{array}{c}\mathcal{H}\equiv\{\,|x\rangle\mid x\in\mathcal{H}\,\}\\ \mathcal{H}^*\equiv\{\,\langle a|\mid a\in\mathcal{H}\,\}\end{array}}$ and $\boxed{\langle a,x\rangle\equiv\langle a|x\rangle}$

25 $x=\sum_{k=1}^{n}\langle e_k,x\rangle e_k$ becomes $|x\rangle=\sum_{k=1}^{n}|e_k\rangle\langle e_k|x\rangle$

26 and can be written as (*resolution of the identity*) $\boxed{\mathbb{I}=\sum_{k=1}^{n}|e_k\rangle\langle e_k|}$

27 $\begin{array}{c}A:\mathcal{H}\longrightarrow\mathcal{H}\\ A|x\rangle=|a\rangle\langle b|x\rangle\end{array}$ can be written as $\boxed{A=|a\rangle\langle b|}$

28 $\begin{array}{c}P_u:\mathcal{H}\longrightarrow\mathcal{H}\\ P_u|x\rangle=|u\rangle\langle u|x\rangle\end{array}$ can be written as $\boxed{P_u=|u\rangle\langle u|}$

■ **Remarks.**

29 $\boxed{\begin{array}{c}\mathfrak{B}=\{|e_1\rangle,|e_2\rangle,...,|e_n\rangle\}\\ \text{orthonormal basis}\end{array}} \Leftrightarrow$
- $\langle e_j|e_k\rangle=\delta_{jk}$ and
- $\sum_{k=1}^{n}|e_k\rangle\langle e_k|=\mathbb{I}$.

30 $\boxed{\begin{array}{c}\mathfrak{B}=\{|e_1\rangle,|e_2\rangle,...,|e_n\rangle\}\\ \text{orthonormal basis}\end{array}} \Rightarrow \boxed{\begin{array}{c}\mathfrak{B}^*=\{\langle e_1|,\langle e_2|,...,\langle e_n|\}\\ \text{dual basis } (\langle e_j|e_k\rangle=\delta_{jk})\end{array}}$

■ **Theorem.** Let $\{|e_1\rangle,|e_2\rangle,...,|e_n\rangle\}$ be orthonormal basis.

31 $\text{tr}\,|a\rangle\langle b|=\sum_{k=1}^{n}\langle e_k|a\rangle\langle b|e_k\rangle=\sum_{k=1}^{n}\langle b|e_k\rangle\langle e_k|a\rangle=\langle b|a\rangle.$

32 $\mathbb{I}=\sum_{k=1}^{n}|e_k\rangle\langle e_k| \;\Rightarrow\; \begin{cases}|x\rangle=\mathbb{I}|x\rangle=\sum_{k=1}^{n}|e_k\rangle\langle e_k|x\rangle,\\ \langle x|=\langle x|\mathbb{I}=\sum_{k=1}^{n}\langle x|e_k\rangle\langle e_k|.\end{cases}$

■ **Theorem.**

33 Direct sum of two Hilbert spaces over the same field: $\mathcal{H}\oplus\mathcal{K}=\{\,(x,y)\mid x\in\mathcal{H},\ x\in\mathcal{K}\,\}$ is a Hilbert space with $\langle(x,y),(x',y')\rangle=\langle x,x'\rangle+\langle y,y'\rangle$. $\dim\mathcal{H}\oplus\mathcal{K}=\dim\mathcal{H}+\dim\mathcal{K}$.

■ **Definition.** *Conjugate transpose* of a matrix

$\boxed{34}$ $\mathbf{A}=\begin{pmatrix} a_1^1 & a_2^1 & \cdots & a_n^1 \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^m & a_2^m & \cdots & a_n^m \end{pmatrix}$ is $\mathbf{A}^\dagger=\begin{pmatrix} \bar a_1^1 & \bar a_1^2 & \cdots & \bar a_1^m \\ \bar a_2^1 & \bar a_2^2 & \cdots & \bar a_2^m \\ \vdots & \vdots & \ddots & \vdots \\ \bar a_n^1 & \bar a_n^2 & \cdots & \bar a_n^m \end{pmatrix}$.

■ **Definition.** A matrix with complex entries

$\boxed{35}$ $\mathbf{A}=\begin{pmatrix} a_1^1 & \cdots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^n & \cdots & a_n^n \end{pmatrix}$ is $\begin{cases} \bullet \text{ a } \textit{Hermitian matrix} \text{ if } \mathbf{A}^\dagger=\mathbf{A}. \\ \bullet \text{ a } \textit{unitary matrix} \text{ if } \mathbf{A}^\dagger\mathbf{A}=\mathbf{I}. \end{cases}$

■ **Definition.** A matrix with real entries

$\boxed{36}$ $\mathbf{A}=\begin{pmatrix} a_1^1 & \cdots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^n & \cdots & a_n^n \end{pmatrix}$ is $\begin{cases} \bullet \text{ a } \textit{symmetric matrix} \text{ if } \mathbf{A}^T=\mathbf{A}. \\ \bullet \text{ an } \textit{orthogonal matrix} \text{ if } \mathbf{A}^T\mathbf{A}=\mathbf{I}. \end{cases}$

■ **Theorem.** Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{K}$.

$\boxed{37}$ $\boxed{\text{Change of basis matrix between two orthonormal bases of } \mathcal{H}}$ is $\begin{cases} \bullet \text{ an orthogonal matrix if } \mathbb{K}=\mathbb{R} \\ \bullet \text{ a unitary matrix if } \mathbb{K}=\mathbb{C}. \end{cases}$

■ **Theorem.** *Orthogonal projector on a subspace $\mathcal{K}\subset\mathcal{H}$.*

$\boxed{38}$ $P_{\mathcal{K}}x\overset{\text{def}}{=}x^{\parallel}$ for $x=x^{\parallel}+x^{\perp}$, $\mathcal{H}=\mathcal{K}\oplus\mathcal{K}^{\perp}$; $\{e_1,e_2...,e_m\}$ orthonormal basis in $\mathcal{K}$, $\{e_1,e_2...,e_m, e_{m+1},...,e_n\}$ orthonormal basis in $\mathcal{H}$ $\Rightarrow$

- $P_{\mathcal{K}}=\sum_{k=1}^{m}|e_k\rangle\langle e_k|$,
- $P_{\mathcal{K}^\perp}=\sum_{k=m+1}^{n}|e_k\rangle\langle e_k|$,
- $P_{\mathcal{K}}+P_{\mathcal{K}^\perp}=\mathbb{I}_{\mathcal{H}}$,
- $P_{\mathcal{K}}\,P_{\mathcal{K}^\perp}=0$.

E SELF-ADJOINT (HERMITIAN) OPERATORS

■ **Theorem.**

$\boxed{1}$ $\mathcal{H}$, $\mathcal{K}$ Hilbert spaces, over the same field. $A:\mathcal{H}\to\mathcal{K}$ linear operator $\Rightarrow$ There exists a unique operator $A^\dagger:\mathcal{K}\to\mathcal{H}$ satisfying $\boxed{\langle A^\dagger x,y\rangle=\langle x,Ay\rangle}$ for any $x\in\mathcal{K}$, $y\in\mathcal{H}$.

■ **Definition.**

$\boxed{2}$ The *adjoint* of $A:\mathcal{H}\to\mathcal{K}$ is $A^\dagger:\mathcal{K}\to\mathcal{H}$ satisfying $\boxed{\langle A^\dagger x,y\rangle=\langle x,Ay\rangle \quad \forall x\in\mathcal{K},\ \forall y\in\mathcal{H}.}$

■ **Theorem.**

$\boxed{3}$ $A,B:\mathcal{H}\to\mathcal{H}$ linear operators, $\lambda\in\mathbb{K}$ $\Rightarrow$ $(A+B)^\dagger=A^\dagger+B^\dagger$, $(\lambda A)^\dagger=\bar\lambda\,A^\dagger$, $(AB)^\dagger=B^\dagger\,A^\dagger$, $(A^\dagger)^\dagger=A$.

$\boxed{4}$ Matrix of $A^\dagger:\mathcal{H}\to\mathcal{H}$ with respect to an orthonormal basis $\mathfrak{B}$ is the conjugate transpose of the matrix of $A:\mathcal{H}\to\mathcal{H}$ with respect to the basis $\mathfrak{B}$.

$\boxed{5}$ The linear form corresponding to $A|x\rangle$ is $\langle x|A^\dagger$.

$\boxed{6}$ $(|a\rangle\langle b|)^\dagger=|b\rangle\langle a|$.

■ **Theorem.** The vector space of matrices

$\boxed{7}$ $$\mathcal{M}_{mxn}(\mathbb{K})=\left\{\mathbf{A}=\begin{pmatrix} a_1^1 & a_2^1 & \cdots & a_n^1 \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^m & a_2^m & \cdots & a_n^m \end{pmatrix}\ \middle|\ a_j^k\in\mathbb{K}\right\}$$

is a Hilbert space with $\boxed{\langle\mathbf{A},\mathbf{B}\rangle=\operatorname{tr}(\mathbf{A}^\dagger\mathbf{B})=\sum_{j,k}\bar a_j^k\,b_j^k}$ *Hilbert–Schmidt* inner product

■ **Theorem.** The space of linear maps

$\boxed{8}$ $\mathcal{L}(\mathcal{H},\mathcal{K})=\{A:\mathcal{H}\to\mathcal{K}\mid A \text{ is a linear map}\}$ is a Hilbert space with $(A+B)x=Ax+Bx$, $(\lambda A)x=\lambda\,Ax$ and $\boxed{\langle A,B\rangle=\operatorname{tr}(A^\dagger B)}$

■ **Definition.**

$\boxed{9}$ A linear operator $A:\mathcal{H}\to\mathcal{H}$ is a *self-adjoint operator* (*Hermitian operator*) if $\boxed{\langle Ax,y\rangle=\langle x,Ay\rangle \quad \forall x,y\in\mathcal{H}.}$

Notation : $\boxed{\mathcal{A}(\mathcal{H})=\{A:\mathcal{H}\to\mathcal{H}\mid \textit{Hermitian operator}\}}$

■ **Theorem.**

$\boxed{10}$ $\boxed{A\in\mathcal{A}(\mathcal{H}) \iff A=A^\dagger}$

$\boxed{11}$ $A,B\in\mathcal{A}(\mathcal{H})$, $\lambda\in\mathbb{R}$ $\Rightarrow$ $A+B\in\mathcal{A}(\mathcal{H})$, $\lambda A\in\mathcal{A}(\mathcal{H})$.

$\boxed{12}$ $\mathcal{A}(\mathcal{H})$ is
- real Hilbert space with $\langle A,B\rangle=\operatorname{tr}(AB)$,
- $\dim_{\mathbb{R}}\mathcal{A}(\mathcal{H})=(\dim\mathcal{H})^2$.

$\boxed{13}$ For $A,B\in\mathcal{A}(\mathcal{H})$: $\boxed{AB\in\mathcal{A}(\mathcal{H}) \iff AB=BA.}$

$\boxed{14}$ If $A\in\mathcal{A}(\mathcal{H})$ is invertible, then $A^{-1}\in\mathcal{A}(\mathcal{H})$.

■ **Theorem.** Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{K}$.

$\boxed{15}$ $A\in\mathcal{A}(\mathcal{H})\iff$ Matrix of A with respect to an orthonormal basis is:
- a symmetric matrix if $\mathbb{K}=\mathbb{R}$.
- a Hermitian matrix if $\mathbb{K}=\mathbb{C}$.

$\boxed{16}$ $A\in\mathcal{A}(\mathcal{H})\Rightarrow$
- Eigenvalues of A are real;
- Eigenvectors of A corresponding to distinct eigenvalues are orthogonal.

$\boxed{17}$ $A\in\mathcal{A}(\mathcal{H})\Rightarrow$ All the roots of the characteristic polynomial of A are real.

$\boxed{18}$ $A\in\mathcal{A}(\mathcal{H})\Rightarrow$ There exists an orthonormal basis $\mathfrak{B}$ in $\mathcal{H}$ such that the matrix of A with respect to $\mathfrak{B}$ is a diagonal matrix.

$\boxed{19}$ $\mathbf{A}\in\mathcal{M}_{n\times n}(\mathbb{C})$ Hermitian matrix $\Rightarrow$ there exists a unitary matrix $\mathbf{S}$ such that $\mathbf{S}^\dagger\mathbf{A}\mathbf{S}$ is a diagonal matrix

$\boxed{20}$ $\mathbf{A}\in\mathcal{M}_{n\times n}(\mathbb{R})$ symmetric matrix $\Rightarrow$ there exists an orthogonal matrix $\mathbf{S}$ such that $\mathbf{S}^T\mathbf{A}\mathbf{S}$ is a diagonal matrix

$\boxed{21}$ $A\in\mathcal{A}(\mathcal{H})$, $B\in\mathcal{A}(\mathcal{H})$ $\Rightarrow$ $AB=BA\iff$ There exists an orthonormal basis consisting in common eigenvectors of A and B

$\boxed{22}$ *Orthogonal projectors.* For $P\in\mathcal{L}(\mathcal{H},\mathcal{H})$:
$\boxed{\text{There exists a subspace } \mathcal{K}\subseteq\mathcal{H} \text{ such that } P=\mathcal{P}_{\mathcal{K}}} \iff \boxed{P^\dagger=P=P^2}$

■

$\boxed{23}$ $A\in\mathcal{A}(\mathcal{H})$, $\langle e_j|e_k\rangle=\delta_{jk}$, $\mathbb{I}=\sum_{k=1}^{n}|e_k\rangle\langle e_k|$, $A=\sum_{k=1}^{n}\lambda_k|e_k\rangle\langle e_k|$, $f:\{\lambda_1,\lambda_2,...,\lambda_n\}\to\mathbb{C}$ $\mapsto$ $f(A)=\sum_{k=1}^{n}f(\lambda_k)|e_k\rangle\langle e_k|$, $e^{A}=\sum_{k=1}^{n}e^{\lambda_k}|e_k\rangle\langle e_k|$, $\sqrt{A}=\sum_{k=1}^{n}\sqrt{\lambda_k}|e_k\rangle\langle e_k|$, for $\lambda_k\geq 0$.

■ **Definition.** For $A,B\in\mathcal{A}(\mathcal{H})$:

$\boxed{24}$ $A\leq B$ if $\boxed{\langle x,Ax\rangle\leq\langle x,Bx\rangle, \text{ for any } x\in\mathcal{H}.}$

$\boxed{25}$ A is called a *positive* operator if $A\geq 0$.

■ **Theorem.**

$\boxed{26}$ $A=A^\dagger=\sum_{k=1}^{n}\lambda_k|e_k\rangle\langle e_k|\geq 0 \iff \lambda_k\geq 0$, for any k.

$\boxed{27}$ P orthogonal projector $\Rightarrow$ $0\leq P\leq\mathbb{I}$.

F UNITARY TRANSFORMATIONS

■ **Definition.** Let $\mathcal{H}$, $\mathcal{K}$ be complex Hilbert spaces.

1. A linear map $A:\mathcal{H}\to\mathcal{K}$ is a *unitary transformation* if $\boxed{\langle Ax,Ay\rangle=\langle x,y\rangle}$ $\forall x,y\in\mathcal{H}$.
2. A linear map $A:\mathcal{H}\to\mathcal{H}$ is a *unitary operator* if $\boxed{\langle Ax,Ay\rangle=\langle x,y\rangle}$ $\forall x,y\in\mathcal{H}$.

Notation $\boxed{U(\mathcal{H})=\{\,A:\mathcal{H}\to\mathcal{H}\mid \textit{unitary operator}\,\}}$.

■ **Theorem.** Let $\mathcal{H}$ be a complex Hilbert space.

3. $A\in U(\mathcal{H}) \;\Rightarrow\; ||Ax||=||x||$, for any $x\in\mathcal{H}$.
4. $A,B\in U(\mathcal{H}) \;\Rightarrow\; AB\in U(\mathcal{H})$.
5. $A\in U(\mathcal{H}) \;\Rightarrow\;$ A is bijective and $A^{-1}\in U(\mathcal{H})$.
6. $\boxed{A:\mathcal{H}\to\mathcal{H}\text{ unitary operator}} \Leftrightarrow \boxed{A^\dagger A=\mathbb{I}} \Leftrightarrow \boxed{A\text{ bijective and }A^{-1}=A^\dagger}$
7. $\boxed{U(\mathcal{H})\text{ is a group}}$
8. $A\in U(\mathcal{H}) \Leftrightarrow$ Matrix of A with respect to an orthonormal basis is a unitary matrix.
9. $A\in U(\mathcal{H}) \Rightarrow$
   - λ eigenvalue of $A \Rightarrow |\lambda|=1$;
   - Eigenvectors of A corresponding to distinct eigenvalues are orthogonal.
10. $A\in U(\mathcal{H}) \Rightarrow$ There exists an orthonormal basis $\mathfrak{B}$ in $\mathcal{H}$ such that the matrix of A with respect to $\mathfrak{B}$ is a diagonal matrix.
11. $\mathbf{A}\in\mathcal{M}_{n\times n}(\mathbb{C})$ unitary matrix $\Rightarrow$ there exists a unitary matrix $\mathbf{S}$ such that $\mathbf{S}^\dagger\mathbf{A}\mathbf{S}$ is a diagonal matrix

■ **Definition.**
Unitary representation of a group G in the space $\mathcal{H}$ = any morphism of groups $\varrho:G\longrightarrow U(\mathcal{H})$.

G ORTHOGONAL TRANSFORMATIONS

■ **Definition.** Let $\mathcal{H}$, $\mathcal{K}$ be real Hilbert spaces.

1. A linear map $A:\mathcal{H}\to\mathcal{K}$ is an *orthogonal transformation* if $\boxed{\langle Ax,Ay\rangle=\langle x,y\rangle}$ $\forall x,y$.
2. A linear map $A:\mathcal{H}\to\mathcal{H}$ is an *orthogonal operator* if $\boxed{\langle Ax,Ay\rangle=\langle x,y\rangle}$ $\forall x,y$.

Notation $\boxed{O(\mathcal{H})=\{\,A:\mathcal{H}\to\mathcal{H}\mid \textit{orthogonal operator}\,\}}$.

■ **Theorem.** Let $\mathcal{H}$ be a real Hilbert space.

3. $A\in O(\mathcal{H}) \;\Rightarrow\; ||Ax||=||x||$, for any $x\in\mathcal{H}$.
4. $A,B\in O(\mathcal{H}) \;\Rightarrow\; AB\in O(\mathcal{H})$.
5. $A\in O(\mathcal{H}) \;\Rightarrow\;$ A is bijective and $A^{-1}\in O(\mathcal{H})$

■ **Theorem.** Let $\mathcal{H}$ be a real Hilbert space.

6. $\boxed{A:\mathcal{H}\to\mathcal{H}\text{ orthogonal operator}} \Leftrightarrow \boxed{A^{T}A=\mathbb{I}} \Leftrightarrow \boxed{A\text{ bijective and }A^{-1}=A^{T}}$
7. $\boxed{O(\mathcal{H})\text{ is a group.}}$
8. $A\in O(\mathcal{H}) \Leftrightarrow$ Matrix of A with respect to an orthonormal basis is an orthogonal matrix.
9. $A\in O(\mathcal{H}) \Rightarrow$
   - λ eigenvalue of $A \Rightarrow \lambda=\pm 1$;
   - Eigenvectors of A corresponding to distinct eigenvalues are orthogonal.

■ **Remark.** $A\in O(\mathcal{H}) \;\not\Rightarrow\; A$ is diagonalizable.

■ **Definition.**
Orthogonal representation of a group G in the space $\mathcal{H}$ = any morphism of groups $\varrho:G\longrightarrow O(\mathcal{H})$.

H TENSOR PRODUCT OF HILBERT SPACES

■ **Definition.**
Tensor product of $\mathbf{A}=\begin{pmatrix}a_0^0 & a_1^0\\ a_0^1 & a_1^1\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}b_0^0 & b_1^0\\ b_0^1 & b_1^1\end{pmatrix}$ is

1. $$\mathbf{A}\otimes\mathbf{B}=\begin{pmatrix}a_0^0B & a_1^0B\\ a_0^1B & a_1^1B\end{pmatrix}=\begin{pmatrix}a_0^0b_0^0 & a_0^0b_1^0 & a_1^0b_0^0 & a_1^0b_1^0\\ a_0^0b_0^1 & a_0^0b_1^1 & a_1^0b_0^1 & a_1^0b_1^1\\ a_0^1b_0^0 & a_0^1b_1^0 & a_1^1b_0^0 & a_1^1b_1^0\\ a_0^1b_0^1 & a_0^1b_1^1 & a_1^1b_0^1 & a_1^1b_1^1\end{pmatrix}.$$

■ **Remark.**

2. Any $\mathbb{K}$-Hilbert space $\mathcal{H}$ of dimension n can be identified with a space of functions $\boxed{\mathcal{H}\equiv\{\,\varphi:\{0,1,...,n-1\}\to\mathbb{K}\,\},\ \text{with } \langle\varphi,\psi\rangle=\sum_{k=0}^{n-1}\overline{\varphi(k)}\,\psi(k).}$

■ **Definition** (in a particular representation).
The *tensor product* of the Hilbert spaces
$\mathcal{H}_1=\{\,\varphi:\{0,1,...,n-1\}\to\mathbb{K}\,\}$,
$\mathcal{H}_2=\{\,\psi:\{0,1,...,m-1\}\to\mathbb{K}\,\}$ is

3. $$\boxed{\begin{aligned}&\mathcal{H}_1\otimes\mathcal{H}_2=\{\,\Phi:\{0,1,...,n-1\}\times\{0,1,...,m-1\}\to\mathbb{K}\,\}\\ &\text{with } \langle\Phi,\Psi\rangle=\sum_{j=0}^{n-1}\sum_{k=0}^{m-1}\overline{\Phi(j,k)}\,\Psi(j,k).\end{aligned}}$$

4. For any $\varphi\in\mathcal{H}_1$ and $\psi\in\mathcal{H}_2$, the map $\varphi\otimes\psi:\{0,1,...,n-1\}\times\{0,1,...,m-1\}\to\mathbb{K}$, $\boxed{\varphi\otimes\psi(j,k)=\varphi(j)\,\psi(k)}$ belongs to $\mathcal{H}_1\otimes\mathcal{H}_2$

■ **Theorem.**

5. $(\varphi_1+\varphi_2)\otimes\psi=\varphi_1\otimes\psi+\varphi_2\otimes\psi$
   $\varphi\otimes(\psi_1+\psi_2)=\varphi\otimes\psi_1+\varphi\otimes\psi_2$
   $\lambda(\varphi\otimes\psi)=(\lambda\varphi)\otimes\psi=\varphi\otimes(\lambda\psi)$
6. $\langle\varphi_1\otimes\psi_1,\varphi_2\otimes\psi_2\rangle=\langle\varphi_1,\varphi_2\rangle\,\langle\psi_1,\psi_2\rangle$.
7. $\{\eta_0,\eta_1,...,\eta_{n-1}\}$ orthonormal basis of $\mathcal{H}_1$, $\{\vartheta_0,\vartheta_1,...,\vartheta_{m-1}\}$ orthonormal basis of $\mathcal{H}_2$ $\Rightarrow$ $\left\{|\eta_j\vartheta_k\rangle\equiv|\eta_j\rangle\otimes|\vartheta_k\rangle \;\middle|\; \begin{matrix}0\leq j\leq n-1\\ 0\leq k\leq m-1\end{matrix}\right\}$ is an orthonormal basis of $\mathcal{H}_1\otimes\mathcal{H}_2$
   $\boxed{\dim\mathcal{H}_1\otimes\mathcal{H}_2=\dim\mathcal{H}_1\,\dim\mathcal{H}_2}$
8. Canonical bases of $\mathcal{H}$ and $\mathcal{H}_1\otimes\mathcal{H}_2$ are
   $\boxed{\{|k\rangle\equiv|\delta_k\rangle\mid 0\leq k\leq n-1\}}$, where $\delta_k(j)=\delta_{kj}$.
   $\boxed{\{|jk\rangle\equiv|j\rangle\otimes|k\rangle\mid 0\leq j\leq n-1,\ 0\leq k\leq m-1\}}$.
9. Matrix of an operator $T:\mathcal{H}_1\otimes\mathcal{H}_2\to\mathcal{H}_1\otimes\mathcal{H}_2$ is $\boxed{\mathbf{T}=(T^{ik}_{j\ell}=\langle ik|T|j\ell\rangle)}$
10. $A:\mathcal{H}_1\to\mathcal{H}_1$, $B:\mathcal{H}_2\to\mathcal{H}_2$ linear operators $\Rightarrow$ $A\otimes B:\mathcal{H}_1\otimes\mathcal{H}_2\longrightarrow\mathcal{H}_1\otimes\mathcal{H}_2$ $\boxed{(A\otimes B)(\varphi\otimes\psi)=(A\varphi)\otimes(B\psi)}$ defines a linear operator.
11. The matrix of $A\otimes B$ is the tensor product of the matrices of A and B
12. The *adjoint* of $A\otimes B$ is $\boxed{(A\otimes B)^\dagger=A^\dagger\otimes B^\dagger}$
13. $|\varphi_1\psi_1\rangle\langle\varphi_2\psi_2|=|\varphi_1\rangle\langle\varphi_2|\otimes|\psi_1\rangle\langle\psi_2|$.

■ **Definition** *Partial traces* of $T\in\mathcal{L}(\mathcal{H}_1\otimes\mathcal{H}_2)$ are:

14. the only map $\mathrm{tr}_1T:\mathcal{H}_2\to\mathcal{H}_2$ satisfying $\mathrm{tr}\,((\mathbb{I}\otimes B)T)=\mathrm{tr}\,(B\,\mathrm{tr}_1\,T)$, for any $B\in\mathcal{L}(\mathcal{H}_2)$,
15. the only map $\mathrm{tr}_2T:\mathcal{H}_1\to\mathcal{H}_1$ satisfying $\mathrm{tr}\,((A\otimes\mathbb{I})T)=\mathrm{tr}\,(A\,\mathrm{tr}_2\,T)$, for any $A\in\mathcal{L}(\mathcal{H}_1)$.

■ **Theorem.**

16. $\langle k|\mathrm{tr}_1T|\ell\rangle=\sum_{j=0}^{n-1}T^{jk}_{j\ell}$, $\quad\langle i|\mathrm{tr}_2T|j\rangle=\sum_{k=0}^{m-1}T^{ik}_{jk}$.

■ **Theorem** (*Schmidt decomposition*). For $\Phi\in\mathcal{H}_1\otimes\mathcal{H}_2$,

17. there exist $p_0\geq p_1\geq...\geq p_{r-1}>0$, $\{\varphi_0,\varphi_1,...,\varphi_{n-1}\}$ $\{\psi_0,\psi_1,...,\psi_{m-1}\}$ orthonormal bases such that $\boxed{\Phi=\sum_{k=0}^{r-1}\sqrt{p_k}\,|\varphi_k\psi_k\rangle}$ where $r=\mathrm{rank}(\Phi^{jk})$.

I QUADRATIC FORMS

■ **Definition.**

1 A *bilinear form* on the $\mathbb{K}$-vector space $\mathcal{V}$ is $\boxed{\text{a map } \boxed{g:\mathcal{V}\times\mathcal{V}\longrightarrow\mathbb{K}} \text{ satisfying: } g(\alpha x+\beta y,z)=\alpha g(x,z)+\beta g(y,z),\ g(x,\alpha y+\beta z)=\alpha g(x,y)+\beta g(x,z)}$

■ **Theorem.** Let $\mathfrak{B}=\{v_1,v_2,...,v_n\}$ be a basis in $\mathcal{V}$.
If $g:\mathcal{V}\times\mathcal{V}\to\mathbb{K}$ is a bilinear form, $x=x^j v_j$, $y=y^k v_k$ and $\boxed{g_{jk}=g(v_j,v_k)}$, then $\boxed{g(x,y)=g_{jk}x^jy^k}$, that is

2 $$g(x,y)=\begin{pmatrix}x^1 & x^2 & ... & x^n\end{pmatrix}\begin{pmatrix}g_{11} & g_{12} & \cdots & g_{1n}\\ g_{21} & g_{22} & \cdots & g_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ g_{n1} & g_{n2} & \cdots & g_{nn}\end{pmatrix}\begin{pmatrix}y^1\\ y^2\\ \vdots\\ y^n\end{pmatrix}.$$

■ **Definition.** The matrix of g with respect to $\mathfrak{B}$ is

3 $$\mathbf{G}=\begin{pmatrix}g_{11} & \cdots & g_{1n}\\ \vdots & \ddots & \vdots\\ g_{n1} & \cdots & g_{nn}\end{pmatrix}=\begin{pmatrix}g(v_1,v_1) & \cdots & g(v_1,v_n)\\ \vdots & \ddots & \vdots\\ g(v_n,v_1) & \cdots & g(v_n,v_n)\end{pmatrix}.$$

■ **Theorem.** Let $g:\mathcal{V}\times\mathcal{V}\to\mathbb{K}$ be a bilinear form.

4 $\mathfrak{B}=\{v_1,...,v_n\}$, $\mathfrak{B}'=\{v'_1,...,v'_n\}$ bases in $\mathcal{V}$, $g_{jk}=g(v_j,v_k)$, $g'_{jk}=g(v'_j,v'_k)$, $v'_k=\alpha^j_k v_j$ $\Rightarrow$ $\boxed{g'_{jk}=\alpha^\ell_j\alpha^m_k g_{\ell m}}$, that is $\boxed{\mathbf{G}'=\mathbf{S}^T\mathbf{G}\mathbf{S}}$.

■ **Definition.**

5 Bilinear form $g:\mathcal{V}\times\mathcal{V}\to\mathbb{K}$ is *symmetric* if $g(x,y)=g(y,x)$, $\forall x,y\in\mathcal{V}$.

6 $Q:\mathcal{V}\to\mathbb{K}$ is a *quadratic form* if there exists a symmetric bilinear form $g:\mathcal{V}\times\mathcal{V}\to\mathbb{K}$ such that $\boxed{Q(x)=g(x,x)}$.

■ **Remark.**

The symmetric bilinear form g corresponding to a quadratic form Q is uniquely determined

7 $$g(x,y)=\tfrac{1}{2}\left(\,Q(x+y)-Q(x)-Q(y)\,\right),$$

and the *matrix of* Q in a basis is the matrix of g.

8 $$Q(x)=g_{jk}x^jx^k=\begin{pmatrix}x^1 & ... & x^n\end{pmatrix}\begin{pmatrix}g_{11} & \cdots & g_{1n}\\ \vdots & \ddots & \vdots\\ g_{n1} & \cdots & g_{nn}\end{pmatrix}\begin{pmatrix}x^1\\ \vdots\\ x^n\end{pmatrix}.$$

■ **Theorem** (*Gauss*).

By using a change of basis $\mathfrak{B}\mapsto\mathfrak{B}'$, the matrix of a quadratic form can be reduced to a diagonal matrix

9 $$\mathbf{G}\mapsto\mathbf{G}'=\mathbf{S}^T\mathbf{G}\mathbf{S}=\begin{pmatrix}g'_{11} & 0 & \cdots & 0\\ 0 & g'_{22} & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & g'_{nn}\end{pmatrix}.$$

In the basis $\mathfrak{B}'$, Q has the *reduced form*

10 $$Q(x)=g'_{11}(x'^1)^2+g'_{22}(x'^2)^2+\cdots+g'_{nn}(x'^n)^2.$$

■ **Theorem** (*Jacobi*). If $Q:\mathcal{V}\to\mathbb{K}$ is a quadratic form

$$\mathbf{G}=\begin{pmatrix}g_{11} & g_{12} & \cdots & g_{1n}\\ g_{21} & g_{22} & \cdots & g_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ g_{n1} & g_{n2} & \cdots & g_{nn}\end{pmatrix}$$

is its matrix with respect to a basis $\mathfrak{B}=\{v_1,v_2,...,v_n\}$,

$$\begin{cases}\Delta_1=g_{11}\neq 0,\\ \Delta_2=\begin{vmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{vmatrix}\neq 0,\\ \cdots\cdots\cdots\\ \Delta_n=\begin{vmatrix}g_{11} & g_{12} & \cdots & g_{1n}\\ g_{21} & g_{22} & \cdots & g_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ g_{n1} & g_{n2} & \cdots & g_{nn}\end{vmatrix}\neq 0\end{cases}$$

11 then There exists a basis $\mathfrak{B}'$ such that the expression of Q in terms of $\mathfrak{B}'$ is
$$\boxed{Q(x)=\tfrac{1}{\Delta_1}{x'_1}^2+\tfrac{\Delta_1}{\Delta_2}{x'_2}^2+\cdots+\tfrac{\Delta_{n-1}}{\Delta_n}{x'_n}^2}$$

■ **Theorem.**

12 $\mathcal{H}$ real Hilbert space, $Q:\mathcal{H}\to\mathbb{R}$ quadratic form, $\mathbf{G}=\begin{pmatrix}g_{11} & g_{12} & \cdots & g_{1n}\\ g_{21} & g_{22} & \cdots & g_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ g_{n1} & g_{n2} & \cdots & g_{nn}\end{pmatrix}$ matrix of Q with respect to an orthonormal basis $\mathfrak{B}$ $\Rightarrow$ There exists an orthonormal basis $\mathfrak{B}'$ such that the expression of Q in terms of $\mathfrak{B}'$ is $\boxed{Q(x)=\lambda_1{x'_1}^2+\lambda_2{x'_2}^2+\cdots+\lambda_n{x'_n}^2}$ where $\lambda_1,\lambda_2,...,\lambda_n$ are the eigenvalues of $\mathbf{G}$, that is the roots of the equation $\det(\mathbf{G}-\lambda\mathbf{I})=0$.

J CONICS

■ **Definition.**

We describe the geometric plane by using Cartesian coordinates.

1 A *conic* is the set of all the solutions $(x,y)\in\mathbb{R}^2$ of an equation $\boxed{a_{11}\,x^2+2\,a_{12}\,xy+a_{22}\,y^2+2a_{10}\,x+2a_{20}\,y+a_{00}=0}$, where at least one of a_{11}, a_{12}, a_{22} is non-null.

■ **Theorem.**

Under a change of the system of axes

2 $$\begin{cases}x=x_0+x'\cos\alpha-y'\sin\alpha\\ y=y_0+x'\sin\alpha+y'\cos\alpha\end{cases}$$

the Eq. $\boxed{J1}$ *transforms into an equation of the same form, and*

3 $$I=a_{11}+a_{22},\quad \delta=\begin{vmatrix}a_{11} & a_{12}\\ a_{12} & a_{22}\end{vmatrix},\quad \Delta=\begin{vmatrix}a_{11} & a_{12} & a_{10}\\ a_{12} & a_{22} & a_{20}\\ a_{10} & a_{20} & a_{00}\end{vmatrix}$$

remain invariant.

■ **Theorem.**

4 The point (x_0,y_0) is a center of symmetry of the conic $\boxed{J1}$ $\Leftrightarrow\begin{cases}a_{11}\,x_0+a_{12}\,y_0+a_{10}=0\\ a_{12}\,x_0+a_{22}\,y_0+a_{20}=0.\end{cases}$

■ **Theorem.** If $\delta\neq 0$, then:

5 The conic $\boxed{J1}$ has only one center of symmetry, namely $(x_0,y_0)=\left(\frac{1}{\delta}\begin{vmatrix}-a_{10} & a_{12}\\ -a_{20} & a_{22}\end{vmatrix},\ \frac{1}{\delta}\begin{vmatrix}a_{11} & -a_{10}\\ a_{12} & -a_{20}\end{vmatrix}\right)$.

6 After the translation with (x_0,y_0), the equation becomes $a_{11}\,x^2+2\,a_{12}\,xy+a_{22}\,y^2+\frac{\Delta}{\delta}=0.$

7 The orientation of axes can be chosen such that the equation to become $\boxed{\lambda_1\,{x'}^2+\lambda_2\,{y'}^2+\frac{\Delta}{\delta}=0}$, where λ_1, λ_1 are the roots of $\begin{vmatrix}a_{11}-\lambda & a_{12}\\ a_{12} & a_{22}-\lambda\end{vmatrix}=0.$

■ **Theorem.**

8 $\delta=0$, $a_{11}\neq 0$, $\Delta=0$ $\Rightarrow$ There exists a change of system of axes such that $\boxed{J1}$ to become $\boxed{a'_{22}{y'}^2+2\,a'_{20}y'+a'_{00}=0}$.

9 $\delta=0$, $a_{11}\neq 0$, $\Delta\neq 0$ $\Rightarrow$ There exists a change of system of axes such that $\boxed{J1}$ to become $\boxed{{y'}^2=2p\,x'}$, where $p=\sqrt{-\frac{\Delta}{I^3}}$.

■

10

Case	Conic
$\delta<0$ and $\Delta\neq 0$	hyperbola
$\delta>0$ and $I\Delta<0$	ellipse
$\delta>0$ and $I\Delta>0$	empty set
$\delta>0$ and $\Delta=0$	a point
$\delta<0$ and $\Delta=0$	pair of intersecting lines
$\delta=0$ and $\Delta=0$	pair of parallel lines, a line or empty set
$\delta=0$ and $\Delta\neq 0$	parabola

■ **Reduced forms** ($a, b \in (0,\infty)$ are parameters).

11

- Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$;
- Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0$;
- Parabola $y^2-2ax=0$;
- Two intersecting lines $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$;
- Two parallel lines $x^2-a^2=0$;
- A line $x^2=0$;
- A point $\frac{x^2}{a^2}+\frac{y^2}{b^2}=0$;
- Empty set $\frac{x^2}{a^2}+\frac{y^2}{b^2}+1=0$; $x^2+a^2=0$.

K QUADRICS

■ **Definition.**

1 A *quadric* is the set of all the solutions $(x,y,z)\in\mathbb{R}^3$ of an equation

$$\boxed{\begin{aligned} a_{11}\,x^2+a_{22}\,y^2+a_{33}\,z^2+2\,a_{12}\,xy \\ +2\,a_{13}\,xz+2\,a_{23}\,yz+2a_{10}\,x \\ +2a_{20}\,y+2a_{30}\,z+a_{00}=0 \end{aligned}},$$

where at least one of a_{11}, a_{22}, a_2, a_{12}, a_{13}, a_{23} is non-null.

■ **Theorem.**

Under a change of Cartesian coordinates the Eq. $\boxed{K1}$ *transforms into an equation of the same form, and*

2

$$I=a_{11}+a_{22}+a_{33},$$

$$J=\begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix}+\begin{vmatrix} a_{11} & a_{13} \\ a_{13} & a_{33} \end{vmatrix}+\begin{vmatrix} a_{22} & a_{23} \\ a_{23} & a_{33} \end{vmatrix}$$

$$\delta=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix},$$

$$\Delta=\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{10} \\ a_{12} & a_{22} & a_{23} & a_{20} \\ a_{13} & a_{23} & a_{33} & a_{30} \\ a_{10} & a_{20} & a_{30} & a_{00} \end{vmatrix}$$

remain invariant.

■ **Reduced forms** ($a, b, c \in (0,\infty)$ are parameters).

3

- Ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$;
- Elliptic hyperboloid of one sheet $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}-1=0$;
- Elliptic hyperboloid of two sheets $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}-1=0$;
- Elliptic paraboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-z=0$;
- Hyperbolic paraboloid $\frac{x^2}{a^2}-\frac{y^2}{b^2}-z=0$;
- Elliptic cone $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$;
- Elliptic cylinder $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$;
- Hyperbolic cylinder $\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0$;
- Parabolic cylinder $y^2-2ax=0$;
- Two intersecting planes $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$;
- Two parallel planes $x^2-a^2=0$;
- A plane $x^2=0$;
- A line $\frac{x^2}{a^2}+\frac{y^2}{b^2}=0$;
- A point $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$;
- Empty set $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}+1=0$; $\frac{x^2}{a^2}+\frac{y^2}{b^2}+1=0$; $x^2+a^2=0$.

L LINEAR DIFFERENTIAL EQUATIONS

■ For any $a_0\neq 0, a_1, ..., a_n\in\mathbb{R}$, by denoting $D^k=\frac{\mathrm{d}^k}{\mathrm{d}x^k}$ we get

1 $\boxed{\begin{aligned} Ly&=a_0y^{(n)}+\cdots+a_{n-1}y'+a_ny \\ &=(a_0D^n+\cdots+a_{n-1}D+a_n)y=P(D)y \end{aligned}}$ $P(\lambda)=a_0\lambda^n+\cdots+a_n$ is the characteristic polynomial

■ **Linear equation with constant coefficients**

2 *Theorem:* Solutions $y:\mathbb{R}\to\mathbb{R}$ of $\boxed{Ly=0}$ form a real vector space of dimension n.

■ **General solution of** $Ly=0$.

3 $y=c_1\,y_1+...+c_n\,y_n$ is the *general solution* $\Leftrightarrow$ $\{y_1,...,y_n\}$ is a basis in the vector space of solutions

■ **Particular solutions of** $Ly=0$**.**

4 $\boxed{y(x)=\mathrm{e}^{\lambda x}\text{ is a solution of }Ly=0}\Leftrightarrow\boxed{P(\lambda)=0}$

5 In the case $\lambda=\alpha+\beta\mathrm{i}$, $\boxed{\mathrm{e}^{(\alpha+\beta\mathrm{i})x}\overset{\text{def}}{=}\mathrm{e}^{\alpha x}\cos\beta x+\mathrm{i}\,\mathrm{e}^{\alpha x}\sin\beta x}$

6 $P(\alpha+\beta\mathrm{i})=0\Rightarrow\begin{cases} y_1(x)=\mathrm{e}^{\alpha x}\cos\beta x \\ y_2(x)=\mathrm{e}^{\alpha x}\sin\beta x\end{cases}$ linearly independent solutions of $Ly=0$.

7 λ root of P of multiplicity k $\Rightarrow\begin{cases} y_1(x)=\mathrm{e}^{\lambda x} \\ y_2(x)=x\,\mathrm{e}^{\lambda x} \\ \dots\dots \\ y_k(x)=x^{k-1}\,\mathrm{e}^{\lambda x}\end{cases}$ are linearly independent solutions of $Ly=0$.

■ **Method of the variation of parameters**.

8 $\boxed{\begin{gathered} y=c_1\,y_1+...+c_n\,y_n \\ \text{general solution} \\ \text{of}\quad Ly=0\end{gathered}}\Rightarrow\boxed{\begin{gathered} Ly=f\text{ admits a particular} \\ \text{solution of the form} \\ y(x)=c_1(x)\,y_1(x)+...+c_n(x)\,y_n(x)\end{gathered}}$

The functions $c_1(x),...,c_n(x)$ can be obtained by solving the system

$$\begin{cases} c_1'(x)\,y_1(x)+...+c_n'(x)\,y_n(x)=0 \\ c_1'(x)\,y_1'(x)+...+c_n'(x)\,y_n'(x)=0 \\ \dots\dots\dots \\ c_1'(x)\,y_1^{(n-2)}(x)+...+c_n'(x)\,y_n^{(n-2)}(x)=0 \\ c_1'(x)\,y_1^{(n-1)}(x)+...+c_n'(x)\,y_n^{(n-1)}(x)=\frac{f(x)}{a_0}.\end{cases}$$

■ **Linear non-homogeneous equation**.

9 $\boxed{\begin{gathered}\text{general solution} \\ \text{of}\ \ Ly=f\end{gathered}}=\boxed{\begin{gathered}\text{general solution} \\ \text{of}\ \ Ly=0\end{gathered}}+\boxed{\begin{gathered}\text{a particular solution} \\ \text{of}\ \ Ly=f.\end{gathered}}$

■ **Euler's equation.**

10 $\boxed{a_0x^ny^{(n)}+...+a_{n-1}xy'+a_ny=0}\xrightarrow[x=\mathrm{e}^t]{\text{change}}$ linear equation with constant coefficients

■ **Systems of linear equations with constant coefficients.**

11 $\boxed{Y'=AY+F}$ $Y=\begin{pmatrix} y_1 \\ \vdots \\ y_n\end{pmatrix}$, $F=\begin{pmatrix} f_1 \\ \vdots \\ f_n\end{pmatrix}$, $A=\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn}\end{pmatrix}$.

12 *Theorem:* Solutions $Y:\mathbb{R}\to\mathbb{R}^n$ of $\boxed{Y'=AY}$ form a real vector space of dimension n.

■ **Wronski matrix**

13 $Y_1=\begin{pmatrix} y_{11} \\ \vdots \\ y_{n1}\end{pmatrix},...,Y_n=\begin{pmatrix} y_{1n} \\ \vdots \\ y_{nn}\end{pmatrix}$ form a basis $\Leftrightarrow W=\begin{pmatrix} y_{11} & \dots & y_{1n} \\ \vdots & & \vdots \\ y_{n1} & \dots & y_{nn}\end{pmatrix}$ has $\det W\neq 0$

■ **Particular solutions of** $Y'=AY$**.**

14 $Y=w\mathrm{e}^{\lambda x}=\begin{pmatrix} w_1 \\ \vdots \\ w_n\end{pmatrix}\mathrm{e}^{\lambda x}$ is a solution of $Y'=AY$ $\Leftrightarrow\boxed{A\begin{pmatrix} w_1 \\ \vdots \\ w_n\end{pmatrix}=\lambda\begin{pmatrix} w_1 \\ \vdots \\ w_n\end{pmatrix}}$

15 $\lambda=\alpha+\beta\mathrm{i}\ \Rightarrow\ Y_1(x)=\mathfrak{Re}(w\mathrm{e}^{\lambda x}),\ Y_2(x)=\mathfrak{Im}(w\mathrm{e}^{\lambda x})$ solutions.

■ **General solution** of $Y'=AY$ can be written as

16 $Y=c_1Y_1+\cdots+c_nY_n=W\cdot C,$ where $C=\begin{pmatrix} c_1 \\ \vdots \\ c_n\end{pmatrix}$.

■ **Method of the variation of parameters**.

17 $\boxed{\begin{gathered} Y=W\cdot C\text{ general} \\ \text{solution of}\quad Y'=AY\end{gathered}}\Rightarrow\boxed{\begin{gathered} Y'=AY+F\text{ admits a} \\ \text{solution }Y(x)=W\cdot C(x)\end{gathered}}$ with $C'=W^{-1}F$

DESCRIPTION OF QUANTUM SYSTEMS WITH FINITE-DIMENSIONAL HILBERT SPACE

From a mathematical point of view:
Quantum system $\equiv$ a complex Hilbert space $\mathcal{H}$.

M AXIOMS OF QUANTUM MECHANICS

■ **Definition.** *Closed system* = a perfectly isolated system.
Open system = an imperfectly isolated system.

1 A *ray* in $\mathcal{H}$ is a set of the form $\{ \lambda|\psi\rangle \ | \ 0\neq\lambda\in\mathbb{C} \}$ described by a representative $|\psi\rangle$ with $||\psi||=\sqrt{\langle\psi|\psi\rangle}=1$.

Normalized vectors $e^{i\vartheta}|\psi\rangle$, with $\vartheta\in\mathbb{R}$, represent the same ray.

■ *Axioms of quantum mechanics* (for closed quantum systems).

- *Axiom 1.* **Pure state** = ray $|\psi\rangle$ in a Hilbert space $\mathcal{H}$
- *Axiom 2.* **Observable** = a self-adjoint operator $A\in\mathcal{A}(\mathcal{H})$.
- *Axiom 3.* **Measurement.** The measurement of an observable $A:\mathcal{H}\to\mathcal{H}$ with the spectral decomposition

2 $\boxed{A=\sum a_k E_k}$ $\left(E_k = \text{the orthogonal projector onto the eigenspace of } a_k\right)$

prepares an eigenstate of A, and the observer learns the value of the corresponding eigenvalue.
If the quantum state *just prior to the measurement* is $|\psi\rangle$, then the *outcome* a_k is obtained with *a priori probability*

3 $\boxed{prob(a_k)=\| E_k|\psi\rangle \|^2= \langle\psi|E_k|\psi\rangle=\mathrm{tr}(E_k\,|\psi\rangle\langle\psi|)}$.

If a_k is attained, then the quantum state *just after the measurement* is $\frac{E_k|\psi\rangle}{\|E_k|\psi\rangle\|}$, that is $|\psi\rangle$ collapses to $\frac{E_k|\psi\rangle}{\|E_k|\psi\rangle\|}$.
If many identically prepared systems are measured, each described by the state $|\psi\rangle$, then the expectation value of the outcomes is

4 $\boxed{\langle A\rangle=\sum a_k\langle\psi|E_k|\psi\rangle=\langle\psi|A|\psi\rangle=\mathrm{tr}(A\,|\psi\rangle\langle\psi|)}$.

- *Axiom 4.* **Dynamics.** The *time evolution* of a closed system is described by unitary operators $U(t',t):\mathcal{H}\to\mathcal{H}$,
$|\psi(t')\rangle=U(t',t)\,|\psi(t)\rangle$.
The infinitesimal time evolution is described by the

5 *Schrödinger equation* $\boxed{\mathrm{i}\,\frac{d}{dt}|\psi(t)\rangle=H(t)\,|\psi(t)\rangle}$

where $H(t)$ is the *Hamiltonian* of the system.

- *Axiom 5.* **Composite systems.**
$\boxed{\begin{array}{l}\mathcal{H}_A=\text{Hilbert space of system A}\\ \mathcal{H}_B=\text{Hilbert space of system B}\end{array}} \Rightarrow \boxed{\text{Hilbert space of the composite system is } \mathcal{H}_A\otimes\mathcal{H}_B.}$
If the system A is prepared in the state $|\psi\rangle_A$ and B in the state $|\psi\rangle_B$, then the composite system state is the product $|\psi\rangle_A\otimes|\psi\rangle_B$.

6 $\langle\psi_1|\psi_2\rangle$ = amplitude probability that ψ_2 is in state ψ_1.

7 *Measurement in the orthonormal eigenbasis of A*
If the eigenvalues of A are distinct, then $A=\sum a_k|\psi_k\rangle\langle\psi_k|$,
$E_k=|\psi_k\rangle\langle\psi_k|$ and $prob(a_k)=|\langle\psi_k|\psi\rangle|^2$.

N DENSITY OPERATORS

■
1 A pure state $|\psi\rangle$ can alternatively be described by using the orthogonal *projector corresponding to* $|\psi\rangle$ $\boxed{\begin{array}{l}E_\psi:\mathcal{H}\to\mathcal{H},\\ E_\psi=|\psi\rangle\langle\psi|\end{array} \text{ satisfying } \begin{array}{r}E_\psi\geq 0,\\ \mathrm{tr}\,E_\psi=1.\end{array}}$

■ **Definition.**
2 A *density operator* is a linear operator $\boxed{\varrho:\mathcal{H}\to\mathcal{H} \text{ satisfying } \begin{array}{r}\varrho\geq 0,\\ \mathrm{tr}\,\varrho=1.\end{array}}$

3 *Quantum state* of an open system $\equiv$ a density operator $\varrho:\mathcal{H}\longrightarrow\mathcal{H}$.

$\mathcal{D}(\mathcal{H})=\boxed{\text{Set of all the quantum states}}\equiv\boxed{\text{Set of all the density operators}}$.

4 A *mixed state* $\equiv$ a state which is not a pure state.

5 *Purity* of a state ϱ is $\boxed{\mathrm{tr}\,\varrho^2}$ satisfying $\frac{1}{\dim\mathcal{H}}\leq\mathrm{tr}\,\varrho^2\leq 1$.

6 *Fidelity* of two states is $F(\varrho_1,\varrho_2)=\left(\mathrm{tr}\sqrt{\varrho_1^{\frac{1}{2}}\,\varrho_2\,\varrho_1^{\frac{1}{2}}}\right)^2$.

■ **Theorem.**

7 Two probabilistic representations of a density operator ϱ are related through a unitary transform (see $\mathfrak{B}$7 at pag. 21)
$\boxed{\begin{array}{l}\varrho=\sum p_j|\psi_j\rangle\langle\psi_j|\\ \quad=\sum q_k|\varphi_k\rangle\langle\varphi_k|\end{array}}$ $0\leq p_j,q_k\leq 1$, $\sum p_j=\sum q_k=1$.
$\Updownarrow$
$\boxed{\text{There exists a unitary matrix } (u_{jk}) \text{ such that } \sqrt{p_j}|\psi_j\rangle=\sum_k u_{jk}\sqrt{q_k}|\varphi_k\rangle}$

8 $\varrho\in\mathcal{D}(\mathcal{H}) \quad\Rightarrow\quad 0\leq\varrho\leq\mathbb{I}$.

9 *Expectation value* of an observable A in a state ϱ is $\boxed{\langle A\rangle\equiv\langle A\rangle_\varrho=\mathrm{tr}(A\varrho)=\mathrm{tr}(\varrho A).}$

10 $\boxed{\varrho \text{ is a pure state}}\Leftrightarrow\boxed{\varrho^2=\varrho}\Leftrightarrow\boxed{\mathrm{tr}\,\varrho^2=1}\Leftrightarrow\boxed{||\varrho||=1}$.

11 $\boxed{\varrho \text{ is a mixed state}}\Leftrightarrow\boxed{\varrho^2\neq\varrho}\Leftrightarrow\boxed{\mathrm{tr}\,\varrho^2<1}\Leftrightarrow\boxed{||\varrho||<1}$.

12 $\left.\begin{array}{r}\varrho\in\mathcal{D}(\mathcal{H})\\ K\in U(\mathcal{H})\end{array}\right\} \Rightarrow \quad K^\dagger\varrho K\in\mathcal{D}(\mathcal{H})$.

13 $\mathcal{D}(\mathcal{H})$ is a *convex set*, that is $\boxed{\left.\begin{array}{r}\varrho_1,\varrho_2\in\mathcal{D}(\mathcal{H})\\ 0\leq\lambda\leq 1\end{array}\right\}\Rightarrow(1-\lambda)\varrho_1+\lambda\varrho_2\in\mathcal{D}(\mathcal{H})}$.

14 $\boxed{\varrho \text{ is a pure state}}\Leftrightarrow\boxed{\begin{array}{l}\varrho \text{ is an } \textit{extremal point} \text{ of } \mathcal{D}(\mathcal{H}):\\ \left.\begin{array}{r}\varrho=(1-\lambda)\varrho_1+\lambda\varrho_2,\\ \varrho_1,\varrho_2\in\mathcal{D}(\mathcal{H}),\\ 0\leq\lambda\leq 1\end{array}\right\}\Rightarrow\varrho_1=\varrho_2=\varrho.\end{array}}$

O QUBITS (quantum bits)

■ **Definition.**
Qubit = quantum system with a two-dimensional Hilbert space $\mathcal{H}$.

■ **Remarks:**
By choosing an orthonormal basis $\{|0\rangle,|1\rangle\}$ in $\mathcal{H}$, $\mathcal{H}=\{\alpha|0\rangle+\beta|1\rangle \mid \alpha,\beta\in\mathbb{C}\}$ can be identified with

1 $\boxed{\begin{array}{c}\mathbb{C}^2=\left\{|\psi\rangle=\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \,\middle|\, \alpha,\beta\in\mathbb{C}^2\right\}\equiv\{\psi:\{0,1\}\to\mathbb{C}\},\\ \langle\psi_1|\psi_2\rangle=\bar\alpha_1\beta_1+\bar\alpha_2\beta_2=\sum_{k=0}^{1}\overline{\psi_1(k)}\,\psi_2(k).\end{array}}$

2 Computational state basis is $|0\rangle\equiv|\uparrow\rangle=\begin{pmatrix}1\\ 0\end{pmatrix}$, $|1\rangle\equiv|\downarrow\rangle=\begin{pmatrix}0\\ 1\end{pmatrix}$.

3 The "bra - ket" correspondence $\langle\psi|\equiv(\,\bar\alpha\quad\bar\beta\,) \quad\Leftrightarrow\quad |\psi\rangle\equiv\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$.

■ **Theorem.**

- In the complex Hilbert space of linear operators

4 $\mathcal{L}(\mathbb{C}^2)=\left\{A:\mathbb{C}^2\to\mathbb{C}^2 \,\middle|\, A \text{ is linear operator}\right\}$, $\langle A,B\rangle=\mathrm{tr}(A^\dagger B)$,

the *Pauli operators* (several notations are presented)

5 $\boxed{\begin{array}{ll}\sigma_0\equiv\mathbb{I}=\begin{pmatrix}1&0\\ 0&1\end{pmatrix}, & \sigma_1\equiv\sigma_x\equiv X=\begin{pmatrix}0&1\\ 1&0\end{pmatrix},\\ \sigma_2\equiv\sigma_y\equiv Y=\begin{pmatrix}0&-\mathrm{i}\\ \mathrm{i}&0\end{pmatrix}, & \sigma_3\equiv\sigma_z\equiv Z=\begin{pmatrix}1&0\\ 0&-1\end{pmatrix}\end{array}}$

form an orthogonal basis: $\langle\sigma_j,\sigma_k\rangle=2\delta_{jk}\mathbb{I}$, and
$\sigma_j^\dagger=\sigma_j$, $\sigma_1^2=\sigma_2^2=\sigma_3^2=\mathbb{I}$, $\sigma_j\sigma_k+\sigma_k\sigma_j=0$ for $j\neq k$.

6 $A\in\mathcal{L}(\mathbb{C}^2)\Rightarrow\boxed{\begin{array}{l}\text{there exist } r_0,r_1,r_2,r_3\in\mathbb{C} \text{ such that}\\ A=\frac{1}{2}(r_0\,\sigma_0+r_1\,\sigma_1+r_2\,\sigma_2+r_3\,\sigma_3)\\ \quad=\frac{1}{2}\begin{pmatrix}r_0+r_3 & r_1-\mathrm{i}r_2\\ r_1+\mathrm{i}r_2 & r_0-r_3\end{pmatrix}\end{array}}$

7 Eigenvalues of $A\in\mathcal{L}(\mathbb{C}^2)$ are $\lambda_{1,2}=\frac{r_0\pm\sqrt{r_1^2+r_2^2+r_3^2}}{2}$.

8 $A\in\mathcal{L}(\mathbb{C}^2) \quad\Rightarrow\quad \mathrm{tr}\,A=r_0$.

- In the real Hilbert space of Hermitian operators

9 $\mathcal{A}(\mathbb{C}^2)=\{A:\mathbb{C}^2\to\mathbb{C}^2 \mid A^\dagger=A\}$, $\langle A,B\rangle=\mathrm{tr}(AB)$, the Pauli operators form also an orthogonal basis.

10 $A\in\mathcal{A}(\mathbb{C}^2)\Rightarrow$ $\boxed{\begin{array}{l}\text{there exist } r_0\in\mathbb{R},\ r=(r_1,r_2,r_3)\in\mathbb{R}^3\\ \text{such that } A=\frac{1}{2}(r_0\,\mathbb{I}+r\cdot\sigma)\\ \qquad\qquad =\frac{1}{2}\begin{pmatrix} r_0+r_3 & r_1-\mathrm{i}r_2\\ r_1+\mathrm{i}r_2 & r_0-r_3\end{pmatrix}\end{array}}$

11 Eigenvalues of $A\in\mathcal{A}(\mathbb{C}^2)$ are $\lambda_{1,2}=\frac{r_0\pm||r||}{2}$.

12 $A\in\mathcal{A}(\mathbb{C}^2) \quad\Rightarrow\quad \boxed{A\geq 0 \ \Leftrightarrow\ ||r||\leq r_0}$.

13 $\varrho(r)=\frac{1}{2}(r_0\,\mathbb{I}+r\cdot\sigma)$ is a density operator $\Leftrightarrow \begin{cases} r_0=1\\ ||r||\leq 1.\end{cases}$

14 The set of all the density operators is $\boxed{\mathcal{D}(\mathbb{C}^2)=\{\,\varrho=\frac{1}{2}(\mathbb{I}+r\cdot\sigma)\ \mid\ ||r||\leq 1\}}$.

15 Eigenvalues of $\varrho\in\mathcal{D}(\mathbb{C}^2)$ are $\lambda_{1,2}=\frac{1\pm||\mathbf{r}||}{2}$.

16 Purity of $\varrho\in\mathcal{D}(\mathbb{C}^2)$ is $\frac{1+||r||^2}{2}\in\left[\frac{1}{2},1\right]$.

17 *Bloch sphere* $\Omega_3=\{r\in\mathbb{R}^3 \mid ||r||\leq 1\}$

- $\Omega_3\to\mathcal{D}(\mathbb{C}^2)$: $r\mapsto\varrho(r)=\frac{1}{2}(\mathbb{I}+r\cdot\sigma)$ is a one-to-one map.
- r corresponds to a pure state $\Leftrightarrow ||r||=1$.

18 $\boxed{|\psi\rangle \text{ is a pure state}}$ $\Rightarrow$ there exist $\theta\in[0,\pi]$, $\varphi\in[0,2\pi)$ such that $\boxed{|\psi\rangle=\cos\frac{\theta}{2}|0\rangle+\sin\frac{\theta}{2}\mathrm{e}^{\mathrm{i}\varphi}|1\rangle}$ (a phase factor $\mathrm{e}^{\mathrm{i}\eta}$ is ignored)

$\Downarrow$

$\boxed{\varrho=|\psi\rangle\langle\psi|=\frac{1}{2}\begin{pmatrix}1+\cos\theta & \sin\theta\,\mathrm{e}^{-\mathrm{i}\phi}\\ \sin\theta\,\mathrm{e}^{\mathrm{i}\phi} & 1+\cos\theta\end{pmatrix}}$

corresponds to

$r=(\sin\theta\,\cos\phi,\ \sin\theta\,\sin\phi,\ \cos\theta)\in\partial\Omega_3$.

19 Orthogonal pure states correspond to diametrically opposite points of $\partial\Omega_3=\{r\in\mathbb{R}^3 \mid ||r||=1\}$.

20 Spectral decomposition of Pauli operators

$$\sigma_x=|{\uparrow_x}\rangle\langle{\uparrow_x}|-|{\downarrow_x}\rangle\langle{\downarrow_x}|,\ \text{where}\ \begin{cases}|{\uparrow_x}\rangle\equiv|+\rangle=\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\\ |{\downarrow_x}\rangle\equiv|-\rangle=\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\end{cases}$$

$$\sigma_y=|{\uparrow_y}\rangle\langle{\uparrow_y}|-|{\downarrow_y}\rangle\langle{\downarrow_y}|,\ \text{where}\ \begin{cases}|{\uparrow_y}\rangle=\frac{1}{\sqrt{2}}(|0\rangle+\mathrm{i}|1\rangle)\\ |{\downarrow_y}\rangle=\frac{1}{\sqrt{2}}(|0\rangle-\mathrm{i}|1\rangle)\end{cases}$$

$$\sigma_z=|{\uparrow_z}\rangle\langle{\uparrow_z}|-|{\downarrow_z}\rangle\langle{\downarrow_z}|,\ \text{where}\ \begin{cases}|{\uparrow_z}\rangle=|0\rangle\\ |{\downarrow_z}\rangle=|1\rangle\end{cases}$$

21 $A^2=\mathbb{I} \ \Rightarrow\ \boxed{\mathrm{e}^{\mathrm{i}\theta A}=\cos\theta\,\mathbb{I}+\mathrm{i}\,\sin\theta\,A}$, for any $\theta\in\mathbb{R}$.

22 $\sigma_x^2=\mathbb{I} \ \Rightarrow\ \mathrm{e}^{-\mathrm{i}\frac{\theta}{2}\sigma_x}=\begin{pmatrix}\cos\frac{\theta}{2} & -\mathrm{i}\,\sin\frac{\theta}{2}\\ -\mathrm{i}\,\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}$.

23 $\sigma_y^2=\mathbb{I} \ \Rightarrow\ \mathrm{e}^{-\mathrm{i}\frac{\theta}{2}\sigma_y}=\begin{pmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}$.

24 $\sigma_z^2=\mathbb{I} \ \Rightarrow\ \mathrm{e}^{-\mathrm{i}\frac{\theta}{2}\sigma_z}=\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\frac{\theta}{2}} & 0\\ 0 & \mathrm{e}^{\mathrm{i}\frac{\theta}{2}}\end{pmatrix}$.

25 Through $\Omega_3\to\mathcal{D}(\mathbb{C}^2): r\mapsto\varrho(r)=\frac{1}{2}(\mathbb{I}+r\cdot\sigma)$, the transform $\mathcal{D}(\mathbb{C}^2)\to\mathcal{D}(\mathbb{C}^2):\varrho\mapsto\mathrm{e}^{-\mathrm{i}\frac{\theta}{2}\sigma_x}\varrho\,\mathrm{e}^{\mathrm{i}\frac{\theta}{2}\sigma_x}$ corresponds to the rotation of Ω_3 around Ox

$$\Omega_3\to\Omega_3:\begin{pmatrix}r_1\\ r_2\\ r_3\end{pmatrix}\mapsto\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}r_1\\ r_2\\ r_3\end{pmatrix}.$$

26 The transform $\mathcal{D}(\mathbb{C}^2)\to\mathcal{D}(\mathbb{C}^2):\varrho\mapsto\mathrm{e}^{-\mathrm{i}\frac{\theta}{2}\sigma_y}\varrho\,\mathrm{e}^{\mathrm{i}\frac{\theta}{2}\sigma_y}$ corresponds to the rotation of Ω_3 around Oy

$$\Omega_3\to\Omega_3:\begin{pmatrix}r_1\\ r_2\\ r_3\end{pmatrix}\mapsto\begin{pmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{pmatrix}\begin{pmatrix}r_1\\ r_2\\ r_3\end{pmatrix}.$$

27 The transform $\mathcal{D}(\mathbb{C}^2)\to\mathcal{D}(\mathbb{C}^2):\varrho\mapsto\mathrm{e}^{-\mathrm{i}\frac{\theta}{2}\sigma_z}\varrho\,\mathrm{e}^{\mathrm{i}\frac{\theta}{2}\sigma_z}$ corresponds to the rotation of Ω_3 around Oz

$$\Omega_3\to\Omega_3:\begin{pmatrix}r_1\\ r_2\\ r_3\end{pmatrix}\mapsto\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}r_1\\ r_2\\ r_3\end{pmatrix}.$$

28 For any $U\in U(\mathbb{C}^2)$, there exist $\theta,\alpha,\beta,\gamma\in\mathbb{R}$ such that

$$U=\mathrm{e}^{\mathrm{i}\theta}\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}} & 0\\ 0 & \mathrm{e}^{\mathrm{i}\frac{\alpha}{2}}\end{pmatrix}\begin{pmatrix}\cos\frac{\beta}{2} & -\sin\frac{\beta}{2}\\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\frac{\gamma}{2}} & 0\\ 0 & \mathrm{e}^{\mathrm{i}\frac{\gamma}{2}}\end{pmatrix}.$$

29 For any $U\in U(2)\equiv U(\mathbb{C}^2)$, the transform $\mathcal{D}(\mathbb{C}^2)\to\mathcal{D}(\mathbb{C}^2):\varrho\mapsto U\varrho\,U^\dagger$ corresponds to a rotation of the Bloch sphere Ω_3.

■ Bit versus Qubit

Bit 30

1. Indivisible unit of classical information.
2. It describes the state of a two-state classical system.
3. It takes one of the possible values 0 and 1.
4. There exists only one observable.
5. We can measure a bit without disturbing it.
6. We can obtain the entire information it encodes by a single measurement.

Qubit 31

1′. indivisible unit of quantum information;

2′. It describes the state of a quantum system with a two-dimensional Hilbert space $\mathcal{H}$.

3′. It takes any of the values

$$|\psi\rangle=a|0\rangle+b|1\rangle=\cos\tfrac{\theta}{2}\mathrm{e}^{\mathrm{i}\varphi_0}|0\rangle+\sin\tfrac{\theta}{2}\mathrm{e}^{\mathrm{i}\varphi_1}|1\rangle=\mathrm{e}^{\mathrm{i}\varphi_0}\left(\cos\tfrac{\theta}{2}|0\rangle+\sin\tfrac{\theta}{2}\mathrm{e}^{\mathrm{i}\varphi}|1\rangle\right)$$

where $\{|0\rangle,|1\rangle\}$ is an orthonormal basis of $\mathcal{H}$, and $\varphi=\varphi_1-\varphi_0$. The physically irrelevant phase factor $\mathrm{e}^{\mathrm{i}\varphi_0}$ can be ignored.

4′. There exist more than one observable.

5′. After a measurement, the state before the measurement $|\psi\rangle$ collapses to a known state which, in general, is different from $|\psi\rangle$.

6′. If the state $|\psi\rangle=a|0\rangle+b|1\rangle$ of a qubit is unknown, then there is no way to determine a and b with a single measurement.

32 If we measure a qubit in the state $|\psi\rangle=a|0\rangle+b|1\rangle$ in the eigenbasis $\{|0\rangle,|1\rangle\}$ of σ_z, then $|\psi\rangle$ collapses to $|0\rangle$ with the probability $|a|^2=\cos^2\frac{\theta}{2}$, and to $|1\rangle$ with the probability $|b|^2=\sin^2\frac{\theta}{2}$.

33 The state before the measurement $|\psi\rangle=a|0\rangle+b|1\rangle$ is a *coherent superposition* of $|0\rangle$ and $|1\rangle$. It is not a *probabilistic mixture* of $|0\rangle$ with the probability $|a|^2$ and $|1\rangle$ with the probability $|b|^2$ because, the state $|\psi\rangle$ depends not only on $|a|$ and $|b|$, but also on the relative phase φ of a and b.

34 The result of certain measurements depends on the phase factor $\mathrm{e}^{\mathrm{i}\varphi}$ (there exists quantum interference). If we measure $|\psi\rangle$ in the eigenbasis $\{|+\rangle,|-\rangle\}$ of σ_x, then the state before the measurement $|\psi\rangle$ collapses to $|+\rangle$ with probability $|a+b|^2/2=\left|\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\mathrm{e}^{\mathrm{i}\varphi}\right|^2/2$ and $|-\rangle$ with probability $|a-b|^2/2=\left|\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\mathrm{e}^{\mathrm{i}\varphi}\right|^2/2$.

P COMPOSITE QUANTUM SYSTEMS

■ Definitions and notations:

$\mathcal{H}_1 \equiv \mathcal{H}_A$ = Hilbert space of the main quantum system.
$\mathcal{H}_2 \equiv \mathcal{H}_B$ = Hilbert space of a reference quantum system.
$\{|j\rangle_1\} \equiv \{|j\rangle_A\}$ – orthonormal basis of $\mathcal{H}_1 \equiv \mathcal{H}_A$
$\{|k\rangle_2\} \equiv \{|k\rangle_B\}$ – orthonormal basis of $\mathcal{H}_2 \equiv \mathcal{H}_B$

1 $$|\varphi\psi\rangle \equiv |\varphi\rangle_1 \otimes |\psi\rangle_2 \equiv |\varphi\rangle_A \otimes |\psi\rangle_B$$

2 $$\boxed{\langle\varphi_1\psi_1|\varphi_2\psi_2\rangle = \langle\varphi_1|\varphi_2\rangle\,\langle\psi_1|\psi_2\rangle}$$
$$\langle ik|j\ell\rangle = \delta_{ij}\,\delta_{k\ell}$$

Pure state of the composite system $|\Phi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ $\boxed{|\Phi\rangle = \sum_{i,k} \Phi^{ik}|ik\rangle}$ $\langle\Phi|\Phi\rangle = \sum_{i,k} |\Phi^{ik}|^2 = 1.$

Quantum state of the composite system $\varrho : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2$ $\boxed{\varrho = \sum_{i,j,k,\ell} \varrho^{ik}_{j\ell}|ik\rangle\langle j\ell|}$ $\varrho \geq 0$, $\mathrm{tr}\,\varrho = 1.$

3 $${}_1\langle a| : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_2, \quad {}_2\langle b| : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1 \qquad \boxed{\begin{array}{c} {}_1\langle a|\varphi\psi\rangle = \langle a|\varphi\rangle\,|\psi\rangle \\ {}_2\langle b|\varphi\psi\rangle = \langle b|\psi\rangle\,|\varphi\rangle \end{array}}$$

4 $$\langle ab| : \mathcal{H}_1 \to \mathcal{H}_2^*, \quad \langle ab| : \mathcal{H}_2 \to \mathcal{H}_1^* \qquad \boxed{\begin{array}{c} \langle ab|\varphi\rangle_1 = \langle a|\varphi\rangle\,\langle b| \\ \langle ab|\psi\rangle_2 = \langle b|\psi\rangle\,\langle a| \end{array}}$$

For $\mathcal{H}_1 \otimes \mathcal{H}_2 \xrightarrow{T} \mathcal{H}_1 \otimes \mathcal{H}_2$ we define the operators:

5 $$\boxed{\begin{array}{l} {}_1\langle a|T : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_2 : \ |\varphi\psi\rangle \mapsto {}_1\langle a|T|\varphi\psi\rangle \\ {}_2\langle a|T : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1 : \ |\varphi\psi\rangle \mapsto {}_2\langle a|T|\varphi\psi\rangle \\ T|b\rangle_1 : \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2 : \ |\psi\rangle \mapsto T|b\psi\rangle \\ T|b\rangle_2 : \mathcal{H}_1 \to \mathcal{H}_1 \otimes \mathcal{H}_2 : \ |\varphi\rangle \mapsto T|\varphi b\rangle \end{array}}$$

6 $$\left.\begin{array}{c} A : \mathcal{H}_1 \to \mathcal{H}_1 \\ B : \mathcal{H}_2 \to \mathcal{H}_2 \\ \text{linear operators} \end{array}\right\} \Rightarrow \begin{array}{c} A \otimes B : \mathcal{H}_1 \otimes \mathcal{H}_2 \longrightarrow \mathcal{H}_1 \otimes \mathcal{H}_2 \\ \boxed{(A \otimes B)(\varphi \otimes \psi) = (A\varphi) \otimes (B\psi)} \\ \text{defines a linear operator.} \end{array}$$

7 For $\mathcal{H}_1 \otimes \mathcal{H}_2 \xrightarrow{T} \mathcal{H}_1 \otimes \mathcal{H}_2$
$$\boxed{\begin{array}{c} \mathrm{tr}_1 T : \mathcal{H}_2 \to \mathcal{H}_2, \\ \mathrm{tr}_2 T : \mathcal{H}_1 \to \mathcal{H}_1 \end{array}} \qquad \boxed{\begin{array}{c} \mathrm{tr}\,T = \sum_{j,k} \langle jk|T|jk\rangle \\ \mathrm{tr}_1 T = \sum_j {}_1\langle j|T|j\rangle_1, \\ \mathrm{tr}_2 T = \sum_k {}_2\langle k|T|k\rangle_2, \\ \mathrm{tr}\,T = \mathrm{tr}_1\,\mathrm{tr}_2\,T = \mathrm{tr}_2\,\mathrm{tr}_1\,T \end{array}}$$

■ Fundamental formulas:

8 $$\mathrm{tr}(|\varphi\rangle\langle\psi|) = \langle\psi|\varphi\rangle$$

9 $$\boxed{\begin{array}{c} {}_1\langle i|j\ell\rangle = \delta_{ij}\,|\ell\rangle_2 \\ {}_2\langle k|j\ell\rangle = \delta_{k\ell}\,|j\rangle_1 \end{array}} \qquad \boxed{\begin{array}{c} \langle ik|j\rangle_1 = \delta_{ij}\ {}_2\langle k| \\ \langle ik|\ell\rangle_2 = \delta_{k\ell}\ {}_1\langle i| \end{array}}$$

10 $$\boxed{\sum_j {}_2\langle j|\varphi\psi\rangle \otimes |j\rangle_2 = |\varphi\psi\rangle}$$

11 $$\begin{array}{c} {}_1\langle\varphi_1|A \otimes B|\varphi_2\rangle_1 = \langle\varphi_1|A|\varphi_2\rangle\,B, \\ {}_2\langle\psi_1|A \otimes B|\psi_2\rangle_2 = \langle\psi_1|B|\psi_2\rangle\,A. \end{array}$$

12 $$\left.\begin{array}{c} A : \mathcal{H}_1 \to \mathcal{H}_1 \\ B : \mathcal{H}_2 \to \mathcal{H}_2 \end{array}\right\} \Rightarrow \boxed{\begin{array}{c} \mathrm{tr}_1(A \otimes B) = (\mathrm{tr}\,A)\,B, \\ \mathrm{tr}_2(A \otimes B) = (\mathrm{tr}\,B)\,A. \end{array}}$$

13 $$|\varphi_1\psi_1\rangle\langle\varphi_2\psi_2| = |\varphi_1\rangle\langle\varphi_2| \otimes |\psi_1\rangle\langle\psi_2|.$$

14 $$\begin{array}{c} \mathrm{tr}_1(|\varphi_1\psi_1\rangle\langle\varphi_2\psi_2|) = \langle\varphi_2|\varphi_1\rangle\ |\psi_1\rangle\langle\psi_2|, \\ \mathrm{tr}_2(|\varphi_1\psi_1\rangle\langle\varphi_2\psi_2|) = \langle\psi_2|\psi_1\rangle\ |\varphi_1\rangle\langle\varphi_2|. \end{array}$$

15 $$\begin{array}{c} \mathcal{H}_1 \otimes \mathcal{H}_2 \xrightarrow{T} \mathcal{H}_1 \otimes \mathcal{H}_2 \\ \boxed{T^{ik}_{j\ell} = \langle ik|T|j\ell\rangle} \end{array} \Rightarrow \boxed{\begin{array}{c} (\mathrm{tr}_1 T)^k_\ell = {}_2\langle k|\mathrm{tr}_1 T|\ell\rangle_2 = \sum_j T^{jk}_{j\ell} \\ (\mathrm{tr}_2 T)^i_j = {}_1\langle i|\mathrm{tr}_2 T|j\rangle_1 = \sum_k T^{ik}_{jk} \end{array}}$$

16 $$\varrho = \sum_{i,j,k,\ell} \varrho^{ik}_{j\ell}|ik\rangle\langle j\ell| \ \Rightarrow \ \boxed{\mathrm{tr}_1\,\varrho = \sum_{j,k,\ell} \varrho^{jk}_{j\ell}|k\rangle\langle\ell|} \quad \boxed{\mathrm{tr}_2\,\varrho = \sum_{i,j,k} \varrho^{ik}_{jk}|i\rangle\langle j|}$$

17 $$\mathcal{H}_1 \otimes \mathcal{H}_2 \xrightarrow{T} \mathcal{H}_1 \otimes \mathcal{H}_2 \ \Rightarrow \boxed{\sum_{k,\ell} {}_2\langle k|T|\ell\rangle_2 \otimes |k\rangle_2\,{}_2\langle\ell| = T}.$$

18 $$\mathcal{H} \xrightarrow{A} \mathcal{H} \quad \mapsto \quad \mathrm{tr}(|\varphi\rangle\langle\psi|\,A) = \langle\psi|A|\varphi\rangle$$

19 $$\begin{array}{c} \mathrm{tr}_2((\mathbb{I} \otimes (|b\rangle\langle b|))T) = {}_2\langle b|T|b\rangle_2 \\ \mathrm{tr}_2(T(\mathbb{I} \otimes (|b\rangle\langle b|))) = {}_2\langle b|T|b\rangle_2 \end{array}$$

20 $$\mathrm{tr}((A \otimes B)T) = \mathrm{tr}(A\ \mathrm{tr}_2((\mathbb{I} \otimes B)T))$$

21 $$\mathrm{tr}((A \otimes B)T) = \mathrm{tr}(B\ \mathrm{tr}_1((A \otimes \mathbb{I})T))$$

22 $$\mathrm{tr}((A \otimes B)T) = \mathrm{tr}((A \otimes \mathbb{I})(\mathbb{I} \otimes B)T)$$

23 $$\boxed{\mathrm{tr}((A \otimes \mathbb{I})\varrho) = \mathrm{tr}(A\ \mathrm{tr}_2\,\varrho)}$$

24 $$\boxed{\mathrm{tr}((\mathbb{I} \otimes B)\varrho) = \mathrm{tr}(B\ \mathrm{tr}_1\,\varrho)}$$

25 $$\begin{array}{c} \mathcal{H}_1 \otimes \mathcal{H}_2 \xrightarrow{\varrho} \mathcal{H}_1 \otimes \mathcal{H}_2 \\ \text{density operator} \end{array} \Rightarrow \boxed{\begin{array}{c} \mathrm{tr}_1\,\varrho : \mathcal{H}_2 \to \mathcal{H}_2, \\ \mathrm{tr}_2\,\varrho : \mathcal{H}_1 \to \mathcal{H}_1 \\ \text{density operators} \end{array}}$$

26 For $\varrho : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$,
$$\varrho = \begin{pmatrix} \varrho^{00}_{00} & \varrho^{00}_{01} & \varrho^{00}_{10} & \varrho^{00}_{11} \\ \varrho^{01}_{00} & \varrho^{01}_{01} & \varrho^{01}_{10} & \varrho^{01}_{11} \\ \varrho^{10}_{00} & \varrho^{10}_{01} & \varrho^{10}_{10} & \varrho^{10}_{11} \\ \varrho^{11}_{00} & \varrho^{11}_{01} & \varrho^{11}_{10} & \varrho^{11}_{11} \end{pmatrix} \Rightarrow \begin{array}{l} \mathrm{tr}_1\,\varrho = \begin{pmatrix} \varrho^{00}_{00} & \varrho^{00}_{01} \\ \varrho^{01}_{00} & \varrho^{01}_{01} \end{pmatrix} + \begin{pmatrix} \varrho^{10}_{10} & \varrho^{10}_{11} \\ \varrho^{11}_{10} & \varrho^{11}_{11} \end{pmatrix}, \\ \mathrm{tr}_2\,\varrho = \begin{pmatrix} \mathrm{tr}\begin{pmatrix} \varrho^{00}_{00} & \varrho^{00}_{01} \\ \varrho^{01}_{00} & \varrho^{01}_{01} \end{pmatrix} & \mathrm{tr}\begin{pmatrix} \varrho^{00}_{10} & \varrho^{00}_{11} \\ \varrho^{01}_{10} & \varrho^{01}_{11} \end{pmatrix} \\ \mathrm{tr}\begin{pmatrix} \varrho^{10}_{00} & \varrho^{10}_{01} \\ \varrho^{11}_{00} & \varrho^{11}_{01} \end{pmatrix} & \mathrm{tr}\begin{pmatrix} \varrho^{10}_{10} & \varrho^{10}_{11} \\ \varrho^{11}_{10} & \varrho^{11}_{11} \end{pmatrix} \end{pmatrix}. \end{array}$$

■ Schmidt form:

27 Given $\boxed{\begin{array}{l} |\Phi\rangle = \sum_{kj} \Phi^{kj}|\varphi_k\rangle \otimes |\psi_j\rangle, \\ \text{with } \{\varphi_k\} \text{ chosen such that} \\ \mathrm{tr}_2|\Phi\rangle\langle\Phi| = \sum_{k=0}^{r-1} p_k|\varphi_k\rangle\langle\varphi_k| \end{array}}$

28 $$\sum_j \Phi^{kj}\,\overline{\Phi^{\ell j}} \overset{\Downarrow}{=} p_k\,\delta_{k\ell}.$$

29 By choosing $\boxed{|\eta_k\rangle = \frac{1}{\sqrt{p_k}} \sum_j \Phi^{kj}|\psi_j\rangle}$

30 we get $\boxed{\boxed{|\Phi\rangle = \sum_{k=0}^{r-1} \sqrt{p_k}|\varphi_k\rangle \otimes |\eta_k\rangle}}$

■ Purification: **mixed state = tr_2(pure state)**

31 Given $\boxed{\begin{array}{l} \varrho : \mathbb{C}^n \to \mathbb{C}^n, \ \varrho = \sum_{k=0}^{n-1} p_k|\varphi_k\rangle\langle\varphi_k|, \\ \{|\varphi_k\rangle\} \text{ orthonormal basis in } \mathbb{C}^n \end{array}}$

32 by choosing $\boxed{\begin{array}{c} \{|\psi_k\rangle\} \text{ orthonormal basis in } \mathbb{C}^n, \\ |\Phi\rangle = \sum_{k=0}^{n-1} \sqrt{p_k}\,|\varphi_k\rangle \otimes |\psi_k\rangle \end{array}}$

33 we get $\boxed{\boxed{\varrho = \mathrm{tr}_2|\Phi\rangle\langle\Phi|}}$

$\boxed{\#}$ ADDITIONAL DEFINITIONS/RESULTS

$\boxed{\#1}$ **Exercise**.

For a $\mathbb{K}$-vector space $\mathcal{V}$, $A\in\mathcal{L}(\mathcal{V})$ and $f:\mathcal{V}\to\mathbb{K}$:

$$\boxed{\begin{array}{r}Av=f(v)\,v,\\ \text{for any } v\end{array}} \Rightarrow \boxed{\begin{array}{l}f \text{ is a constant function,}\\ \quad f(v)=\lambda,\ \text{that is } A=\lambda\mathbb{I}\end{array}}$$

$\boxed{\#2}$ **Theorem** (*Singular value decomposition*).

$$\mathbf{A}\in\mathcal{M}_{n\times m}(\mathbb{C}) \Rightarrow \begin{array}{c}\text{there}\\ \text{exist}\end{array} \begin{array}{c}\mathbf{U}\in U(n),\\ \mathbf{V}\in U(m),\\ \mathbf{D}\in\mathcal{M}_{n\times m}(\mathbb{C})\\ \text{diagonal}\end{array} \begin{array}{c}\text{such}\\ \text{that}\end{array} \boxed{\mathbf{A}=\mathbf{UDV}}$$

$\boxed{\#3}$ **Theorem** (*Polar decomposition*).

$$\mathbf{A}\in\mathcal{M}_{n\times n}(\mathbb{C}) \Rightarrow \begin{array}{c}\text{there}\\ \text{exist}\end{array} \begin{array}{c}\mathbf{U}\in U(n),\\ \mathbf{P}\in\mathcal{M}_{n\times m}(\mathbb{C})\\ \text{with } \mathbf{P}\geq 0\end{array} \begin{array}{c}\text{such}\\ \text{that}\end{array} \boxed{\mathbf{A}=\mathbf{UP}}$$

$$\mathbf{A}\in\mathcal{M}_{n\times n}(\mathbb{C}) \Rightarrow \begin{array}{c}\text{there}\\ \text{exist}\end{array} \begin{array}{c}\mathbf{U}\in U(n),\\ \mathbf{P}\in\mathcal{M}_{n\times m}(\mathbb{C})\\ \text{with } \mathbf{P}\geq 0\end{array} \begin{array}{c}\text{such}\\ \text{that}\end{array} \boxed{\mathbf{A}=\mathbf{PU}}$$

$\boxed{\#4}$ **Definition.**

A linear operator $A:\mathcal{H}\to\mathcal{H}$ is a *bounded operator* if there exists a constant $C\in(0,\infty)$ such that $||Ax||\leq C\,||x||,\ \forall\, x\in\mathcal{H}$.

$\boxed{\#5}$ **Theorem**.

$$\mathcal{H} \text{ is finite-dimensional} \ \Rightarrow \boxed{\begin{array}{c}\text{Any linear operator}\\ A:\mathcal{H}\to\mathcal{H}\\ \text{is a } \textit{bounded operator}.\end{array}}$$

$\boxed{\#6}$ **Definition.**

A *tight frame* in a Hilbert space $\mathcal{H}$ is a set of vectors $\{u_1,u_2,\ldots,u_\ell\}$ such that $\boxed{\sum_{j=1}^{\ell}|u_j\rangle\langle u_j|=\mathbb{I}_{\mathcal{H}}}$

$\boxed{\#7}$ **Remark**.

Any orthonormal basis is a tight frame.

$\boxed{\#8}$ **Exercise**

$$\boxed{(\mathrm{e}^{A})^\dagger=\mathrm{e}^{A^\dagger}}.$$

$\boxed{\#9}$ **Exercise**.

$$\boxed{A\in\mathcal{A}(\mathcal{H}) \quad \Rightarrow \mathrm{e}^{\mathrm{i}A}\in U(\mathcal{H})}.$$

$\boxed{\#10}$ **Theorem** Discrete (finite) *Fourier transform*.

$$\begin{array}{c}F:\mathbb{C}^n\longrightarrow\mathbb{C}^n\\ \varphi\mapsto F[\varphi]\end{array} \quad \boxed{F[\varphi](k)=\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{n}kj}\varphi(j)}$$

defined by using the identification $\{\varphi:\{0,1,\ldots,n-1\}\to\mathbb{C}\}\equiv\mathbb{C}^n$

$$\varphi\equiv(\varphi(0),\varphi(1),\ldots,\varphi(n-1)),$$

is a unitary transformation. Its inverse is

$$F^{-1}=F^\dagger,\ \text{where} \ \boxed{F^\dagger[\varphi](k)=\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}\mathrm{e}^{\frac{2\pi\mathrm{i}}{n}kj}\varphi(j)}$$

$\boxed{\#11}$ **Theorem** Alternative definition of $\mathcal{H}_1\otimes\mathcal{H}_2$.

$\mathcal{H}_1\otimes\mathcal{H}_2$ can be defined as $\boxed{\text{the space of all the anti-linear operators } T:\mathcal{H}_2\to\mathcal{H}_1.}$

$\boxed{\#12}$ **Theorem**.

A tensor product of tight frames is a tight frame.

$\boxed{\#13}$ **Remark**.

The superposition (liniar combination) of states

$$|\psi\rangle=\sum_{k=1}^{n}c_k|\psi_k\rangle$$

is a pure state with the density operator

$$\varrho=|\psi\rangle\langle\psi|=\sum_{k=1}^{n}|c_k|^2|\psi_k\rangle\langle\psi_k|+\sum_{k\neq j}c_k\,\bar{c}_j\,|\psi_k\rangle\langle\psi_j|.$$

The second sum contains the *interference terms*.

$\boxed{\#14}$ **Exercise**.

Representations as a probabilistic ensemble of pure states of the maximally mixed state of a qubit:

$$\varrho=\begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix} \Rightarrow$$

- $\varrho=\frac{1}{2}|0\rangle\langle 0|+\frac{1}{2}|1\rangle\langle 1|,$
- $\varrho=\frac{1}{2}|+\rangle\langle +|+\frac{1}{2}|-\rangle\langle -|,$
- $\varrho=\frac{1}{2}|\uparrow_y\rangle\langle\uparrow_y|+\frac{1}{2}|\downarrow_y\rangle\langle\downarrow_y|,$
- $\varrho=\frac{1}{4}|0\rangle\langle 0|+\frac{1}{4}|1\rangle\langle 1| +\frac{1}{4}|+\rangle\langle +|+\frac{1}{4}|-\rangle\langle -|,$
- etc.

$\boxed{\#15}$ **Exercise**.

$\boxed{\text{A mixed state } \varrho \text{ of any quantum system can be realised as an } \textit{ensemble} \text{ of orthogonal pure states.}}$

$\boxed{\#16}$ **Exercise**.

In the Schrödinger picture, the time evolution of the density operator

$$\varrho=\sum_{k=0}^{d-1}p_k|\psi_k\rangle\langle\psi_k|$$

for a closed system is described by

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\varrho=[H,\varrho],$$

where H is the Hamiltonian of the system.

MAIN PROOFS

$\boxed{A}$ VECTOR SPACES

$\boxed{A16}$

$$\alpha_1v_1+\alpha_2v_2+\cdots+\alpha_nv_n=0$$
$$\Downarrow\ \begin{array}{l}\alpha_n=0 \text{ since}\\ \alpha_n\neq0\Rightarrow v_n=-\frac{\alpha_1}{\alpha_n}v_1-\ldots-\frac{\alpha_{n-1}}{\alpha_n}v_{n-1}\end{array}$$
$$\alpha_1v_1+\alpha_2v_2+\cdots+\alpha_{n-1}v_{n-1}=0$$
$$\Downarrow\ \begin{array}{l}\alpha_{n-1}=0 \text{ since}\\ \alpha_{n-1}\neq0\Rightarrow v_{n-1}=-\frac{\alpha_1}{\alpha_{n-1}}v_1-\ldots-\frac{\alpha_{n-2}}{\alpha_{n-1}}v_{n-2}\end{array}$$
$$\alpha_1v_1+\alpha_2v_2+\cdots+\alpha_{n-2}v_{n-2}=0$$
$$\Downarrow$$
....
$$\Downarrow$$
$$\alpha_1v_1=0$$
$$\Downarrow v_1\neq0$$
$$\alpha_1=0.$$

$\boxed{A18}$

$$x_1v_1+x_2v_2+\cdots+x_nv_n=x_1'v_1+x_2'v_2+\cdots+x_n'v_n$$
$$\Downarrow$$
$$(x_1-x_1')v_1+(x_2-x_2')v_2+\cdots+(x_n-x_n')v_n=0$$
$$\Downarrow$$
$$x_1-x_1'=x_2-x_2'=\cdots=x_n-x_n'=0$$
$$\Downarrow$$
$$x_1=x_1',\quad x_2=x_2',\quad \ldots,\quad x_n=x_n'.$$

$\boxed{A19}$

We repeat as many times as necessary the procedure of eliminating the first vector which is a linear combination of the vectors written before it (see $\boxed{A16}$).

$\boxed{A21}$

Step 1 : $\mathcal{V}=\operatorname{span}\{w_1,w_2,\ldots,w_k\}$
$$\Downarrow$$
$v_1=\alpha_1w_1+\alpha_2w_2+\cdots+\alpha_nw_n.$

There exists $\alpha_{i_1}\neq0$ and consequently $\mathcal{S}_1=\{v_1,w_1,w_2,\ldots,w_k\}\backslash\{w_{i_1}\}$ spans $\mathcal{V}$.

Step 2 : $\mathcal{V}=\operatorname{span}\mathcal{S}_1$
$$\Downarrow$$
$$v_2=\beta_1v_1+\sum_{i\neq i_1}\gamma_iw_i.$$

There exists $\gamma_{i_2}\neq0$ and consequently $\mathcal{S}_2=\{v_1,v_2,w_1,w_2,\ldots,w_k\}\backslash\{w_{i_1},w_{i_2}\}$ spans $\mathcal{V}$.

Step 3 : $\mathcal{V}=\operatorname{span}\mathcal{S}_2$
$$\Downarrow$$
$$v_3=av_1+bv_2+\sum_{i\notin\{i_1,i_2\}}c_iw_i.$$

There exists $c_{i_3}\neq0$ and consequently $\mathcal{S}_3=\{v_1,v_2,v_3,w_1,\ldots,w_k\}\backslash\{w_{i_1},w_{i_2},w_{i_2}\}$ spans $\mathcal{V}$.

After at most n steps we can arrive at $\tilde{\mathcal{S}}=\{v_1,\ldots,v_n,w_1,\ldots,w_k\}\backslash\{w_{i_1},\ldots,w_{i_n}\}$ spans $\mathcal{V}$.

Therefore $n\leq k$.

From $\tilde{\mathcal{S}}$ we can extract a basis by using $\boxed{A19}$.

$\boxed{A22}$

$$\left.\begin{array}{l}\{v_1,v_2,\ldots,v_n\} \text{ basis of } \mathcal{V}\\ \{v_1',v_2',\ldots,v_m'\} \text{ basis of } \mathcal{V}\end{array}\right\}\ \overset{\boxed{A21}}{\Rightarrow}\ \begin{cases}n\leq m\\ m\leq n.\end{cases}$$

$\boxed{A29}$

We extend $\{v_1,v_2,\ldots,v_n\}$ basis of $\mathcal{W}_1\cap\mathcal{W}_2$ up to
$$\left.\begin{array}{l}\{v_1,v_2,\ldots,v_n,w_1,w_2,\ldots,w_m\} \text{ basis of } \mathcal{W}_1\\ \{v_1,v_2,\ldots,v_n,u_1,u_2,\ldots,u_k\} \text{ basis of } \mathcal{W}_2\end{array}\right\}$$
$$\Downarrow$$
$$\{v_1,v_2,\ldots,v_n,w_1,w_2,\ldots,w_m,u_1,u_2,\ldots,u_k\}$$
is a basis of $\mathcal{W}_1+\mathcal{W}_2$.

$$\alpha_1v_1+\ldots+\alpha_nv_n+\beta_1w_1+\ldots+\beta_mw_m+\gamma_1u_1+\ldots+\gamma_ku_k=0$$
$$\Downarrow$$
$$\alpha_1v_1+\ldots+\alpha_nv_n+\beta_1w_1+\ldots+\beta_mw_m=-\gamma_1u_1-\ldots-\gamma_ku_k$$
$$\Downarrow$$
$$-\gamma_1u_1-\ldots-\gamma_ku_k\in\mathcal{W}_1\cap\mathcal{W}_2$$
$$\Downarrow$$
$$-\gamma_1u_1-\ldots-\gamma_ku_k=\delta_1v_1+\ldots+\delta_nv_n$$
$$\Downarrow$$
$$\gamma_1=\ldots=\gamma_k=\delta_1=\ldots=\delta_n=0 \text{ etc.}$$

$\boxed{A30}$

$$\left.\begin{array}{c}x\in\mathcal{W}_1\cap\mathcal{W}_2\\ x\neq0\end{array}\right\}\ \Rightarrow\ x=x+0=0+x.$$
$$\left.\begin{array}{c}\mathcal{W}_1\cap\mathcal{W}_2=\{0\}\\ w_1+w_2=x=w_1'+w_2'\end{array}\right\}\ \Rightarrow\ w_1-w_1'=w_2-w_2'=0.$$

$\boxed{A34}$

$$\left.\begin{array}{l}\{v_1,v_2,\ldots,v_n\} \text{ basis of } \mathcal{V}_1\\ \{w_1,w_2,\ldots,w_k\} \text{ basis of } \mathcal{V}_2\end{array}\right\}$$
$$\Downarrow$$
$$\{(v_1,0),(v_2,0),\ldots,(v_n,0),(0,w_1),(0,w_2),\ldots,(0,w_k)\}$$
is a basis of $\mathcal{V}_1\times\mathcal{V}_2$.

$\boxed{A35}$

$\{v_1,v_2,\ldots,v_n\}$ basis of complex space $\mathcal{V}$
$$\Downarrow$$
$\{v_1,\mathrm{i}v_1,v_2,\mathrm{i}v_2,\ldots,v_n,\mathrm{i}v_n\}$ basis of real space $\mathcal{V}$

$\boxed{A36}$

$\{v_1,v_2,\ldots,v_n\}$ basis of real space $\mathcal{V}$
$$\Downarrow$$
$\{(v_1,0),(v_2,0),\ldots,(v_n,0)\}$ basis of complex space $\mathcal{V}^{\mathbb{C}}$

$\boxed{A37}$

$$\left.\begin{array}{l}\{v_1,v_2,\ldots,v_n\} \text{ basis of } \mathcal{W}\\ \{v_1,v_2,\ldots,v_n,w_1,w_2,\ldots,w_k\} \text{ basis of } \mathcal{V}\end{array}\right\}$$
$$\Downarrow$$
$\{\hat{w}_1,\hat{w}_2,\ldots,\hat{w}_k\}$ is a basis of $\mathcal{V}/\mathcal{W}$.

$\boxed{A45}$

$$\left.\begin{array}{r}v_j'=\alpha_j^kv_k\\ v_k=\beta_k^jv_j'\end{array}\right\}\ \Rightarrow\ \begin{array}{r}v_j'=\alpha_j^k\beta_k^mv_m'\\ v_k=\beta_k^m\alpha_m^kv_k\end{array}\ \Rightarrow\ \begin{array}{r}\beta_k^m\alpha_j^k=\delta_j^m\\ \alpha_m^j\beta_k^m=\delta_k^j.\end{array}$$

$\boxed{A46}$

$$\left.\begin{array}{r}v_j'=\alpha_j^kv_k\\ x=x^kv_k=x'^jv_j'\\ v_k=\beta_k^jv_j'\end{array}\right\}\ \Rightarrow\ \begin{array}{r}x^kv_k=x'^j\alpha_j^kv_k\\ x'^jv_j'=x^k\beta_k^jv_j'\end{array}\ \Rightarrow\ \begin{array}{r}x'^j=\beta_k^jx^k\\ x^k=\alpha_j^kx'^j.\end{array}$$

$\boxed{B}$ LINEAR MAPS AND MATRICES

$\boxed{B5}$

$$\left.\begin{array}{l}\{v_1,v_2,\ldots,v_k\} \text{ basis of } \operatorname{Ker}A\\ \{v_1,v_2,\ldots,v_k,v_{k+1},\ldots,v_n\} \text{ basis of } \mathcal{V}\end{array}\right\}$$
$$\Downarrow$$
$\{Av_{k+1},Av_{k+2},\ldots,Av_n\}$ is a basis of $\operatorname{Im}A$.

$\boxed{B8}$

$$\alpha x+\beta y=\alpha AA^{-1}x+\beta AA^{-1}y=A(\alpha A^{-1}x+\beta A^{-1}y)$$
$$\Downarrow$$
$$A^{-1}(\alpha x+\beta y)=\alpha A^{-1}x+\beta A^{-1}y.$$

$\boxed{B9}$

$$\left.\begin{array}{l}A:\mathcal{V}\longrightarrow\mathcal{W} \text{ isomorphism}\\ \{v_1,v_2,\ldots,v_n\} \text{ basis of } \mathcal{V}\end{array}\right\}\ \Rightarrow\ \{Av_1,Av_2,\ldots,Av_n\} \text{ basis of } \mathcal{W}.$$
$$\left.\begin{array}{l}\{v_1,v_2,\ldots,v_n\} \text{ basis of } \mathcal{V}\\ \{w_1,w_2,\ldots,w_n\} \text{ basis of } \mathcal{W}\end{array}\right\}\ \Rightarrow\ \begin{array}{l}A:\mathcal{V}\longrightarrow\mathcal{W}\\ A(x^kv_k)=x^kw_k\\ \text{is isomorphism.}\end{array}$$

$\boxed{B13}$

$$\left.\begin{array}{r}Av_k=a_k^jv_j\\ Av_k'=a'^j_kv_j'\\ v_k'=\alpha_k^jv_j\\ v_j=\beta_j^kv_k'\end{array}\right\}\ \Rightarrow\ \begin{array}{l}a'^j_kv_j'=Av_k'=A(\alpha_k^mv_m)=\alpha_k^m\,Av_m\\ \qquad\quad=\alpha_k^m\,a_m^iv_i=\alpha_k^m\,a_m^i\beta_i^jv_j'\\ \qquad\Downarrow\\ \quad a'^j_k=\beta_i^j\,a_m^i\,\alpha_k^m.\end{array}$$

$\boxed{B14}$

$$c^j_k w_j = Cu_k = (BA)u_k = B(Au_k) = B(a^m_k v_m) = a^m_k\, Bv_m = a^m_k\, b^j_m w_j.$$

$\boxed{B17}$

$$\det(\mathbf{A}'-\lambda\mathbf{I}) = \det(\mathbf{S}^{-1}\mathbf{A}\mathbf{S}-\lambda\mathbf{I}) = \det\mathbf{S}^{-1}(\mathbf{A}-\lambda\mathbf{I})\mathbf{S} = \tfrac{1}{\det\mathbf{S}}\det(\mathbf{A}-\lambda\mathbf{I})\det\mathbf{S} = \det(\mathbf{A}-\lambda\mathbf{I}).$$

$\boxed{B18,19}$

$$P(\lambda) = (-1)^n\lambda^n + (-1)^{n-1}\ \operatorname{tr}\mathbf{A}\,\lambda^{n-1} + \cdots + \det\mathbf{A}.$$

$\boxed{B22}$

$$\lambda \text{ eigenvalue } \Leftrightarrow \begin{cases} (a^1_1-\lambda)x^1 + a^1_2 x^2 + \cdots + a^1_n x^n = 0 \\ a^2_1 x^1 + (a^2_2-\lambda)x^2 + \cdots + a^2_n x^n = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a^n_1 x^1 + a^n_2 x^2 + \cdots + (a^n_n-\lambda)x^n = 0 \end{cases}$$

admits a solution $(x^1, x^2, ..., x^n) \neq (0, 0, ..., 0)$.

$\boxed{B25}$

$$\mathbf{A} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \quad\Leftrightarrow\quad Av_k = \lambda_k v_k.$$

$\boxed{B26}$ (Particular case $n=3$.)

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$$
$$\Downarrow A$$
$$\alpha_1\lambda_1 v_1 + \alpha_2\lambda_2 v_2 + \alpha_3\lambda_3 v_3 = 0$$
$$\Downarrow$$
$$\alpha_1(\lambda_1-\lambda_3)v_1 + \alpha_2(\lambda_2-\lambda_3)v_2 = 0$$
$$\Downarrow A$$
$$\alpha_1\lambda_1(\lambda_1-\lambda_3)v_1 + \alpha_2\lambda_2(\lambda_2-\lambda_3)v_2 = 0$$
$$\Downarrow$$
$$\alpha_1(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)v_1 = 0$$
$$\Downarrow$$
$$\alpha_1 = \alpha_2 = \alpha_3 = 0.$$

$\boxed{B30}$ "$\Rightarrow$"

$$\mathbf{A} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \Rightarrow P(\lambda) = (\lambda_1-\lambda)(\lambda_2-\lambda)\cdots(\lambda_n-\lambda).$$

"$\Leftarrow$" (Particular case of two eigenvalues λ_1 and λ_2 of multiplicities 2 and 3)

$$\mathcal{V}_{\lambda_1} = \operatorname{span}\{v_1, v_2\}$$
$$\mathcal{V}_{\lambda_2} = \operatorname{span}\{w_1, w_2, w_3\}$$
$$\Downarrow$$

$$\boxed{\begin{array}{c} \alpha_1 v_1 + \alpha_2 v_2 + \beta_1 w_1 + \beta_2 w_2 + \beta_3 w_3 = 0 \\ \Downarrow A \\ \lambda_1(\alpha_1 v_1 + \alpha_2 v_2) + \lambda_2(\beta_1 w_1 + \beta_2 w_2 + \beta_3 w_3) = 0 \\ \Downarrow \\ (\lambda_1-\lambda_2)(\alpha_1 v_1 + \alpha_2 v_2) = 0 \\ \Downarrow \\ \alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \beta_3 = 0 \\ \Downarrow \\ \{v_1, v_2, w_1, w_2, w_3\} \text{ is a basis of } \mathcal{V}. \end{array}}$$

C TENSORS

$\boxed{C2}$

$$\begin{array}{c}\{v_1, v_2, ..., v_n\} \\ \text{basis of } \mathcal{V}\end{array} \Rightarrow \boxed{\begin{array}{c}\{v^1, v^2, ..., v^n\}, \\ v^j : \mathcal{V} \longrightarrow \mathbb{K}, \\ v^j(x^k v_k) = x^j\end{array}} \text{ is a basis of } \mathcal{V}^*.$$

$\boxed{C6,7,8}$

$$\left.\begin{array}{r} v'_j = \alpha^k_j v_k \\ v_k = \beta^j_k v'_j \end{array}\right\} \Rightarrow \begin{array}{l} v'_j = \alpha^k_j \beta^m_k v'_m \\ v_k = \beta^m_k \alpha^k_m v_k \end{array} \Rightarrow \boxed{\begin{array}{c} \beta^m_k \alpha^k_j = \delta^m_j \\ \alpha^j_m \beta^m_k = \delta^j_k. \end{array}}$$

$$\left.\begin{array}{r} x = x^i v_i = x'^j v'_j \\ \varphi = \varphi_m v^m = \varphi'_k v'^k \end{array}\right\} \Rightarrow \boxed{\begin{array}{r} x^i = v^i(x) \\ x'^j = v'^j(x) \end{array}} \quad \boxed{\begin{array}{r} \varphi_m = \varphi(v_m) \\ \varphi'_k = \varphi(v'_k). \end{array}}$$

$$\left.\begin{array}{r} Av_k = a^j_k v_j \\ Av'_k = a'^j_k v'_j \end{array}\right\} \Rightarrow \begin{array}{l} a'^j_k v'_j = Av'_k = A(\alpha^m_k v_m) = \alpha^m_k\, Av_m \\ \qquad = \alpha^m_k\, a^i_m v_i = \alpha^m_k\, a^i_m \beta^j_i v'_j \end{array}$$
$$\Downarrow$$
$$\boxed{a'^j_k = \beta^j_i\, a^i_m\, \alpha^m_k}.$$

$\boxed{C11}$ (particular case $p=1$, $q=2$)

$T : \mathcal{V}^* \times \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{K}$ three linear map

$$\Downarrow$$
$$\begin{aligned} T'^{j_1}_{k_1 k_2} &= T(v'^{j_1}, v'_{k_1}, v'_{k_2}) \\ &= T(\beta^{j_1}_{i_1} v^{i_1}, \alpha^{m_1}_{k_1} v_{m_1}, \alpha^{m_2}_{k_2} v_{m_2}) \\ &= \beta^{j_1}_{i_1} \alpha^{m_1}_{k_1} \alpha^{m_2}_{k_2} T(v^{i_1}, v_{m_1}, v_{m_2}) \\ &= \beta^{j_1}_{i_1} \alpha^{m_1}_{k_1} \alpha^{m_2}_{k_2} T^{i_1}_{m_1 m_2}. \end{aligned}$$

Conversely, $\quad T'^{j_1}_{k_1 k_2} = \beta^{j_1}_{i_1} \alpha^{m_1}_{k_1} \alpha^{m_2}_{k_2} T^{i_1}_{m_1 m_2}$

$$\Downarrow$$
$$\begin{aligned} T'^{j_1}_{k_1 k_2} \varphi'_{j_1} x'^{k_1} y'^{k_2} &= \beta^{j_1}_{i_1} \alpha^{m_1}_{k_1} \alpha^{m_2}_{k_2} T^{i_1}_{m_1 m_2} \alpha^{a_1}_{j_1} \beta^{k_1}_{b_1} \beta^{k_2}_{b_2} \varphi_{a_1} x^{b_1} y^{b_2} \\ &= T^{i_1}_{m_1 m_2} \varphi_{i_1} x^{m_1} y^{m_2} \end{aligned}$$
$$\Downarrow$$

the three linear map defined by using a basis

$$T : \mathcal{V}^* \times \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{K}, \quad T(\varphi, x, y) = T^{i_1}_{m_1 m_2} \varphi_{i_1} x^{m_1} y^{m_2}$$

does not depend on the basis we chose.

$\boxed{C12}$ (particular case $p=1$, $q=2$)

$$\begin{aligned} T'^{j_1}_{k_1 k_2} + S'^{j_1}_{k_1 k_2} &= \beta^{j_1}_{i_1} \alpha^{m_1}_{k_1} \alpha^{m_2}_{k_2} T^{i_1}_{m_1 m_2} + \beta^{j_1}_{i_1} \alpha^{m_1}_{k_1} \alpha^{m_2}_{k_2} S^{i_1}_{m_1 m_2} \\ &= \beta^{j_1}_{i_1} \alpha^{m_1}_{k_1} \alpha^{m_2}_{k_2} (T^{i_1}_{m_1 m_2} + S^{i_1}_{m_1 m_2}). \end{aligned}$$

$\boxed{C14}$ (particular case $p=1$, $q=2$, $r=1$, $s=1$)

$$\begin{aligned} T'^{j_1}_{k_1 k_2}\ S'^{j_2}_{k_3} &= \beta^{j_1}_{i_1} \alpha^{m_1}_{k_1} \alpha^{m_2}_{k_2} T^{i_1}_{m_1 m_2} \beta^{j_2}_{i_2} \alpha^{m_3}_{k_3} S^{i_2}_{m_3} \\ &= \beta^{j_1}_{i_1} \alpha^{m_1}_{k_1} \alpha^{m_2}_{k_2} \beta^{j_2}_{i_2} \alpha^{m_3}_{k_3} (T^{i_1}_{m_1 m_2}\ S^{i_2}_{m_3}). \end{aligned}$$

$\boxed{C15}$ (particular case $p=2$, $q=2$, $r=1$, $s=2$)

$$\begin{aligned} S'^{j_2}_{k_1} &= \sum_{m=1}^{n} T'^{m j_2}_{k_1 m} = \sum_{m=1}^{n} \beta^m_{i_1} \beta^{j_2}_{i_2} \alpha^{\ell_1}_{k_1} \alpha^{\ell_2}_m T^{i_1 i_2}_{\ell_1 \ell_2} \\ &= \sum_{i_1=1}^{n} \beta^{j_2}_{i_2} \alpha^{\ell_1}_{k_1} T^{i_1 i_2}_{\ell_1 i_1} = \beta^{j_2}_{i_2} \alpha^{\ell_1}_{k_1} \sum_{i_1=1}^{n} T^{i_1 i_2}_{\ell_1 i_1} \\ &= \beta^{j_2}_{i_2} \alpha^{\ell_1}_{k_1} S^{i_2}_{\ell_1}. \end{aligned}$$

D FINITE-DIMENSIONAL HILBERT SPACES

$\boxed{D5}$

Case $y=0$: $\quad |\langle x, 0\rangle| = 0 = ||x||\ ||0||.$

Case $y\neq 0$: $\quad x = \lambda\, y \ \Rightarrow \langle x, y\rangle = \bar{\lambda}\langle y, y\rangle \Rightarrow \ \lambda = \frac{\langle y, x\rangle}{\langle y, y\rangle}.$

$$\text{For } \lambda = \tfrac{\langle y, x\rangle}{\langle y, y\rangle} \quad \boxed{\begin{array}{c} ||x-\lambda\, y||^2 \geq 0 \\ \Updownarrow \\ \langle x-\lambda\, y, x-\lambda\, y\rangle \geq 0 \\ \Updownarrow \\ \langle x, x\rangle\langle y, y\rangle \geq |\langle x, y\rangle|^2 \\ \Updownarrow \\ ||x||\ ||y|| \geq |\langle x, y\rangle| \end{array}} \quad \boxed{\begin{array}{c} x = \lambda\, y \\ \Uparrow \\ \langle x-\lambda\, y, x-\lambda\, y\rangle = 0 \\ \Uparrow \\ \langle x, x\rangle\langle y, y\rangle = |\langle x, y\rangle|^2 \\ \Uparrow \\ ||x||\ ||y|| = |\langle x, y\rangle| \end{array}}$$

$\boxed{D12}$

$$\sum_{k=1}^{n} \alpha_k v_k = 0$$
$$\Downarrow$$
$$\alpha_m = \sum_{k=1}^{n} \alpha_k\, \langle v_m, v_k\rangle = \left\langle v_m, \sum_{k=1}^{n} \alpha_k v_k \right\rangle = 0.$$

$\boxed{D13}$

$$P_w x = \lambda w$$
$$\Downarrow$$
$$\langle w, x-\lambda w\rangle = 0 \quad\Rightarrow\quad \lambda = \tfrac{\langle w, x\rangle}{\langle w, w\rangle}.$$

$\boxed{D15}$

$$\begin{aligned} \langle w_1, w_2\rangle &= \left\langle w_1, v_2 - \tfrac{\langle w_1, v_2\rangle}{\langle w_1, w_1\rangle}\, w_1 \right\rangle \\ &= \langle w_1, v_2\rangle - \tfrac{\langle w_1, v_2\rangle}{\langle w_1, w_1\rangle}\, \langle w_1, w_1\rangle = 0 \text{ etc.} \end{aligned}$$

$\boxed{D17}$

$$x=\sum_{m=1}^{n}\alpha_m e_m$$
$$\Downarrow$$
$$\langle e_k,x\rangle=\left\langle e_k,\sum_{m=1}^{n}\alpha_m e_m\right\rangle=\sum_{m=1}^{n}\alpha_m\langle e_k,e_m\rangle=\alpha_k.$$
$$\langle x,y\rangle=\left\langle\sum_{k=1}^{n}\langle e_k,x\rangle e_k,y\right\rangle=\sum_{k=1}^{n}\langle x,e_k\rangle\langle e_k,y\rangle.$$
$$||x||=\sum_{k=1}^{n}\langle x,e_k\rangle\langle e_k,x\rangle=\sum_{k=1}^{n}|\langle e_k,x\rangle|^2.$$

$\boxed{D18}$

$\{v_1,v_2,...,v_m\}$ basis of span $\mathcal{M}$

$\downarrow$ extended up to

$\{v_1,v_2,...,v_m,v_{m+1},...,v_n\}$ basis of $\mathcal{H}$

$\boxed{D16}\downarrow$ Gram-Schmidt orthonormalization

$$\boxed{\begin{array}{l}\{u_1,u_2,...,u_m,u_{m+1},...,u_n\}\text{ orthonormal basis of }\mathcal{H}\\ \{u_1,u_2,...,u_m\}\text{ orthonormal basis of span }\mathcal{M}\end{array}}$$
$$\Downarrow$$

$\{u_{m+1},...,u_n\}$ orthonormal basis of $\mathcal{M}^\perp$.

$\boxed{D19}$

$$x\in\mathcal{K}\cap\mathcal{K}^\perp\ \Rightarrow\ \langle x,x\rangle=0\ \Rightarrow\ x=0.$$
$$\boxed{D18}\ \Rightarrow\ \dim\mathcal{K}\oplus\mathcal{K}^\perp=\dim\mathcal{H}.$$

$\boxed{D20}$

$$\langle x,v\rangle=\langle y,v\rangle\ \Rightarrow\ \langle x-y,v\rangle=0\quad\text{for any } v\in\mathcal{H}$$
$$\Downarrow v=x-y$$
$$\langle x-y,x-y\rangle=0$$
$$\Downarrow$$
$$x-y=0.$$

$\boxed{D21}$

$\{e_1,e_2,...,e_n\}$ orthonormal basis of $\mathcal{H}$

$$\Downarrow$$
$$\begin{aligned}\varphi(x)&=\varphi\Big(\sum_{k=1}^{n}\langle e_k,x\rangle e_k\Big)\\&=\sum_{k=1}^{n}\langle e_k,x\rangle\varphi(e_k)\\&=\sum_{k=1}^{n}\langle\overline{\varphi(e_k)}e_k,x\rangle\\&=\left\langle\sum_{k=1}^{n}\overline{\varphi(e_k)}e_k,x\right\rangle\quad\Rightarrow\quad a=\sum_{k=1}^{n}\overline{\varphi(e_k)}e_k.\end{aligned}$$

$\boxed{D37}$

$\{e_1,e_2,...,e_n\}$
$\{e'_1,e'_2,...,e'_n\}$
orthonormal
bases of $\mathcal{H}$

$$\mathbf{S}=\begin{pmatrix}\alpha_1^1&\alpha_2^1&\cdots&\alpha_n^1\\\alpha_1^2&\alpha_2^2&\cdots&\alpha_n^2\\\vdots&\vdots&\ddots&\vdots\\\alpha_1^n&\alpha_2^n&\cdots&\alpha_n^n\end{pmatrix}$$

$$e'_k=\alpha_k^j e_j\ \Rightarrow\ \alpha_k^j=\langle e_j,e'_k\rangle$$
$$\Downarrow$$
$$\begin{aligned}\sum_{j=1}^{n}\bar\alpha_k^j\alpha_m^j&=\sum_{j=1}^{n}\overline{\langle e_j,e'_k\rangle}\langle e_j,e'_m\rangle\\&=\sum_{j=1}^{n}\langle e'_k,e_j\rangle\langle e_j,e'_m\rangle\\&=\left\langle e'_k,\sum_{j=1}^{n}\langle e_j,e'_m\rangle e_j\right\rangle\\&=\langle e'_k,e'_m\rangle=\delta_{km}\ \Rightarrow\ S^\dagger S=\mathbb{I}.\end{aligned}$$

$\boxed{D38}$

$$|x\rangle\in\mathcal{H}\ \Rightarrow\ |x\rangle=\sum_{k=1}^{m}|e_k\rangle\langle e_k|x\rangle+\sum_{k=m+1}^{n}|e_k\rangle\langle e_k|x\rangle$$
$$\Downarrow$$
$$|x\rangle=P_{\mathcal{K}}|x\rangle+P_{\mathcal{K}^\perp}|x\rangle\in\mathcal{K}\oplus\mathcal{K}^\perp$$
$$\Downarrow$$
$$\mathbb{I}_{\mathcal{H}}=P_{\mathcal{K}}+P_{\mathcal{K}^\perp}.$$

$\boxed{\mathbf{E}}$ SELF-ADJOINT (HERMITIAN) OPERATORS

$\boxed{E1}$ Uniqueness of $A^\dagger$

$$\begin{array}{l}\langle A_1^\dagger x,y\rangle=\langle x,Ay\rangle\\\langle A_2^\dagger x,y\rangle=\langle x,Ay\rangle\end{array}\ \Rightarrow\ \langle A_1^\dagger x,y\rangle=\langle A_2^\dagger x,y\rangle,\quad\forall x,y$$
$$\Downarrow$$
$$A_1^\dagger x=A_2^\dagger x,\quad\forall x.$$

Existence of $A^\dagger$

$\{e_1,e_2,...,e_n\}$ orthonormal basis of $\mathcal{H}$

$$\Downarrow$$
$$\begin{aligned}\langle A^\dagger x,y\rangle=\langle x,Ay\rangle&=\left\langle x,A\sum_{k=1}^{n}\langle e_k,y\rangle e_k\right\rangle\\&=\left\langle x,\sum_{k=1}^{n}\langle e_k,y\rangle Ae_k\right\rangle\\&=\sum_{k=1}^{n}\langle e_k,y\rangle\,\langle x,Ae_k\rangle\\&=\left\langle\sum_{k=1}^{n}\langle Ae_k,x\rangle\,e_k,y\right\rangle\end{aligned}$$
$$\Downarrow$$
$$A^\dagger x=\sum_{k=1}^{n}\langle Ae_k,x\rangle\,e_k.$$

$\boxed{E3}$

$$\begin{aligned}\langle(A+B)^\dagger x,y\rangle&=\langle x,(A+B)y\rangle\\&=\langle x,Ay\rangle+\langle x,By\rangle\\&=\langle A^\dagger x,y\rangle+\langle B^\dagger x,y\rangle,\quad\forall x,y\\&=\langle(A^\dagger+B^\dagger)x,y\rangle,\quad\forall x,y\end{aligned}$$
$$\Downarrow$$
$$(A+B)^\dagger x=(A^\dagger+B^\dagger)x,\quad\forall x.$$

$$\begin{aligned}\langle(\lambda A)^\dagger x,y\rangle&=\langle x,(\lambda A)y\rangle\\&=\lambda\langle x,Ay\rangle\\&=\lambda\langle A^\dagger x,y\rangle\\&=\langle\bar\lambda\,A^\dagger x,y\rangle,\quad\forall x,y.\end{aligned}$$
$$\Downarrow$$
$$(\lambda A)^\dagger x=\bar\lambda\,A^\dagger x,\quad\forall x.$$

$$\begin{aligned}\langle(AB)^\dagger x,y\rangle&=\langle x,ABy\rangle\\&=\langle A^\dagger x,By\rangle\\&=\langle B^\dagger A^\dagger x,y\rangle,\quad\forall x,y.\end{aligned}$$
$$\Downarrow$$
$$(AB)^\dagger x=B^\dagger A^\dagger x,\quad\forall x.$$

$$\begin{aligned}\langle(A^\dagger)^\dagger x,y\rangle&=\langle x,A^\dagger y\rangle\\&=\overline{\langle A^\dagger y,x\rangle}\\&=\overline{\langle y,Ax\rangle}\\&=\langle Ax,y\rangle,\quad\forall x,y.\end{aligned}$$
$$\Downarrow$$
$$(A^\dagger)^\dagger x=Ax,\quad\forall x.$$

$\boxed{E4}$ Let $\{e_1,e_2,...,e_n\}$be an orthonormal basis of $\mathcal{H}$

$$Ae_j=a_j^k e_k\qquad\qquad A^\dagger e_j={a^\dagger}_j^k e_k$$
$$\Downarrow\qquad\qquad\qquad\Downarrow$$
$$a_j^k=\langle e_k,Ae_j\rangle\qquad\qquad{a^\dagger}_j^k=\langle e_k,A^\dagger e_j\rangle=\overline{\langle A^\dagger e_j,e_k\rangle}=\overline{\langle e_j,Ae_k\rangle}=\bar a_k^j.$$

$\boxed{E5}$

$$\langle Ax|y\rangle=\langle x|A^\dagger y\rangle=\langle x|A^\dagger|y\rangle,\quad\forall y$$
$$\Downarrow$$
$$\langle Ax|=\langle x|A^\dagger.$$

$\boxed{E6}$

$$\langle(|a\rangle\langle b|)^\dagger x,y\rangle=\langle x,(|a\rangle\langle b|)y\rangle=\langle x|a\rangle\langle b|y\rangle.$$
$$\langle(|b\rangle\langle a|)x,y\rangle=\langle(\langle a|x\rangle b,y\rangle=\langle x|a\rangle\langle b|y\rangle.$$
$$\Downarrow$$
$$\langle(|a\rangle\langle b|)^\dagger x,y\rangle=\langle(|b\rangle\langle a|)x,y\rangle,\quad\forall x,y.$$

$\boxed{E12}$

$$\{e_1, e_2, ..., e_n\} \text{ orthonormal basis of } \mathcal{H}$$
$$\Downarrow$$
$$\{|e_k\rangle\langle e_k| \mid 1\le k\le n\} \cup \{|e_k\rangle\langle e_m|+|e_m\rangle\langle e_k| \mid 1\le k,m\le n\}$$
$$\cup\{\mathrm{i}|e_k\rangle\langle e_m|-\mathrm{i}|e_m\rangle\langle e_k| \mid 1\le k,m\le n\}$$
is a basis in the real Hilbert space $\mathcal{A}(\mathcal{H})$.

$\boxed{E13}$

"$\Rightarrow$" $\quad AB=(AB)^\dagger=B^\dagger A^\dagger=BA.$

"$\Leftarrow$" $\quad (AB)^\dagger=B^\dagger A^\dagger=BA=-AB.$

$\boxed{E14}$

$$\mathbb{I}=\mathbb{I}^\dagger=(AA^{-1})^\dagger=(A^{-1})^\dagger A^\dagger=(A^{-1})^\dagger A \;\Rightarrow\; (A^{-1})^\dagger=A^{-1}.$$

$\boxed{E15}$

Let $\{e_1, e_2, ..., e_n\}$ be an orthonormal basis of $\mathcal{H}$.

"$\Rightarrow$" $\quad \boxed{\begin{array}{c} Ae_k=a_k^j e_j \\ \Downarrow \\ a_k^j=\langle e_j, Ae_k\rangle \end{array}} \;\Rightarrow\; \begin{array}{l} a_k^j=\langle e_j, Ae_k\rangle=\langle Ae_j, e_k\rangle \\ \qquad =\overline{\langle e_k, Ae_j\rangle}=\bar{a}_j^k. \end{array}$

"$\Leftarrow$"

$$Ax=A(\sum_{j=1}^n x^j e_j)=\sum_{j=1}^n x^j\, Ae_j=\sum_{j,k=1}^n x^j\, a_j^k e_k=\sum_{j,k=1}^n x^j\, \bar{a}_k^j e_k$$
$$=\sum_{j,k=1}^n x^j \langle Ae_k, e_j\rangle e_k=\sum_{k=1}^n \langle Ae_k, x\rangle e_k=\sum_{k=1}^n \langle e_k, Ax\rangle e_k=Ax.$$

$\boxed{E16}$

If $\begin{array}{c} Ax=\lambda x \\ x\neq 0 \end{array}$ then $\begin{array}{c} \langle Ax,x\rangle=\langle x,Ax\rangle \\ \Downarrow \\ \langle \lambda x,x\rangle=\langle x,\lambda x\rangle \\ \Downarrow \\ \bar\lambda=\lambda. \end{array}$

If $\begin{array}{c} Ax=\lambda x \\ Ay=\mu y \\ \lambda\neq\mu \end{array}$ then $\begin{array}{c} \langle Ax,y\rangle=\langle x,Ay\rangle \\ \Downarrow \\ \langle \lambda x,y\rangle=\langle x,\mu y\rangle \\ \Downarrow \\ \lambda\langle x,y\rangle=\mu\langle x,y\rangle \\ \Downarrow \\ \langle x,y\rangle=0. \end{array}$

$\boxed{E17}$

Let $\{e_1, e_2, ..., e_n\}$ be an orthonormal basis of the space $\mathcal{H}$.

$$Ae_j=a_j^k e_k.$$

The characteristic equation
$$\begin{vmatrix} a_1^1-\lambda & a_2^1 & \cdots & a_n^1 \\ a_1^2 & a_2^2-\lambda & \cdots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^n & a_2^n & \cdots & a_n^n-\lambda \end{vmatrix}=0$$
admits in $\mathbb{C}$ at least one root λ. For this root λ, the system
$$\sum_{k=1}^n a_k^j x^k=\lambda\, x^j, \quad j\in\{1,2,...,n\}$$
admits in $\mathbb{C}^n$ a solution $(x^1,x^2,...,x^n)\neq(0,0,...,0)$
$$\Downarrow$$
$$\sum_{k,j=1}^n a_k^j x^k\, \bar{x}^j=\lambda\sum_{j=1}^n x^j\, \bar{x}^j$$
$$\Downarrow$$
$$\lambda=\frac{\sum_{k,j=1}^n a_k^j x^k\, \bar{x}^j}{\sum_{j=1}^n |x^j|^2}=\frac{\sum_{k,j=1}^n \bar{a}_j^k x^k\, \bar{x}^j}{\sum_{j=1}^n |x^j|^2}=\bar\lambda.$$

$\boxed{E18}$ Recurrence with respect to $n=\dim\mathcal{H}$.

<u>*Step 1:*</u>

Statement is true for any $A\in\mathcal{A}(\mathcal{H})$ in the case $\dim\mathcal{H}=1$

<u>*Step 2:*</u>

Assume it is true for any $B\in\mathcal{A}(\mathcal{K})$ with $\dim\mathcal{K}=n-1$.

Let $A\in\mathcal{A}(\mathcal{H})$ with $\dim\mathcal{H}=n$.

There exist $\left\{\begin{array}{l} \lambda_1\in\mathbb{R} \text{ and } u_1\in\mathcal{H} \\ \quad \text{with } ||u_1||=1, \\ \det(A-\lambda_1\mathbb{I})=0 \end{array}\right.$ such that $Au_1=\lambda_1 u_1$.

$$x\in\mathcal{H}_1 \overset{\text{def}}{=} \{\, x\in\mathcal{H} \mid \langle x,u_1\rangle=0\,\}$$
$$\Downarrow$$
$\langle Ax,u_1\rangle=\langle x,Au_1\rangle=\lambda_1\langle x,u_1\rangle=0 \;\Rightarrow\; Ax\in\mathcal{H}_1.$

$B=A_{|\mathcal{H}_1}:\mathcal{H}_1\longrightarrow\mathcal{H}_1$ is self-adjoint and $\dim\mathcal{H}_1=n-1$.

There exists an orthonormal basis $\{u_2,u_3,...,u_n\}$ of $\mathcal{H}_1$ consisting of eigenvectors of B.

$$\Downarrow$$

$\{u_1,u_2,u_3,...,u_n\}$ is an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of A.

$\boxed{E19}$ The Hermitian matrix $\mathbf{A}\in\mathcal{M}_{n\times n}(\mathbb{C})$ is the matrix relative to the canonical basis of an operator $A\in\mathcal{A}(\mathbb{C}^n)$ which is diagonalizable.

$\boxed{E20}$

The symmetric matrix $\mathbf{A}\in\mathcal{M}_{n\times n}(\mathbb{R})$ is the matrix relative to the canonical basis of an operator $A\in\mathcal{A}(\mathbb{R}^n)$ which is diagonalizable.

$\boxed{E21}$

"$\Rightarrow$"

$\lambda_1,\,\lambda_2,...,\,\lambda_m$ eigenvalues of B

$\mathcal{H}_k=\{\,x\in\mathcal{H}\mid Bx=\lambda_k x\,\}$ eigenspace of λ_k

$x\in\mathcal{H}_k \;\Rightarrow\; B(Ax)=ABx=\lambda_k Ax \;\Rightarrow Ax\in\mathcal{H}_k$

$A_{|\mathcal{H}_k}:\mathcal{H}_k\to\mathcal{H}_k$ is a self-adjoint operator

$$\Downarrow$$

there exists $\{u_{k1},u_{k2},...,u_{kn_k}\}$ orthonormal basis of $\mathcal{H}_k$ consisting of eigenvectors of $A_{|\mathcal{H}_k}$

$$\Downarrow$$

$\{u_{11},...,u_{1n_1},u_{21},...,u_{2n_2},...,u_{m1},...,u_{mn_m}\}$ is an orthonormal basis of $\mathcal{H}$ consisting of common eigenvectors of A and B.

"$\Leftarrow$"

$\{e_1,e_2,...,e_n\}$ orthonormal basis consisting of common eigenvectors of A and B $\qquad \begin{array}{l} Ae_k=\alpha_k e_k \\ Be_k=\beta_k e_k \end{array}$

$$\Downarrow$$
$$ABx=AB(\sum_{k=1}^n x^k e_k)=\sum_{k=1}^n x^k\alpha_k\beta_k e_k$$
$$=\sum_{k=1}^n x^k\beta_k\alpha_k e_k=BA(\sum_{k=1}^n x^k e_k)=BAx.$$

$\boxed{E22}$

"$\Rightarrow$" *Proof 1*

$\mathcal{H}=\mathcal{K}\oplus\mathcal{K}^\perp$

$\begin{array}{l} x=x^\parallel+x^\perp \\ x^\parallel=x^\parallel+0 \end{array} \;\Rightarrow\; \begin{array}{l} Px=x^\parallel \\ Px^\parallel=x^\parallel \end{array} \;\Rightarrow P^2=P.$

$\begin{array}{l} \langle Px,y\rangle=\langle x^\parallel,y\rangle=\langle x^\parallel,y^\parallel\rangle \\ \langle x,Py\rangle=\langle x,y^\parallel\rangle=\langle x^\parallel,y^\parallel\rangle \end{array} \;\Rightarrow P^\dagger=P.$

"$\Rightarrow$" *Proof 2*

$$\begin{array}{l}\{e_1,e_2,...,e_m\} \text{ orthonormal basis} \\ \qquad\qquad \text{in the subspace } \mathcal{K}\end{array} \Rightarrow P_{\mathcal{K}}=\sum_{k=1}^{m}|e_k\rangle\langle e_k|$$

$$\Rightarrow P_{\mathcal{K}}^2=\sum_{k=1}^{m}|e_k\rangle\langle e_k|\sum_{j=1}^{m}|e_j\rangle\langle e_j|=\sum_{k=1}^{m}|e_k\rangle\langle e_k|=P_{\mathcal{K}},$$

$$P_{\mathcal{K}}=\sum_{k=1}^{m}|e_k\rangle\langle e_k|=P_{\mathcal{K}}^{\dagger}.$$

"$\Leftarrow$"

$$\begin{aligned}\mathcal{K}=\operatorname{Im}P \ \Rightarrow\ \mathcal{K}^{\perp}&=\{\,x\mid\langle x,Py\rangle=0,\ \ \forall y\in\mathcal{K}\,\}\\ &=\{\,x\mid\langle Px,y\rangle=0,\ \ \forall y\in\mathcal{K}\,\}\\ &=\{\,x\mid Px=0\,\}=\operatorname{Ker}P.\end{aligned}$$

$$\begin{aligned}&x\in\mathcal{H}\ \Rightarrow\ P(x-Px)=(P-P^2)x=0\ \Rightarrow\ x-Px\in\mathcal{K}^{\perp}\\ &\quad\Rightarrow\ x=Px+(x-Px)\in\mathcal{K}\oplus\mathcal{K}^{\perp}\ \Rightarrow\ Px=P_{\mathcal{K}}x.\end{aligned}$$

$\boxed{E26}$

$$A\geq 0\ \Rightarrow\ \lambda_k=\langle e_k,Ae_k\rangle\geq 0.$$

$$\left.\begin{array}{l}\lambda_k\geq 0\\ \text{for all } k\end{array}\right\}\Rightarrow\ \langle x,Ax\rangle=\sum_{k=1}^{n}\lambda_k\langle x|e_k\rangle\langle e_k|x\rangle=\sum_{k=1}^{n}\lambda_k|\langle e_k|x\rangle|^2\geq 0.$$

$\boxed{E27}$

$$P^{\dagger}=P=P^2\ \Rightarrow\ \bar{\lambda}_k=\lambda_k=\lambda_k^2\ \Rightarrow\ \lambda_k\in\{0,1\}.$$

$\boxed{\mathbf{F}}$ UNITARY TRANSFORMATIONS

$\boxed{F3}$

$$A\in U(\mathcal{H})\ \Rightarrow\ ||Ax||=\sqrt{\langle Ax,Ax\rangle}=\sqrt{\langle x,x\rangle}=||x||.$$

$\boxed{F4}$

$$A,B\in U(\mathcal{H})\ \Rightarrow\ \langle ABx,ABy\rangle=\langle Bx,By\rangle=\langle x,y\rangle.$$

$\boxed{F5}$

$$\begin{array}{l}A\in U(\mathcal{H})\\ Ax=0\end{array}\Rightarrow\ ||x||=||Ax||=0\ \Rightarrow\ x=0\ \Rightarrow\ \operatorname{Ker}A=\{0\}.$$

$$\dim\mathcal{H}=\dim\operatorname{Ker}A+\dim\operatorname{Im}A\ \Rightarrow\ \operatorname{Im}A=\mathcal{H}.$$

$$\langle x,y\rangle=\langle AA^{-1}x,AA^{-1}y\rangle=\langle A^{-1}x,A^{-1}y\rangle\ \Rightarrow\ A^{-1}\in U(\mathcal{H}).$$

$\boxed{F6}$

$$A\in U(\mathcal{H})\ \Rightarrow\ \langle x,y\rangle=\langle Ax,Ay\rangle=\langle A^{\dagger}Ax,y\rangle,\ \forall x,y$$
$$\Downarrow$$
$$A^{\dagger}Ax=x,\ \forall x.$$
$$\Downarrow$$
$$A^{\dagger}A=\mathbb{I}.$$

$$A^{\dagger}A=\mathbb{I}\ \Rightarrow\ \det A^{\dagger}\,\det A=1\ \Rightarrow\ \det A\neq 0$$
$$\Rightarrow\ \text{there exists } A^{-1} \text{ and }\ A^{\dagger}A=\mathbb{I}\ \Rightarrow\ A^{\dagger}=A^{-1}.$$
$$A^{\dagger}=A^{-1}\ \Rightarrow\ A^{\dagger}A=\mathbb{I}.$$
$$A^{\dagger}A=\mathbb{I}\ \Rightarrow\ \langle Ax,Ay\rangle=\langle A^{\dagger}Ax,y\rangle=\langle x,y\rangle$$
$$\Downarrow$$
$$A\in U(\mathcal{H}).$$

$\boxed{F8}$ Let $\{e_1,e_2,...,e_n\}$ be an orthonormal basis of $\mathcal{H}$.

"$\Rightarrow$"

$$\boxed{\begin{array}{c}Ae_k=a_k^j e_j\\ \Downarrow\\ a_k^j=\langle e_j,Ae_k\rangle\end{array}}\Rightarrow\ \begin{aligned}\sum_{j=1}^{n}\bar{a}_k^j a_m^j&=\sum_{j=1}^{n}\overline{\langle e_j,Ae_k\rangle}\langle e_j,Ae_m\rangle\\ &=\sum_{j=1}^{n}\langle Ae_k,e_j\rangle\langle e_j,Ae_m\rangle\\ &=\left\langle Ae_k,\sum_{j=1}^{n}\langle e_j,Ae_m\rangle e_j\right\rangle\\ &=\langle Ae_k,Ae_m\rangle=\langle e_k,e_m\rangle=\delta_{km}.\end{aligned}$$

"$\Leftarrow$"

$$\begin{aligned}A^{\dagger}Ax&=A^{\dagger}A(\sum_{j=1}^{n}x^j e_j)=\sum_{j=1}^{n}x^j\,A^{\dagger}Ae_j=\sum_{j,k=1}^{n}x^j\,a_j^k A^{\dagger}e_k\\ &=\sum_{j,k=1}^{n}x^j\,\bar{a}_k^j A^{\dagger}e_k=\sum_{j,k=1}^{n}x^j\,\bar{a}_k^j\bar{a}_m^k e_m\\ &=\sum_{j,k=1}^{n}x^j\,\bar{a}_k^j a_k^m e_m=\sum_{j=1}^{n}x^j\,\delta_{jm}e_m=\sum_{j=1}^{n}x^j\,e_j=x.\end{aligned}$$

$\boxed{F9}$

$$\text{If}\quad \begin{array}{c}Ax=\lambda x\\ x\neq 0\end{array}\quad\text{then}\quad \begin{array}{c}\langle Ax,Ax\rangle=\langle x,x\rangle\\ \Downarrow\\ \langle\lambda x,\lambda x\rangle=\langle x,x\rangle\\ \Downarrow\\ \bar{\lambda}\lambda=1.\end{array}$$

$$\text{If}\quad \begin{array}{c}Ax=\lambda x\\ Ay=\mu y\\ \lambda\neq\mu\end{array}\quad\text{then}\quad \begin{array}{c}\langle Ax,Ay\rangle=\langle x,y\rangle\\ \Downarrow\\ \langle\lambda x,\mu y\rangle=\langle x,y\rangle\\ \Downarrow\\ \frac{\mu}{\lambda}\langle x,y\rangle=\langle x,y\rangle\\ \Downarrow\\ \langle x,y\rangle=0.\end{array}$$

$\boxed{F10}$ Recurrence with respect to $n=\dim\mathcal{H}$.

<u>*Step 1:*</u> Statement is true for any $A\in U(\mathcal{H})$ in the case $\dim\mathcal{H}=1$

<u>*Step 2:*</u> Assume it is true for any $B\in U(\mathcal{K})$ with $\dim\mathcal{K}=n-1$.

Let $A\in U(\mathcal{H})$ with $\dim\mathcal{H}=n$.

$$\text{There exist}\ \left\{\begin{array}{r}\lambda_1\in\mathbb{C}\text{ and }u_1\in\mathcal{H}\\ \text{with }||u_1||=1,\\ |\lambda_1|=1,\\ \det(A-\lambda_1\mathbb{I})=0\end{array}\right.\quad\text{such that}\ \ Au_1=\lambda_1u_1.$$

$$x\in\mathcal{H}_1\stackrel{\text{def}}{=}\{\,x\in\mathcal{H}\mid\langle x,u_1\rangle=0\,\}$$
$$\Downarrow$$
$$\langle Ax,u_1\rangle=\bar{\lambda}_1\langle Ax,\lambda_1u_1\rangle=\bar{\lambda}_1\langle Ax,Au_1\rangle=\bar{\lambda}_1\langle x,u_1\rangle=0\ \Rightarrow\ Ax\in\mathcal{H}_1.$$

$B=A_{|\mathcal{H}_1}:\mathcal{H}_1\longrightarrow\mathcal{H}_1$ is a unitary operator and $\dim\mathcal{H}_1=n-1$.

Consequently, there exists an orthonormal basis $\{u_2,u_3,...,u_n\}$ of $\mathcal{H}_1$ consisting of eigenvectors of B.

$$\Downarrow$$

$\{u_1,u_2,u_3,...,u_n\}$ is an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of A.

$\boxed{F11}$ The unitary matrix $\mathbf{A}\in\mathcal{M}_{n\times n}(\mathbb{C})$ is the matrix relative to the canonical basis of an operator $A\in U(\mathbb{C}^n)$ which is diagonalizable.

$\boxed{\mathbf{G}}$ ORTHOGONAL TRANSFORMATIONS

$\boxed{G3}$

$$A\in O(\mathcal{H})\ \Rightarrow\ ||Ax||=\sqrt{\langle Ax,Ax\rangle}=\sqrt{\langle x,x\rangle}=||x||.$$

$\boxed{G4}$

$$A,B\in O(\mathcal{H})\ \Rightarrow\ \langle ABx,ABy\rangle=\langle Bx,By\rangle=\langle x,y\rangle.$$

$\boxed{G5}$

$$\begin{array}{l}A\in O(\mathcal{H})\\ Ax=0\end{array}\Rightarrow\ ||x||=||Ax||=0\ \Rightarrow\ x=0\ \Rightarrow\ \operatorname{Ker}A=\{0\}.$$

$$\dim\mathcal{H}=\dim\operatorname{Ker}A+\dim\operatorname{Im}A\ \Rightarrow\ \operatorname{Im}A=\mathcal{H}.$$

$$\langle x,y\rangle=\langle AA^{-1}x,AA^{-1}y\rangle=\langle A^{-1}x,A^{-1}y\rangle\ \Rightarrow\ A^{-1}\in O(\mathcal{H}).$$

$\boxed{G6}$

$$A\in O(\mathcal{H})\ \Rightarrow\ \langle x,y\rangle=\langle Ax,Ay\rangle=\langle A^{T}Ax,y\rangle,\ \forall x,y$$
$$\Downarrow$$
$$A^{T}Ax=x,\ \forall x.$$
$$\Downarrow$$
$$A^{T}A=\mathbb{I}.$$

$$A^{T}A=\mathbb{I}\ \Rightarrow\ \det A^{T}\,\det A=1\ \Rightarrow\ \det A\neq 0$$
$$\Rightarrow\ \text{there exists } A^{-1} \text{ and }\ A^{T}A=\mathbb{I}\ \Rightarrow\ A^{T}=A^{-1}.$$
$$A^{T}=A^{-1}\ \Rightarrow\ A^{T}A=\mathbb{I}.$$
$$A^{T}A=\mathbb{I}\ \Rightarrow\ \langle Ax,Ay\rangle=\langle A^{T}Ax,y\rangle=\langle x,y\rangle$$
$$\Downarrow$$
$$A\in O(\mathcal{H}).$$

$\boxed{G8}$ Let $\{e_1, e_2, ..., e_n\}$ be an orthonormal basis of $\mathcal{H}$.

"$\Rightarrow$"

$$\boxed{\begin{array}{c} Ae_k = a_k^j e_j \\ \Downarrow \\ a_k^j = \langle e_j, Ae_k\rangle \end{array}} \Rightarrow \quad \begin{aligned} \sum_{j=1}^n a_k^j a_m^j &= \sum_{j=1}^n \langle e_j, Ae_k\rangle\langle e_j, Ae_m\rangle \\ &= \sum_{j=1}^n \langle Ae_k, e_j\rangle\langle e_j, Ae_m\rangle \\ &= \left\langle Ae_k, \sum_{j=1}^n \langle e_j, Ae_m\rangle e_j \right\rangle \\ &= \langle Ae_k, Ae_m\rangle = \langle e_k, e_m\rangle = \delta_{km}. \end{aligned}$$

"$\Leftarrow$"

$$\begin{aligned} A^T Ax = A^T (\sum_{j=1}^n x^j e_j) &= \sum_{j=1}^n x^j A^T Ae_j = \sum_{j,k=1}^n x^j a_j^k A^T e_k \\ &= \sum_{j,k=1}^n x^j a_k^j A^T e_k = \sum_{j,k=1}^n x^j a_k^j a_m^k e_m \\ = \sum_{j,k=1}^n x^j a_k^j a_k^m e_m &= \sum_{j=1}^n x^j \delta_{jm} e_m = \sum_{j=1}^n x^j e_j = x. \end{aligned}$$

$\boxed{G9}$

$$\text{If} \quad \begin{array}{c} Ax = \lambda x \\ x \neq 0 \end{array} \quad \text{then} \quad \begin{array}{c} \langle Ax, Ax\rangle = \langle x, x\rangle \\ \Downarrow \\ \langle \lambda x, \lambda x\rangle = \langle x, x\rangle \\ \Downarrow \\ \lambda^2 = 1. \end{array}$$

$$\text{If} \quad \begin{array}{c} Ax = \lambda x \\ Ay = \mu y \\ \lambda \neq \mu \end{array} \quad \text{then} \quad \begin{array}{c} \langle Ax, Ay\rangle = \langle x, y\rangle \\ \Downarrow \\ \langle \lambda x, \mu y\rangle = \langle x, y\rangle \\ \Downarrow \\ \lambda\mu\langle x, y\rangle = \langle x, y\rangle \\ \Downarrow \\ \langle x, y\rangle = 0. \end{array}$$

$\boxed{\mathbf{H}}$ TENSOR PRODUCT OF HILBERT SPACES

$\boxed{H2}$

- If $\{e_0, e_1, ..., e_{n-1}\}$ is an orthonormal basis of $\mathcal{H}$, then
$$\{\varphi : \{0, 1, ..., n-1\} \to \mathbb{K}\} \longrightarrow \mathcal{H} : \ \varphi \mapsto \sum_{k=0}^{n-1} \varphi(k)\, e_k$$
is an isomorphism which allows us to identify the $\mathbb{K}-$Hilbert space $\mathcal{H}$ with $\{\varphi : \{0, 1, ..., n-1\} \to \mathbb{K}\}$.
- Particularly,
$$\boxed{\begin{array}{l} \{\varphi : \{0, 1, ..., n-1\} \to \mathbb{R}\} \longrightarrow \mathbb{R}^n : \\ \qquad \varphi \mapsto (\varphi(0), \varphi(1), ..., \varphi(n-1)) \end{array}}$$
allows us to identify $\mathbb{R}^n$ with $\{\varphi : \{0, 1, ..., n-1\} \to \mathbb{R}\}$,
$$\boxed{\begin{array}{l} \{\varphi : \{0, 1, ..., n-1\} \to \mathbb{C}\} \longrightarrow \mathbb{C}^n : \\ \qquad \varphi \mapsto (\varphi(0), \varphi(1), ..., \varphi(n-1)) \end{array}}$$
allows us to identify $\mathbb{C}^n$ with $\{\varphi : \{0, 1, ..., n-1\} \to \mathbb{C}\}$.
- The canonical basis of
$$\mathbb{K}^n \equiv \{\varphi : \{0, 1, ..., n-1\} \to \mathbb{K}\}$$
is $\{\delta_0, \delta_1, ..., \delta_{n-1}\}$, where
$$\delta_k : \{0, 1, ..., n-1\} \to \mathbb{K}, \quad \delta_k(j) = \delta_{kj} = \begin{cases} 1 & \text{if } \ j = k \\ 0 & \text{if } \ j \neq k. \end{cases}$$
- By using the shorter notation
$$\boxed{|k\rangle \equiv |\delta_k\rangle},$$
we have
$$\langle k|j\rangle = \delta_{kj}, \qquad \sum_{k=0}^{n-1} |k\rangle\langle k| = \mathbb{I},$$
$$\varphi \in \mathbb{K}^n \ \Rightarrow \ |\varphi\rangle = \sum_{k=0}^{n-1} |k\rangle\langle k|\varphi\rangle = \sum_{k=0}^{n-1} \varphi(k)\, |k\rangle.$$
- The canonical basis of $\mathbb{K}^n \times \mathbb{K}^m$ is
$\{\ |jk\rangle \equiv |j\rangle \otimes |k\rangle \ |\ j \in \{0, 1, ..., n-1\}, \ k \in \{0, 1, ..., m-1\}\ \}$
and
$$|j\rangle \otimes |k\rangle \equiv |\delta_j\rangle \otimes |\delta_k\rangle, \qquad (\delta_j \otimes \delta_k)(i, \ell) = \delta_j(i)\, \delta_k(\ell),$$
$$|\Phi\rangle \in \mathbb{K}^n \otimes \mathbb{K}^m \quad \Rightarrow \quad |\Phi\rangle = \sum_{j=0}^{n-1} \sum_{k=0}^{m-1} \Phi(j, k)\, |jk\rangle.$$
where $\quad \Phi(j, k) = \langle jk|\Phi\rangle$.

$\boxed{H7}$

$$\begin{aligned} \langle \varphi_j \psi_k | \varphi_{j'} \psi_{k'}\rangle &= \sum_{i=0}^{n-1} \sum_{\ell=0}^{m-1} \overline{\varphi_j(i)}\, \overline{\psi_k(\ell)}\, \varphi_{j'}(i)\, \psi_{k'}(\ell) \\ &= \langle \varphi_j | \varphi_{j'}\rangle\, \langle \psi_k | \psi_{k'}\rangle = \delta_{jj'} \delta_{kk'}. \end{aligned}$$
$$\Downarrow$$
$\{\ |\varphi_j \psi_k\rangle \ |\ j \in \{0, 1, ..., n-1\}, \ k \in \{0, 1, ..., m-1\}\ \}$
is an orthonormal system in the space $\mathcal{H}_1 \otimes \mathcal{H}_2$ of dimension nm.

$\boxed{H11}$

$$\begin{aligned} (A \otimes B)^{ik}_{j\ell} &= \langle \eta_i \vartheta_k | A \otimes B | \eta_j \vartheta_\ell\rangle = \langle \eta_i \vartheta_k | (A\eta_j) \otimes (B\vartheta_\ell)\rangle \\ &= \langle \eta_i | A\eta_j\rangle \langle \vartheta_k | B\vartheta_\ell\rangle = A^i_j\, B^k_\ell. \end{aligned}$$

$\boxed{H12}$

$$\begin{aligned} \langle (A \otimes B)^\dagger \varphi_1 \otimes \psi_1, \varphi_2 \otimes \psi_2\rangle &= \langle \varphi_1 \otimes \psi_1, (A \otimes B)\varphi_2 \otimes \psi_2\rangle \\ &= \langle \varphi_1 \otimes \psi_1, (A\varphi_2) \otimes (B\psi_2)\rangle \\ &= \langle \varphi_1, A\varphi_2\rangle \langle \psi_1, B\psi_2\rangle \\ &= \langle A^\dagger \varphi_1, \varphi_2\rangle \langle B^\dagger \psi_1, \psi_2\rangle \\ &= \langle (A^\dagger \varphi_1) \otimes (B^\dagger \psi_1), \varphi_2 \otimes \psi_2\rangle \\ &= \langle (A^\dagger \otimes B^\dagger) \varphi_1 \otimes \psi_1, \varphi_2 \otimes \psi_2\rangle. \end{aligned}$$

$\boxed{H13}$

$$\begin{aligned} \langle \eta_i \vartheta_k | (|\varphi_1 \psi_1\rangle\langle \varphi_2 \psi_2|) | \eta_j \vartheta_\ell\rangle &= \langle \eta_i | \varphi_1\rangle \langle \vartheta_k | \psi_1\rangle \langle \varphi_2 | \eta_j\rangle \langle \psi_2 | \vartheta_\ell\rangle \\ &= \langle \eta_i | \varphi_1\rangle \langle \varphi_2 | \eta_j\rangle \langle \vartheta_k | \psi_1\rangle \langle \psi_2 | \vartheta_\ell\rangle \\ &= \langle \eta_i \vartheta_k | (|\varphi_1\rangle\langle \varphi_2| \otimes |\psi_1\rangle\langle \psi_2|) | \eta_j \vartheta_\ell\rangle. \end{aligned}$$

$\boxed{H14}$

Existence and uniqueness of $\mathrm{tr}_1 T$.
The space $\mathcal{L}(\mathcal{H}_2)$ is a Hilbert space (see $\boxed{E8}$) with
$$\langle B_1, B_2\rangle = \mathrm{tr}(B_1^\dagger B_2),$$
and
$$\mathcal{L}(\mathcal{H}_2) \to \mathbb{C} : B \mapsto \mathrm{tr}\,((\mathbb{I} \otimes B)T)$$
is a linear form. In view of $\boxed{D21}$ there exists $Y \in \mathcal{L}(\mathcal{H}_2)$ uniquely determined such that
$$\mathrm{tr}\,((\mathbb{I} \otimes B)T) = \langle Y, B\rangle, \quad \text{for any} \ \ B \in \mathcal{L}(\mathcal{H}_2),$$
that is
$$\mathrm{tr}\,((\mathbb{I} \otimes B)T) = \mathrm{tr}\,(Y^\dagger B) = \mathrm{tr}\,(BY^\dagger), \quad \text{for any} \ \ B \in \mathcal{L}(\mathcal{H}_2).$$
Therefore, $\mathrm{tr}_1 T = Y^\dagger$ is the unique operator satisfying
$$\mathrm{tr}\,((\mathbb{I} \otimes B)T) = \mathrm{tr}\,(B\, \mathrm{tr}_1 T), \quad \text{for any} \ \ B \in \mathcal{L}(\mathcal{H}_2).$$

$\boxed{H15}$

Existence and uniqueness of $\mathrm{tr}_2 T$.
The space $\mathcal{L}(\mathcal{H}_1)$ is a Hilbert space (see $\boxed{E8}$) with
$$\langle A_1, A_2\rangle = \mathrm{tr}(A_1^\dagger A_2),$$
and
$$\mathcal{L}(\mathcal{H}_1) \to \mathbb{C} : A \mapsto \mathrm{tr}\,((A \otimes \mathbb{I})T)$$
is a linear form. In view of $\boxed{D21}$ there exists $X \in \mathcal{L}(\mathcal{H}_1)$ uniquely determined such that
$$\mathrm{tr}\,((A \otimes \mathbb{I})T) = \langle X, A\rangle, \quad \text{for any} \ \ A \in \mathcal{L}(\mathcal{H}_1),$$
that is
$$\mathrm{tr}\,((A \otimes \mathbb{I})T) = \mathrm{tr}\,(X^\dagger A) = \mathrm{tr}\,(AX^\dagger), \quad \text{for any} \ \ A \in \mathcal{L}(\mathcal{H}_1).$$
Therefore, $\mathrm{tr}_2 T = X^\dagger$ is the unique operator satisfying
$$\mathrm{tr}\,((A \otimes \mathbb{I})T) = \mathrm{tr}\,(A\, \mathrm{tr}_2 T), \quad \text{for any} \ \ A \in \mathcal{L}(\mathcal{H}_1).$$

$\boxed{H16}$

$$\left.\begin{aligned}
&\mathrm{tr}\,(B\,\mathrm{tr}_1 T)=\sum_j (B\,\mathrm{tr}_1 T)^j_j=\sum_{j,\ell} B^j_\ell\,(\mathrm{tr}_1 T)^\ell_j \quad \text{and}\\
&\mathrm{tr}\,((\mathbb{I}\otimes B)T)=\sum_{i,j}((\mathbb{I}\otimes B)T)^{ij}_{ij}=\sum_{i,j}\sum_{k,\ell}(\mathbb{I}\otimes B)^{ij}_{k\ell}T^{k\ell}_{ij}\\
&\qquad\qquad\qquad\;=\sum_{i,j}\sum_{k,\ell}\delta^i_k\,B^j_\ell T^{k\ell}_{ij}=\sum_{j,\ell}B^j_\ell\sum_k T^{k\ell}_{kj}
\end{aligned}\right\}$$

$$\Rightarrow \left\{\begin{array}{l}
\mathrm{tr}_1 T:\mathcal{H}_2\longrightarrow\mathcal{H}_2 \;\text{ defined by } (\mathrm{tr}_1 T)^\ell_j=\sum_k T^{k\ell}_{kj}\\
\text{satisfies } \mathrm{tr}\,((\mathbb{I}\otimes B)T)=\mathrm{tr}\,(B\,\mathrm{tr}_1 T) \text{ for any } B.
\end{array}\right.$$

$$\left.\begin{aligned}
&\mathrm{tr}\,(A\,\mathrm{tr}_2 T)=\sum_j (A\,\mathrm{tr}_2 T)^j_j=\sum_{i,k} A^i_k\,(\mathrm{tr}_2 T)^k_i \quad \text{and}\\
&\mathrm{tr}\,((A\otimes\mathbb{I})T)=\sum_{i,j}((A\otimes\mathbb{I})T)^{ij}_{ij}=\sum_{i,j}\sum_{k,\ell}(A\otimes\mathbb{I})^{ij}_{k\ell}T^{k\ell}_{ij}\\
&\qquad\qquad\qquad\;=\sum_{i,j}\sum_{k,\ell}A^i_k\,\delta^j_\ell T^{k\ell}_{ij}=\sum_{i,k}A^i_k\sum_j T^{kj}_{ij}
\end{aligned}\right\}$$

$$\Rightarrow \left\{\begin{array}{l}
\mathrm{tr}_2 T:\mathcal{H}_1\longrightarrow\mathcal{H}_1 \;\text{ defined by } (\mathrm{tr}_2 T)^k_i=\sum_j T^{kj}_{ij}\\
\text{satisfies } \mathrm{tr}\,((A\otimes\mathbb{I})T)=\mathrm{tr}\,(A\,\mathrm{tr}_2 T) \text{ for any } A.
\end{array}\right.$$

$\boxed{H17}$ *Schmidt decomposition.*

$\{|0\rangle_1,|1\rangle_1,\dots,|n-1\rangle_1\}$ basis of $\mathcal{H}_1$
$\{|0\rangle_2,|1\rangle_2,\dots,|m-1\rangle_2\}$ basis of $\mathcal{H}_2$
$\Phi\in\mathcal{H}_1\otimes\mathcal{H}_2$,

$$\Downarrow$$

$$\begin{aligned}
{}_1\langle j|(\mathrm{tr}_2|\Phi\rangle\langle\Phi|)|k\rangle_1&=\sum_{\ell=0}^{m-1}\langle j\ell|\Phi\rangle\langle\Phi|k\ell\rangle\\
&=\sum_{\ell=0}^{m-1}\overline{\langle k\ell|\Phi\rangle\langle\Phi|j\ell\rangle}=\overline{{}_1\langle k|(\mathrm{tr}_2|\Phi\rangle\langle\Phi|)|j\rangle_1},\\
{}_1\langle\varphi|(\mathrm{tr}_2|\Phi\rangle\langle\Phi|)|\varphi\rangle_1&=\sum_{j,k=0}^{n-1}\sum_{i=0}^{m-1}{}_1\langle\varphi|j\rangle_1\,\langle ji|\Phi\rangle\langle\Phi|ki\rangle\,{}_1\langle k|\varphi\rangle_1\\
&=\sum_{i=0}^{m-1}\langle\varphi\,i|\Phi\rangle\langle\Phi|\varphi\,i\rangle=\sum_{i=0}^{m-1}|\langle\varphi\,i|\Phi\rangle|^2\geq 0
\end{aligned}$$

$$\Downarrow$$

$$\boxed{\mathrm{tr}_2|\Phi\rangle\langle\Phi|:\mathcal{H}_1\to\mathcal{H}_1 \text{ is self-adjoint and positive}}$$

$$\Downarrow$$

> there exists $\{\varphi_0,\varphi_1,\dots,\varphi_{n-1}\}$ orthonormal basis of $\mathcal{H}_1$ such that
> $$\mathrm{tr}_2|\Phi\rangle\langle\Phi|=\sum_{k=0}^{r-1}p_k|\varphi_k\rangle\langle\varphi_k|$$
> and $p_0\geq p_1\geq\ldots\geq p_{r-1}>0$.

$$\Downarrow$$

$$|\Phi\rangle=\sum_{k=0}^{n-1}\sum_{j=0}^{m-1}\Phi^{kj}|\varphi_k\,j\rangle$$

$$\Downarrow$$

$$|\Phi\rangle\langle\Phi|=\sum_{i,k=0}^{n-1}\sum_{j,\ell=0}^{m-1}\Phi^{kj}\,\overline{\Phi^{i\ell}}\,|\varphi_k\,j\rangle\langle\varphi_i\,\ell|$$

$$\Downarrow$$

$$\mathrm{tr}_2|\Phi\rangle\langle\Phi|=\sum_{i,k=0}^{n-1}\sum_{j=0}^{m-1}\Phi^{kj}\,\overline{\Phi^{ij}}\,|\varphi_k\rangle\langle\varphi_i|$$

$$\Downarrow$$

$$\sum_{k=0}^{r-1}p_k|\varphi_k\rangle\langle\varphi_k|=\sum_{i,k=0}^{n-1}\sum_{j=0}^{m-1}\Phi^{kj}\,\overline{\Phi^{ij}}\,|\varphi_k\rangle\langle\varphi_i|$$

$$\Downarrow$$

$$\boxed{\begin{array}{l}
\displaystyle\sum_{j=0}^{m-1}\Phi^{kj}\,\overline{\Phi^{ij}}=p_k\,\delta_{ik}\quad\text{for}\quad 0\leq i,k\leq r-1\\
\displaystyle\sum_{j=0}^{m-1}|\Phi^{kj}|^2=\sum_{j=0}^{m-1}\Phi^{kj}\,\overline{\Phi^{kj}}=0\quad\text{for}\quad k>r-1
\end{array}}$$

$$\Downarrow$$

$$\left\{|\psi_k\rangle=\tfrac{1}{\sqrt{p_k}}\sum_{j=0}^{m-1}\Phi^{kj}\,|j\rangle\;\middle|\;k\in\{0,1,\dots,r-1\}\right\}$$

is an orthonormal set and $\Phi^{kj}=0$ for $k>r-1$

$$\Downarrow$$

$$\begin{aligned}
|\Phi\rangle&=\sum_{k=0}^{n-1}\sum_{j=0}^{m-1}\Phi^{kj}|\varphi_k\,j\rangle\\
&=\sum_{k=0}^{r-1}\sum_{j=0}^{m-1}\Phi^{kj}|\varphi_k\rangle\otimes|j\rangle\\
&=\sum_{k=0}^{r-1}\sqrt{p_k}\,|\varphi_k\rangle\otimes|\psi_k\rangle=\sum_{k=0}^{r-1}\sqrt{p_k}\,|\varphi_k\psi_k\rangle.
\end{aligned}$$

The set $\{\,|\psi_k\rangle\;|\;k\in\{0,1,\dots,r-1\}\,\}$ can be extended up to an orthonormal basis $\{|\psi_0\rangle,|\psi_1\rangle,\dots,|\psi_{m-1}\rangle\}$ of $\mathcal{H}_2$.

$$\left\{|\psi_k\rangle=\tfrac{1}{\sqrt{p_k}}\sum_{j=0}^{m-1}\Phi^{kj}\,|\vartheta_j\rangle\;\middle|\;k\in\{0,1,\dots,r-1\}\right\}$$

is an orthonormal set and $\Phi^{kj}=0$ for $k>r-1$

$$\Downarrow$$

$$\begin{aligned}
|\Phi\rangle&=\sum_{k=0}^{n-1}\sum_{j=0}^{m-1}\Phi^{kj}|\varphi_k\vartheta_j\rangle\\
&=\sum_{k=0}^{r-1}\sum_{j=0}^{m-1}\Phi^{kj}|\varphi_k\rangle\otimes|\vartheta_j\rangle\\
&=\sum_{k=0}^{r-1}\sqrt{p_k}\,|\varphi_k\rangle\otimes|\psi_k\rangle=\sum_{k=0}^{r-1}\sqrt{p_k}\,|\varphi_k\psi_k\rangle.
\end{aligned}$$

The set $\{\,|\psi_k\rangle\;|\;k\in\{0,1,\dots,r-1\}\,\}$ can be extended up to an orthonormal basis $\{|\psi_0\rangle,|\psi_1\rangle,\dots,|\psi_{m-1}\rangle\}$ of $\mathcal{H}_2$.

$\boxed{\text{I}}$ QUADRATIC FORMS

$\boxed{I11}$

Let $\mathfrak{B}=\{e_1,e_2,\dots,e_n\}$.
There exists a basis $\mathfrak{B}'=\{e'_1,e'_2,\dots,e'_n\}$ of the form

$$\begin{aligned}
e'_1&=\alpha_{11}e_1\\
e'_2&=\alpha_{12}e_1+\alpha_{22}e_2\\
&\dots\dots\dots\dots\dots\dots\\
e'_n&=\alpha_{1n}e_1+\alpha_{2n}e_2+\cdots+\alpha_{nn}e_n
\end{aligned}$$

satisfying

$$\left\{\begin{array}{l}
g(e'_j\;,\;e_1)=0\\
g(e'_j\;,\;e_2)=0\\
\dots\dots\dots\dots\dots\\
g(e'_j,e_{j-1})=0\\
g(e'_j\;,\;e_j)=1
\end{array}\right.\qquad\text{for any } j\in\{1,2,\dots,n\}$$

that is, the Cramer system

$$\left\{\begin{array}{l}
g_{11}\alpha_{1j}\;+\;g_{21}\alpha_{2j}\;+\;\cdots\;+\;g_{j1}\alpha_{jj}=0\\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
g_{1j-1}\alpha_{1j}+g_{2j-1}\alpha_{2j}+\cdots+g_{jj-1}\alpha_{jj}=0\\
g_{1j}\alpha_{1j}\;+\;g_{2j}\alpha_{2j}\;+\;\cdots\;+\;g_{jj}\alpha_{jj}=1
\end{array}\right.$$

from which we deduce that

$$\begin{aligned}
g'_{jj}&=g(e'_j,e'_j)=g(e'_j,\alpha_{1j}e_1+\alpha_{2j}e_2+\cdots+\alpha_{jj}e_j)\\
&=\alpha_{1j}\,g(e'_j,e_1)+\alpha_{2j}\,g(e'_j,e_2)+\cdots+\alpha_{j-1j}\,g(e'_j,e_{j-1})\\
&\quad+\alpha_{jj}\,g(e'_j,e_j)=\alpha_{jj}=\Delta_{j-1}/\Delta_j.
\end{aligned}$$

and

$$\begin{aligned}
g'_{jk}&=g(e'_j,e'_k)=g(e'_j,\alpha_{1k}e_1+\alpha_{2k}e_2+\cdots+\alpha_{kk}e_k)\\
&=\alpha_{1k}\,g(e'_j,e_1)+\alpha_{2k}\,g(e'_j,e_2)+\cdots+\alpha_{kk}\,g(e'_j,e_k)=0.
\end{aligned}$$

for $k<j$.

$\boxed{I12}$

Let $\mathfrak{B}=\{e_1,e_2,...,e_n\}$.
The self-adjoint operator

$$A:\mathcal{H}\longrightarrow\mathcal{H},\qquad Ae_i=\sum_{j=1}^{n}g_{ji}\,e_j$$

is diagonalizable

$$\Downarrow$$

there exists an orthonormal basis $\mathfrak{B}'$ such that the matrix of A with respect to $\mathfrak{B}'$ has the form

$$\mathbf{A}'=\begin{pmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & \lambda_n\end{pmatrix}.$$

The change of basis matrix $\mathbf{S}$ from $\mathfrak{B}$ to $\mathfrak{B}'$ is an orthogonal matrix, and

$$\mathbf{A}'=\mathbf{S}^{-1}\mathbf{G}\mathbf{S}=\mathbf{S}^{T}\mathbf{G}\mathbf{S}$$

is also the matrix of Q with respect to $\mathfrak{B}'$.

$\boxed{\mathrm{J}}$ CONICS

$\boxed{J3}$

By using the notations

$$X=\begin{pmatrix}x\\y\end{pmatrix},\qquad B=\begin{pmatrix}a_{10}\\a_{20}\end{pmatrix},\qquad A=\begin{pmatrix}a_{11}&a_{12}\\a_{12}&a_{22}\end{pmatrix},$$

$$X'=\begin{pmatrix}x'\\y'\end{pmatrix},\qquad T=\begin{pmatrix}x_0\\y_0\end{pmatrix},\qquad R=\begin{pmatrix}\cos\alpha&-\sin\alpha\\ \sin\alpha&\cos\alpha\end{pmatrix}$$

the relation

$$f(x,y)=a_{11}\,x^2+2\,a_{12}\,xy+a_{22}\,y^2+2a_{10}\,x+2a_{20}\,y+a_{00}$$

can be written as

$$\begin{aligned}f(x,y)&=(\,x\quad y\quad 1\,)\begin{pmatrix}a_{11}&a_{12}&a_{10}\\a_{12}&a_{22}&a_{20}\\a_{10}&a_{20}&a_{00}\end{pmatrix}\begin{pmatrix}x\\y\\1\end{pmatrix}\\&=(\,X^{T}\quad 1\,)\begin{pmatrix}A&B\\B^{T}&a_{00}\end{pmatrix}\begin{pmatrix}X\\1\end{pmatrix},\end{aligned}$$

and the relation

$$\begin{cases}x=x_0+x'\cos\alpha-y'\sin\alpha\\ y=y_0+x'\sin\alpha+y'\cos\alpha\end{cases}$$

as

$$\begin{pmatrix}x\\y\\1\end{pmatrix}=\begin{pmatrix}\cos\alpha&-\sin\alpha&x_0\\ \sin\alpha&\cos\alpha&y_0\\0&0&1\end{pmatrix}\begin{pmatrix}x'\\y'\\1\end{pmatrix}$$

or

$$\begin{pmatrix}X\\1\end{pmatrix}=\begin{pmatrix}R&T\\0&1\end{pmatrix}\begin{pmatrix}X'\\1\end{pmatrix}.$$

From

$$\begin{aligned}f(x,y)&=(\,X^{T}\quad 1\,)\begin{pmatrix}A&B\\B^{T}&a_{00}\end{pmatrix}\begin{pmatrix}X\\1\end{pmatrix}\\&=(\,X'^{T}\quad 1\,)\begin{pmatrix}R^{T}&0\\T^{T}&1\end{pmatrix}\begin{pmatrix}A&B\\B^{T}&a_{00}\end{pmatrix}\begin{pmatrix}R&T\\0&1\end{pmatrix}\begin{pmatrix}X'\\1\end{pmatrix}\\&=(\,X'^{T}\quad 1\,)\begin{pmatrix}A'&B'\\B'^{T}&a'_{00}\end{pmatrix}\begin{pmatrix}X'\\1\end{pmatrix}\end{aligned}$$

it follows

$$\begin{aligned}\Delta'&=\det\begin{pmatrix}A'&B'\\B'^{T}&a'_{00}\end{pmatrix}\\&=\det\begin{pmatrix}R^{T}&0\\T^{T}&1\end{pmatrix}\det\begin{pmatrix}A&B\\B^{T}&a_{00}\end{pmatrix}\det\begin{pmatrix}R&T\\0&1\end{pmatrix}\\&=\det\begin{pmatrix}A&B\\B^{T}&a_{00}\end{pmatrix}=\Delta\end{aligned}$$

and the relation $A'=R^{T}AR$, that is

$$\begin{pmatrix}a'_{11}&a'_{12}\\a'_{12}&a'_{22}\end{pmatrix}=\begin{pmatrix}\cos\alpha&\sin\alpha\\-\sin\alpha&\cos\alpha\end{pmatrix}\begin{pmatrix}a_{11}&a_{12}\\a_{12}&a_{22}\end{pmatrix}\begin{pmatrix}\cos\alpha&-\sin\alpha\\ \sin\alpha&\cos\alpha\end{pmatrix}$$

leading to $\delta'=\delta$ and $I'=I$.

$\boxed{J4}$

After a translation

$$\begin{cases}x=x_0+x'\\ y=y_0+x',\end{cases}$$

the polynomial

$$f(x,y)=a_{11}\,x^2+2\,a_{12}\,xy+a_{22}\,y^2+2a_{10}\,x+2a_{20}\,y+a_{00}$$

becomes

$$\begin{aligned}\tilde{f}(x',y')=a_{11}\,x'^2+2a_{12}\,x'\,y'+a_{22}\,y'^2+2(a_{11}\,x_0+a_{12}\,y_0\\+a_{10})x'+2(a_{12}\,x_0+a_{22}\,y_0+a_{20})y'+f(x_0,y_0).\end{aligned}$$

DESCRIPTION OF QUANTUM SYSTEMS

$\boxed{\mathrm{M}}$ AXIOMS OF QUANTUM MECHANICS

$\boxed{M3}$ If $\{|\psi_j\rangle\}_{j=1}^{n}$ is an orthonormal basis of $\mathcal{H}$, then

$$\begin{aligned}\mathrm{tr}(E_k\,|\psi\rangle\langle\psi|)&=\sum_{j=1}^{n}\langle\psi_j|E_k|\psi\rangle\langle\psi|\psi_j\rangle\\&=\sum_{j=1}^{n}\langle\psi|\psi_j\rangle\langle\psi_j|E_k|\psi\rangle\\&=\langle\psi|\sum_{j=1}^{n}|\psi_j\rangle\langle\psi_j|E_k|\psi\rangle\\&=\langle\psi|E_k|\psi\rangle\\&=\langle\psi|E_k^2|\psi\rangle\\&=\langle\psi|E_k^{\dagger}E_k|\psi\rangle\\&=||E_k|\psi\rangle||^2.\end{aligned}$$

$\boxed{M4}$

If $\{|\psi_j\rangle\}_{j=1}^{n}$ is an orthonormal basis of $\mathcal{H}$, then

$$\begin{aligned}\mathrm{tr}(A\,|\psi\rangle\langle\psi|)&=\sum_{j=1}^{n}\langle\psi_j|A|\psi\rangle\langle\psi|\psi_j\rangle\\&=\sum_{j=1}^{n}\langle\psi|\psi_j\rangle\langle\psi_j|A|\psi\rangle\\&=\langle\psi|\sum_{j=1}^{n}|\psi_j\rangle\langle\psi_j|A|\psi\rangle\\&=\langle\psi|A|\psi\rangle.\end{aligned}$$

and

$$\begin{aligned}\langle A\rangle&=\sum_{j=1}^{n}a_k\,prob(a_k)\\&=\sum_{j=1}^{n}a_k\,\langle\psi|E_k|\psi\rangle\\&=\langle\psi|\sum_{j=1}^{n}a_k\,E_k|\psi\rangle\\&=\langle\psi|A|\psi\rangle.\end{aligned}$$

$\boxed{M7}$

$$E_k=|\psi_k\rangle\langle\psi_k|$$

$$\Downarrow$$

$$prob(a_k)=\langle\psi|E_k|\psi\rangle=\langle\psi|\psi_k\rangle\langle\psi_k|\psi\rangle=|\langle\psi_k|\psi\rangle|^2.$$

$\boxed{\mathrm{N}}$ DENSITY OPERATORS

$\boxed{N1}$

- $E_\psi^{\dagger}=(|\psi\rangle\langle\psi|)^{\dagger}=|\psi\rangle\langle\psi|=E_\psi$.
- $\langle\varphi|E_\psi|\varphi\rangle=\langle\varphi|\psi\rangle\langle\psi|\varphi\rangle=|\langle\varphi|\psi\rangle|^2\geq 0$.
- If we extend the set $\{|\psi\rangle\}$ up to an orthonormal basis $\{|\psi\rangle,|\varphi_1\rangle,|\varphi_2\rangle,...,|\varphi_{n-1}\rangle\}$ of $\mathcal{H}$, then

$$\begin{aligned}\mathrm{tr}\,E_\psi&=\langle\psi|E_\psi|\psi\rangle+\sum_{j=1}^{n-1}\langle\varphi_j|E_\psi|\varphi_j\rangle\\&=\langle\psi|\psi\rangle\langle\psi|\psi\rangle+\sum_{j=1}^{n-1}\langle\varphi_j|\psi\rangle\langle\psi|\varphi_j\rangle=1.\end{aligned}$$

$\boxed{N5}$ $\varrho^\dagger=\varrho \;\Rightarrow\; \varrho$ is diagonalizable $\varrho=\sum_{k=1}^n p_k|\psi_k\rangle\langle\psi_k|$.

where $\{|\psi_1\rangle,|\psi_2\rangle,...,|\psi_n\rangle\}$ is an orthonormal basis, that is $\langle\psi_j|\psi_k\rangle=\delta_{jk}$, $\sum_{k=1}^n|\psi_k\rangle\langle\psi_k|=\mathbb{I}$.

$$\left.\begin{array}{l}\varrho\geq 0 \;\Rightarrow\; p_k=\langle\psi_k|\varrho|\psi_k\rangle\geq 0,\\ \mathrm{tr}\,\varrho=1 \;\Rightarrow\; \sum_{k=1}^n p_k=\mathrm{tr}\,\varrho=1\end{array}\right\}\;\Rightarrow\; 0\leq p_k\leq 1.$$

$$\Downarrow$$

$$p_k^2\leq p_k.$$

$$\varrho^2=\sum_{k=1}^n p_k^2|\psi_k\rangle\langle\psi_k| \;\Rightarrow\; \mathrm{tr}\,\varrho^2=\sum_{k=1}^n p_k^2\leq\sum_{k=1}^n p_k=1.$$

The minimum value of $\mathrm{tr}\,\varrho^2$ is obtained in the case $p_1=p_2=...=p_n=\frac{1}{n}$, and it is $n\,\frac{1}{n^2}=\frac{1}{n}$.

$\boxed{N7}$ (Case of a qubit).

If r lies on the line segment connecting two points $r',r''\in\partial\Omega_3$, then there exists $\lambda\in(0,1)$ such that $r=(1-\lambda)r'+\lambda r''$, and consequently

$$\begin{aligned}\varrho(r)&=\tfrac{1}{2}(r_0\,\mathbb{I}+((1-\lambda)r'+\lambda r'')\cdot\sigma)\\&=(1-\lambda)\varrho(r')+\lambda\varrho(r'').\end{aligned}$$

$\boxed{N8}$

$$0\leq p_k\leq 1 \;\Rightarrow\; 0\leq\sum_{k=1}^n p_k|\langle\psi_k|\psi\rangle|^2\leq\sum_{k=1}^n|\langle\psi_k|\psi\rangle|^2$$
$$\Downarrow$$
$$0\leq\sum_{k=1}^n p_k\langle\psi|\psi_k\rangle\langle\psi_k|\psi\rangle\leq\sum_{k=1}^n\langle\psi|\psi_k\rangle\langle\psi_k|\psi\rangle$$
$$\Downarrow$$
$$0\leq\langle\psi|\sum_{k=1}^n p_k|\psi_k\rangle\langle\psi_k|\psi\rangle\leq\langle\psi|\sum_{k=1}^n|\psi_k\rangle\langle\psi_k|\psi\rangle$$
$$\Downarrow$$
$$0\leq\langle\psi|\varrho|\psi\rangle\leq\langle\psi|\mathbb{I}|\psi\rangle$$
$$\Downarrow$$
$$0\leq\varrho\leq\mathbb{I}$$

$\boxed{N9}$

$$\varrho=\sum_{k=1}^n p_k|\psi_k\rangle\langle\psi_k| \;\Rightarrow\; \begin{aligned}\langle A\rangle_\varrho&=\sum_{k=1}^n p_k\langle\psi_k|A|\psi_k\rangle\\&=\sum_{k=1}^n p_k\,\mathrm{tr}(A\,|\psi_k\rangle\langle\psi_k|)\\&=\mathrm{tr}(A\sum_{k=1}^n p_k\,|\psi_k\rangle\langle\psi_k|)\\&=\mathrm{tr}(A\,\varrho).\end{aligned}$$

$\boxed{N10,11}$

We have $||\varrho||^2=\langle\varrho,\varrho\rangle=\mathrm{tr}\,\varrho^2$.

- $\varrho=|\psi\rangle\langle\psi| \;\Rightarrow\; \varrho^2=|\psi\rangle\langle\psi|\psi\rangle\langle\psi|=\varrho \;\Rightarrow\; \mathrm{tr}\,\varrho^2=1.$
- $\varrho^2=\varrho \;\Rightarrow\; \sum_{k=1}^n p_k^2|\psi_k\rangle\langle\psi_k|=\sum_{k=1}^n p_k|\psi_k\rangle\langle\psi_k|$
  $\Rightarrow\; p_k\in\{0;1\}$, for any k
  $\Rightarrow$ there exists j such that $p_k=\delta_{kj}$
  $\Rightarrow\; \varrho=|\psi_j\rangle\langle\psi_j|$.
- there exists j such that $0<p_j<1$ $\;\Rightarrow\; p_j^2<p_j \;\Rightarrow\; \mathrm{tr}\,\varrho^2<\mathrm{tr}\,\varrho=1.$

Consequently:

$\mathrm{tr}\,\varrho^2=1 \;\Rightarrow\; p_k\in\{0;1\}$, for any k
$\Rightarrow$ there exists j such that $p_k=\delta_{kj}$
$\Rightarrow\; \varrho=|\psi_j\rangle\langle\psi_j|$.

$\boxed{N12}$

$$\varrho\in\mathcal{D}(\mathcal{H})\Rightarrow\begin{array}{c}\varrho^\dagger=\varrho,\\ \langle\psi|\varrho|\psi\rangle\geq 0,\ \forall\psi\\ \mathrm{tr}\,\varrho=1\end{array}\Rightarrow\begin{array}{c}(K^\dagger\varrho K)^\dagger=K^\dagger\varrho K,\\ \langle\psi|K^\dagger\varrho K|\psi\rangle\geq 0,\ \forall\psi\\ \mathrm{tr}(K^\dagger\varrho K)=\mathrm{tr}(\varrho KK^\dagger)=1.\end{array}$$

$\boxed{N13}$

$((1-\lambda)\varrho_1+\lambda\varrho_2)^\dagger=(1-\lambda)\varrho_1+\lambda\varrho_2,$
$\langle\psi|(1-\lambda)\varrho_1+\lambda\varrho_2|\psi\rangle=(1-\lambda)\langle\psi|\varrho_1|\psi\rangle+\lambda\langle\psi|\varrho_2|\psi\rangle\geq 0,$
$\mathrm{tr}((1-\lambda)\varrho_1+\lambda\varrho_2)=(1-\lambda)\mathrm{tr}\,\varrho_1+\lambda\,\mathrm{tr}\,\varrho_2=1.$

$\boxed{N14}$

"$\Rightarrow$"

Let $\varrho=|\psi\rangle\langle\psi|$, and let $\varrho_1,\varrho_2\in\mathcal{D}(\mathcal{H})$, $\lambda\in[0,1]$ be such that $\varrho=(1-\lambda)\,\varrho_1+\lambda\,\varrho_2$. We have

$$\varrho^2=\varrho\Rightarrow\;\varrho=(1-\lambda)\,\varrho\varrho_1\varrho+\lambda\,\varrho\varrho_2\varrho,$$
$$\Downarrow$$
$$1=(1-\lambda)\,\mathrm{tr}(\varrho\varrho_1\varrho)+\lambda\,\mathrm{tr}(\varrho\varrho_2\varrho).$$

Density operators ϱ_1 and ϱ_2 satisfy the relations

$$\begin{aligned}\mathrm{tr}(\varrho\varrho_i\varrho)=\mathrm{tr}(\varrho^2\varrho_i)=\mathrm{tr}(\varrho\varrho_i)&=\langle\varrho,\varrho_i\rangle\\&\leq|\langle\varrho,\varrho_i\rangle|\leq||\varrho||\,||\varrho||\leq 1\end{aligned}$$

$$\mathrm{tr}(\varrho\varrho_i\varrho)=\textstyle\sum_{j=1}^n\langle\psi_j,\varrho\varrho_i\varrho\psi_j\rangle=\sum_{j=1}^n\langle\varrho\psi_j,\varrho_i\varrho\psi_j\rangle\geq 0,$$

where $\{|\psi_1\rangle,|\psi_2\rangle,...,|\psi_n\rangle\}$ is an orthonormal basis of $\mathcal{H}$.

We have $\mathrm{tr}(\varrho\varrho_1\varrho)=\mathrm{tr}(\varrho\varrho_2\varrho)=1$ because

$$\begin{array}{c}\mathrm{tr}(\varrho\varrho_1\varrho)<1\\ \text{or}\\ \mathrm{tr}(\varrho\varrho_2\varrho)<1\end{array}\;\Rightarrow(1-\lambda)\,\mathrm{tr}(\varrho\varrho_1\varrho)+\lambda\,\mathrm{tr}(\varrho\varrho_2\varrho)<1.$$

Consequently,

$\langle\varrho,\varrho_i\rangle=\mathrm{tr}(\varrho\varrho_i)=\mathrm{tr}(\varrho^2\varrho_i)=\mathrm{tr}(\varrho\varrho_i\varrho)=1.$

Since $\langle\varrho,\varrho\rangle=\mathrm{tr}\,\varrho^2\leq 1$ and $\langle\varrho_i,\varrho_i\rangle=\mathrm{tr}\,\varrho_i^2\leq 1$ in Cauchy-Schwarz inequality

$$|\langle\varrho,\varrho_i\rangle|^2\leq\langle\varrho,\varrho\rangle\,\langle\varrho_i,\varrho_i\rangle$$

we have equality

$$|\langle\varrho,\varrho_i\rangle|^2=\langle\varrho,\varrho\rangle\,\langle\varrho_i,\varrho_i\rangle.$$

This is possible only for $\varrho=c\,\varrho_i$, where c is a constant. But

$$\mathrm{tr}\,\varrho=\mathrm{tr}\,\varrho_i=1\quad\Rightarrow\quad\varrho=\varrho_i.$$

"$\Leftarrow$"

Let ϱ be an extremal point of $\mathcal{D}(\mathcal{H})$ with the spectral decomposition

$$\varrho=\sum_{j=1}^n p_j|\psi_j\rangle\langle\psi_j|=p_1|\psi_1\rangle\langle\psi_1|+\cdots+p_n|\psi_n\rangle\langle\psi_n|.$$

Each p_k must be either 0 or 1 because

$$0<p_k<1$$
$$\Downarrow$$
$$\varrho=p_k\,|\psi_k\rangle\langle\psi_k|+(1-p_k)\sum_{j\neq k}\tfrac{p_j}{1-p_k}|\psi_j\rangle\langle\psi_j|$$

and this is possible only if

$$\varrho=|\psi_k\rangle\langle\psi_k|=\sum_{j\neq k}\tfrac{p_j}{1-p_k}|\psi_j\rangle\langle\psi_j|,$$

and this is impossible since

$$|\psi_k\rangle\langle\psi_k|=\sum_{j\neq k}\tfrac{p_j}{1-p_k}|\psi_j\rangle\langle\psi_j|$$
$$\Downarrow$$
$$|\psi_k\rangle=|\psi_k\rangle\langle\psi_k|\psi_k\rangle=\sum_{j\neq k}\tfrac{p_j}{1-p_k}|\psi_j\rangle\langle\psi_j|\psi_k\rangle=0.$$

$\boxed{\mathbf{O}}$ <u>**QUBITS**</u>

$\boxed{O7}$

Eigenvalues of $A\in\mathcal{L}(\mathbb{C}^2)$ are the roots of the equation

$$\begin{vmatrix}\frac{r_0+r_3}{2}-\lambda & \frac{r_1-\mathrm{i}r_2}{2}\\ \frac{r_1+\mathrm{i}r_2}{2} & \frac{r_0-r_3}{2}-\lambda\end{vmatrix}=0.$$

$\boxed{O12}$

$$A\geq 0\;\Leftrightarrow\;\begin{array}{c}\lambda_1=\frac{r_0+||r||}{2}\geq 0\\ \text{and}\\ \lambda_2=\frac{r_0-||r||}{2}\geq 0\end{array}\;\Leftrightarrow\;||r||\leq r_0.$$

$\boxed{O19}$

The diametrically opposite point corresponding to (ϱ,θ) has the angular coordinates $(\varrho+\pi,\theta-\pi)$. The corresponding pure states

$$\cos\tfrac{\theta}{2}\,|0\rangle+\sin\tfrac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\phi}\,|1\rangle$$
$$\cos\tfrac{\pi-\theta}{2}\,|0\rangle+\sin\tfrac{\pi-\theta}{2}\,\mathrm{e}^{\mathrm{i}(\phi+\pi)}\,|1\rangle$$

are orthogonal.

$\boxed{O21}$

$$A^2=\mathbb{I}\;\Rightarrow\;\mathrm{e}^{\mathrm{i}tA}=I+\tfrac{\mathrm{i}t}{1!}A+\tfrac{(\mathrm{i}t)^2}{2!}A^2+\tfrac{(\mathrm{i}t)^3}{3!}A^3+\ldots$$
$$=\left(1-\tfrac{t^2}{2!}+\tfrac{t^4}{4!}-\tfrac{t^6}{6!}+\ldots\right)\mathbb{I}$$
$$+\mathrm{i}\left(\tfrac{t}{1!}-\tfrac{t^3}{3!}+\tfrac{t^5}{5!}-\tfrac{t^7}{7!}+\ldots\right)A.$$
$$=\cos t\,\mathbb{I}+\mathrm{i}\sin t\,A.$$

$\boxed{O25}$

We have

$$\mathcal{R}_x(t)\begin{pmatrix}1+r_3 & r_1-\mathrm{i}r_2\\ r_1+\mathrm{i}r_2 & 1-r_3\end{pmatrix}\mathcal{R}_x^\dagger(t)=\begin{pmatrix}1+r_3' & r_1'-\mathrm{i}r_2'\\ r_1'+\mathrm{i}r_2' & 1-r_3'\end{pmatrix},$$

where

$$\begin{pmatrix}r_1'\\ r_2'\\ r_3'\end{pmatrix}=\begin{pmatrix}1&0&0\\0&\cos t&-\sin t\\0&\sin t&\cos t\end{pmatrix}\begin{pmatrix}r_1\\ r_2\\ r_3\end{pmatrix}.$$

$\boxed{O26}$

We have

$$\mathcal{R}_y(t)\begin{pmatrix}1+r_3 & r_1-\mathrm{i}r_2\\ r_1+\mathrm{i}r_2 & 1-r_3\end{pmatrix}\mathcal{R}_y^\dagger(t)=\begin{pmatrix}1+r_3' & r_1'-\mathrm{i}r_2'\\ r_1'+\mathrm{i}r_2' & 1-r_3'\end{pmatrix},$$

where

$$\begin{pmatrix}r_1'\\ r_2'\\ r_3'\end{pmatrix}=\begin{pmatrix}\cos t&0&-\sin t\\0&1&0\\\sin t&0&\cos t\end{pmatrix}\begin{pmatrix}r_1\\ r_2\\ r_3\end{pmatrix}.$$

$\boxed{O27}$

We have

$$\mathcal{R}_z(t)\begin{pmatrix}1+r_3 & r_1-\mathrm{i}r_2\\ r_1+\mathrm{i}r_2 & 1-r_3\end{pmatrix}\mathcal{R}_z^\dagger(t)=\begin{pmatrix}1+r_3' & r_1'-\mathrm{i}r_2'\\ r_1'+\mathrm{i}r_2' & 1-r_3'\end{pmatrix},$$

where

$$\begin{pmatrix}r_1'\\ r_2'\\ r_3'\end{pmatrix}=\begin{pmatrix}\cos t&-\sin t&0\\\sin t&\cos t&0\\0&0&1\end{pmatrix}\begin{pmatrix}r_1\\ r_2\\ r_3\end{pmatrix}.$$

$\boxed{O28}$

A transform $U:\mathbb{C}^2\longrightarrow\mathbb{C}^2$ with the matrix

$$U=\begin{pmatrix}u_{00}&u_{01}\\u_{10}&u_{11}\end{pmatrix}$$

is unitary if $U^\dagger U=UU^\dagger=I$, that is

$$\begin{array}{ll}|u_{00}|^2+|u_{01}|^2=1, & |u_{00}|^2+|u_{10}|^2=1,\\ |u_{10}|^2+|u_{11}|^2=1, & |u_{01}|^2+|u_{11}|^2=1,\end{array}\qquad(*)$$

and

$$\bar{u}_{00}u_{10}+\bar{u}_{01}u_{11}=0,\qquad \bar{u}_{00}u_{01}+\bar{u}_{10}u_{11}=0.$$

Matrix U satisfies (*) if and only if it has the form

$$U=\begin{pmatrix}\cos\frac{\beta}{2}\,\mathrm{e}^{\mathrm{i}\frac{a}{2}} & -\sin\frac{\beta}{2}\,\mathrm{e}^{\mathrm{i}\frac{b}{2}}\\ \sin\frac{\beta}{2}\,\mathrm{e}^{\mathrm{i}\frac{c}{2}} & \cos\frac{\beta}{2}\,\mathrm{e}^{\mathrm{i}\frac{d}{2}}\end{pmatrix}.$$

This matrix satisfies the last two conditions if and only if $a-c=b-d$. Consequently, U must have the form

$$U=\begin{pmatrix}\cos\frac{\beta}{2}\,\mathrm{e}^{\mathrm{i}\frac{a}{2}} & -\sin\frac{\beta}{2}\,\mathrm{e}^{\mathrm{i}\frac{b}{2}}\\ \sin\frac{\beta}{2}\,\mathrm{e}^{\mathrm{i}\frac{c}{2}} & \cos\frac{\beta}{2}\,\mathrm{e}^{\mathrm{i}\frac{b+c-a}{2}}\end{pmatrix}.$$

But, this matrix can be written as

$$U=\mathrm{e}^{\mathrm{i}\frac{b+c}{4}}\begin{pmatrix}\cos\frac{\beta}{2}\,\mathrm{e}^{\mathrm{i}\frac{(a-b)+(a-c)}{2}} & -\sin\frac{\beta}{2}\,\mathrm{e}^{\mathrm{i}\frac{(a-c)-(a-b)}{2}}\\ \sin\frac{\beta}{2}\,\mathrm{e}^{-\mathrm{i}\frac{(a-c)-(a-b)}{2}} & \cos\frac{\beta}{2}\,\mathrm{e}^{-\mathrm{i}\frac{(a-b)+(a-c)}{2}}\end{pmatrix}$$
$$=\mathrm{e}^{\mathrm{i}\frac{b+c}{4}}\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}}&0\\0&\mathrm{e}^{\mathrm{i}\frac{\alpha}{2}}\end{pmatrix}\begin{pmatrix}\cos\frac{\beta}{2}&-\sin\frac{\beta}{2}\\\sin\frac{\beta}{2}&\cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\frac{\gamma}{2}}&0\\0&\mathrm{e}^{\mathrm{i}\frac{\gamma}{2}}\end{pmatrix}$$
$$=\mathrm{e}^{\mathrm{i}\theta}\,\mathcal{R}_z(\alpha)\,\mathcal{R}_y(\beta)\,\mathcal{R}_z(\gamma),$$

where $\theta=\frac{b+c}{4}$, $\alpha=\frac{a-b}{2}$ and $\gamma=\frac{a-c}{2}$.

$\boxed{\mathbf{P}}$ COMPOSITE SYSTEMS

$\boxed{P8}$

$$\mathrm{tr}(|\varphi\rangle\langle\psi|)=\sum_j\langle j|\varphi\rangle\langle\psi|j\rangle$$
$$=\sum_j\langle\psi|j\rangle\langle j|\varphi\rangle$$
$$=\langle\psi|\varphi\rangle.$$

$\boxed{P9}$

Direct consequences of $\boxed{\mathfrak{D}3}$ and $\boxed{\mathfrak{D}4}$ (page 11).

$\boxed{P10}$

$$\langle k\ell|\sum_j{}_2\langle j|\varphi\psi\rangle\otimes|j\rangle_2=\sum_k\langle kj|\varphi\psi\rangle\langle\ell|j\rangle$$
$$=\sum_k\langle kj|\varphi\psi\rangle\,\delta_{\ell j}$$
$$=\langle k\ell|\varphi\psi\rangle.$$

$\boxed{P11}$

$$A\otimes B=\sum_{ijk\ell}|ik\rangle\langle ik|A\otimes B|j\ell\rangle\langle j\ell|$$
$$=\sum_{ijk\ell}\langle ik|A\otimes B|j\ell\rangle\,|ik\rangle\langle j\ell|$$
$$=\sum_{ijk\ell}(A\otimes B)^{ik}_{j\ell}\,|ik\rangle\langle j\ell|$$
$$=\sum_{ijk\ell}A^i_j\,B^k_\ell\,|ik\rangle\langle j\ell|$$
$$\Downarrow$$
$${}_1\langle\varphi_1|A\otimes B|\varphi_2\rangle_1=\sum_{ijk\ell}A^i_j\,B^k_\ell\;{}_1\langle\varphi_1|ik\rangle\langle j\ell|\varphi_2\rangle_1$$
$$=\sum_{ijk\ell}A^i_j\,B^k_\ell\,\langle\varphi_1|i\rangle\langle j|\varphi_2\rangle|k\rangle\langle\ell|$$
$$=\sum_{ij}\langle\varphi_1|i\rangle\langle i|A|j\rangle\langle j|\varphi_2\rangle\sum_{k\ell}B^k_\ell\,|k\rangle\langle\ell|$$
$$=\langle\varphi_1|A|\varphi_2\rangle\,B,$$
$${}_2\langle\psi_1|A\otimes B|\psi_2\rangle_2=\sum_{ijk\ell}A^i_j\,B^k_\ell\;{}_2\langle\psi_1|ik\rangle\langle j\ell|\psi_2\rangle_2$$
$$=\sum_{ijk\ell}A^i_j\,B^k_\ell\,\langle\psi_1|k\rangle\langle\ell|\psi_2\rangle|i\rangle\langle j|$$
$$=\sum_{k\ell}\langle\psi_1|k\rangle\langle k|B|\ell\rangle\langle\ell|\psi_2\rangle\sum_{ij}A^i_j\,|i\rangle\langle j|$$
$$=\langle\psi_1|B|\psi_2\rangle\,A$$

$\boxed{P12}$

$$\mathrm{tr}_1(A\otimes B)=\sum_j{}_1\langle j|A\otimes B|j\rangle_1,$$
$$\Downarrow\boxed{\mathfrak{D}9}$$
$$=\sum_j\langle j|A|j\rangle\,B$$
$$=(\mathrm{tr}\,A)\,B.$$
$$\mathrm{tr}_2(A\otimes B)=\sum_k{}_2\langle k|A\otimes B|k\rangle_2,$$
$$\Downarrow\boxed{\mathfrak{D}9}$$
$$=\sum_k\langle k|B|k\rangle\,A$$
$$=(\mathrm{tr}\,B)\,A.$$

$\boxed{P13}$

$$\begin{aligned}\langle ik|(|\varphi_1\rangle\langle\varphi_2|\otimes|\psi_1\rangle\langle\psi_2|)|j\ell\rangle &= \langle ik|(|\varphi_1\rangle\langle\varphi_2|j\rangle\otimes|\psi_1\rangle\langle\psi_2|\ell\rangle)\\ &= \langle i|\varphi_1\rangle\langle\varphi_2|j\rangle\langle k|\psi_1\rangle\langle\psi_2|\ell\rangle\\ &= \langle i|\varphi_1\rangle\langle k|\psi_1\rangle\langle\varphi_2|j\rangle\langle\psi_2|\ell\rangle\\ &= \langle ik|\varphi_1\psi_1\rangle\langle\varphi_2\psi_2|j\ell\rangle.\end{aligned}$$

$\boxed{P14}$

$$\begin{aligned}\mathrm{tr}_1(|\varphi_1\psi_1\rangle\langle\varphi_2\psi_2|) &= \mathrm{tr}_1(|\varphi_1\rangle\langle\varphi_2|\otimes|\psi_1\rangle\langle\psi_2|)\\ &\Downarrow \boxed{\mathfrak{D}13}\\ &= \mathrm{tr}(|\varphi_1\rangle\langle\varphi_2|)\,|\psi_1\rangle\langle\psi_2|)\\ &\Downarrow \boxed{\mathfrak{D}8}\\ &= \langle\psi_2|\psi_1\rangle\ \ |\psi_1\rangle\langle\varphi_2|.\\ \mathrm{tr}_2(|\varphi_1\psi_1\rangle\langle\varphi_2\psi_2|) &= \mathrm{tr}_2(|\varphi_1\rangle\langle\varphi_2|\otimes|\psi_1\rangle\langle\psi_2|)\\ &\Downarrow \boxed{\mathfrak{D}13}\\ &= \mathrm{tr}(|\psi_1\rangle\langle\psi_2|)\,|\varphi_1\rangle\langle\varphi_2|\\ &\Downarrow \boxed{\mathfrak{D}8}\\ &= \langle\varphi_2|\varphi_1\rangle\ \ |\psi_1\rangle\langle\psi_2|,\end{aligned}$$

$\boxed{P15}$

$$\begin{aligned}T=\sum_{ijk\ell}|ik\rangle\langle ik|T|j\ell\rangle\langle j\ell| &= \sum_{ijk\ell}T^{ik}_{j\ell}\,|ik\rangle\langle j\ell|\\ &\Downarrow \boxed{\mathfrak{D}7}\\ \mathrm{tr}_1 T &= \sum_r {}_1\langle r|T|r\rangle_1,\\ &= \sum_r\sum_{ijk\ell}T^{ik}_{j\ell}\ {}_1\langle r|ik\rangle\langle j\ell|r\rangle_1\\ &= \sum_r\sum_{ijk\ell}T^{ik}_{j\ell}\,\delta_{ri}\,\delta_{rj}|k\rangle\langle\ell|\\ &= \sum_{jk\ell}T^{jk}_{j\ell}\,|k\rangle\langle\ell|\\ &\Downarrow\\ \langle k|\mathrm{tr}_1 T|\ell\rangle &= \sum_j T^{jk}_{j\ell}\\ (\mathrm{tr}_1 T)^k_\ell &= \sum_j T^{jk}_{j\ell}\\ T=\sum_{ijk\ell}|ik\rangle\langle ik|T|j\ell\rangle\langle j\ell| &= \sum_{ijk\ell}T^{ik}_{j\ell}\,|ik\rangle\langle j\ell|\\ &\Downarrow \boxed{\mathfrak{D}7}\\ \mathrm{tr}_2 T &= \sum_r {}_2\langle r|T|r\rangle_2,\\ &= \sum_r\sum_{ijk\ell}T^{ik}_{j\ell}\ {}_2\langle r|ik\rangle\langle j\ell|r\rangle_2\\ &= \sum_r\sum_{ijk\ell}T^{ik}_{j\ell}\,\delta_{rk}\,\delta_{r\ell}|i\rangle\langle j|\\ &= \sum_{ijk}T^{ik}_{jk}\,|i\rangle\langle j|\\ &\Downarrow\\ \langle i|\mathrm{tr}_2 T|j\rangle &= \sum_k T^{ik}_{jk}\\ (\mathrm{tr}_2 T)^i_j &= \sum_k T^{ik}_{jk}\end{aligned}$$

$\boxed{P16}$

Direct consequences of $\boxed{\mathfrak{D}15}$ (page 11).

$\boxed{P17}$

$$\begin{aligned}&\langle ab|\sum_{k,\ell}{}_2\langle k|T|\ell\rangle_2\otimes|k\rangle_2\,{}_2\langle\ell|cd\rangle\\ &\quad=\sum_{k,\ell}\langle ak|T|c\ell\rangle\,\langle b|k\rangle\langle\ell|d\rangle\\ &\quad=\sum_{k,\ell}\langle ak|T|c\ell\rangle\,\delta_{bk}\delta_{\ell d}\\ &\quad=\langle ab|T|cd\rangle.\end{aligned}$$

$\boxed{P18}$

$$\begin{aligned}\mathrm{tr}(|\varphi\rangle\langle\psi|A) &= \sum_j\langle j|(|\varphi\rangle\langle\psi|A)|j\rangle\\ &= \sum_j\langle j|\varphi\rangle\langle\psi|\,A|j\rangle\\ &= \sum_j\langle\psi|\,A|j\rangle\langle j|\varphi\rangle\\ &= \langle\psi|A|\varphi\rangle.\end{aligned}$$

$\boxed{P19}$

$$\begin{aligned}(\mathbb{I}\otimes(|b\rangle\langle b|))T &= \sum_{ijk\ell}\sum_{rs}(\mathbb{I}\otimes(|b\rangle\langle b|)^{ik}_{rs}T^{rs}_{j\ell}\,|ik\rangle\langle j\ell|\\ &\Downarrow\\ \mathrm{tr}_2((\mathbb{I}\otimes(|b\rangle\langle b|))T) &= \sum_{ijk}\sum_{rs}(\mathbb{I}\otimes(|b\rangle\langle b|)^{ik}_{rs}T^{rs}_{jk}\,|i\rangle\langle j|\\ &= \sum_{ijk}\sum_{rs}\delta^i_r\langle k|b\rangle\langle b|s\rangle T^{rs}_{jk}\,|i\rangle\langle j|\\ &= \sum_{ijk}\sum_{s}\langle b|s\rangle T^{is}_{jk}\langle k|b\rangle\,|i\rangle\langle j|\\ &= \sum_{ijk}\sum_{s}\langle b|s\rangle\langle is|T|jk\rangle\langle k|b\rangle\,|i\rangle\langle j|\\ &= \sum_{ij}\langle ib|T|jb\rangle\,|i\rangle\langle j|\\ &= \sum_{ij}|i\rangle\langle ib|T|jb\rangle\langle j|\\ &= {}_2\langle b|T|b\rangle_2\end{aligned}$$

$\boxed{P20}$

$$\begin{aligned}&\left.\begin{aligned}A^i_k&=\langle i|A|k\rangle\\ B^m_\ell&=\langle m|B|\ell\rangle\\ T^{k\ell}_{im}&=\langle k\ell|T|im\rangle\end{aligned}\right\}\Rightarrow\ \sum_{ik\ell m}A^i_k\,B^m_\ell\,T^{k\ell}_{im}=\sum_{ik}A^i_k\sum_{j\ell m}\delta^k_j\,B^m_\ell\,T^{j\ell}_{im}\\ &\Downarrow\\ \sum_{ik\ell m}(A\otimes B)^{im}_{k\ell}\,T^{k\ell}_{im} &= \sum_{ik}A^i_k\sum_{j\ell m}(\mathbb{I}\otimes B)^{km}_{j\ell}\,T^{j\ell}_{im}\\ &\Downarrow\\ \sum_{im}((A\otimes B)T)^{im}_{im} &= \sum_{ik}A^i_k\sum_m((\mathbb{I}\otimes B)T)^{km}_{im}\\ &\Downarrow\\ \mathrm{tr}\,((A\otimes B)T) &= \sum_{ik}A^i_k(\mathrm{tr}_2((\mathbb{I}\otimes B)T))^k_i\\ &\Downarrow\\ \mathrm{tr}\,((A\otimes B)T) &= \mathrm{tr}(A\,\mathrm{tr}_2((\mathbb{I}\otimes B)T)).\end{aligned}$$

$\boxed{P21}$

$$\begin{aligned}&\left.\begin{aligned}A^i_k&=\langle i|A|k\rangle\\ B^m_\ell&=\langle m|B|\ell\rangle\\ T^{k\ell}_{im}&=\langle k\ell|T|im\rangle\end{aligned}\right\}\Rightarrow\ \sum_{ik\ell m}A^i_k\,B^m_\ell\,T^{k\ell}_{im}=\sum_{m\ell}B^m_\ell\sum_{ijk}A^i_k\,\delta^\ell_j\,T^{kj}_{im}\\ &\Downarrow\\ \sum_{ik\ell m}(A\otimes B)^{im}_{k\ell}\,T^{k\ell}_{im} &= \sum_{m\ell}B^m_\ell\sum_{ijk}(A\otimes\mathbb{I})^{i\ell}_{kj}\,T^{kj}_{im}\\ &\Downarrow\\ \sum_{im}((A\otimes B)T)^{im}_{im} &= \sum_{m\ell}B^m_\ell\sum_i((A\otimes\mathbb{I})T)^{i\ell}_{im}\\ &\Downarrow\\ \mathrm{tr}\,((A\otimes B)T) &= \sum_{m\ell}B^m_\ell(\mathrm{tr}_1((A\otimes\mathbb{I})T))^\ell_m\\ &\Downarrow\\ \mathrm{tr}\,((A\otimes B)T) &= \mathrm{tr}(B\,\mathrm{tr}_1((A\otimes\mathbb{I})T)).\end{aligned}$$

$\boxed{P22}$

$$\begin{aligned}A\otimes B &= (A\otimes\mathbb{I})(\mathbb{I}\otimes B)\\ &\Downarrow\\ \mathrm{tr}\,((A\otimes B)T) &= \mathrm{tr}\,((A\otimes\mathbb{I})(\mathbb{I}\otimes B)T)\end{aligned}$$

$\boxed{P23}$

Direct consequence of $\boxed{\mathfrak{D}20}$ (page 11).

$\boxed{P24}$

Direct consequence of $\boxed{\mathfrak{D}21}$ (page 11).

$\boxed{\#}$ Additional definitions/results

$\boxed{\#1}$

We have

$$\begin{aligned} A(\alpha v) &= f(\alpha v)\,\alpha v,\\ \alpha\; Av &= f(\alpha v)\,\alpha v,\\ \alpha\; f(v)\,v &= f(\alpha v)\,\alpha v,\\ &\Downarrow \text{ for } \alpha\neq 0 \text{ and } v\neq 0\\ f(\alpha v) &= f(v). \end{aligned}$$

If the vectors v and w are linearly independent, then

$$A(v+w)=Av+Aw$$
$$f(v+w)\,(v+w)=f(v)\,v+f(w)\,w$$
$$\Downarrow$$
$$\begin{cases} f(v+w)=f(v)\\ f(v+w)=f(w)\end{cases}$$
$$\Downarrow$$
$$f(v)=f(w).$$

If $\{v_0,v_1,...,v_{n-1}\}$ is a basis of $\mathcal{V}$, then

$$f(v_0)=f(v_1)=\cdots=f(v_{n-1})$$

and

$$\begin{aligned} A\sum_{j=0}^{n-1}\alpha_j v_j &= \sum_{j=0}^{n-1}\alpha_j\; Av_j\\ &= \sum_{j=0}^{n-1}\alpha_j\; f(v_j)\,v_j\\ &= f(v_0)\sum_{j=0}^{n-1}\alpha_j v_j. \end{aligned}$$

$\boxed{\#5}$

If $\{|e_j\rangle\}$ is an orthonormal basis, then

$$\begin{aligned} ||Ax|| &= ||A(\textstyle\sum|e_j\rangle\langle e_j|x\rangle)||\\ &= \textstyle\sum|\langle e_j|x\rangle|\;||A|e_j\rangle||\\ &\leq \sqrt{\textstyle\sum|\langle e_j|x\rangle|^2}\;\sqrt{\textstyle\sum||A|e_j\rangle||^2}\\ &\leq ||x||\;\sqrt{\textstyle\sum||A|e_j\rangle||^2}\\ &= C\,||x||. \end{aligned}$$

$\boxed{\#8}$

$$\begin{aligned} (\mathrm{e}^A)^\dagger &= \left(\sum_{k=0}^{\infty}\frac{A^k}{k!}\right)^\dagger = \lim_{m\to\infty}\left(\sum_{k=0}^{m}\frac{A^k}{k!}\right)^\dagger\\ &= \lim_{m\to\infty}\sum_{k=0}^{m}\frac{A^{\dagger k}}{k!} = \sum_{k=0}^{\infty}\frac{A^{\dagger k}}{k!} = \mathrm{e}^{A^\dagger}. \end{aligned}$$

$\boxed{\#9}$

$$(\mathrm{e}^{\mathrm{i}A})^\dagger\mathrm{e}^{\mathrm{i}A} = \mathrm{e}^{-\mathrm{i}A^\dagger}\mathrm{e}^{\mathrm{i}A} = \mathrm{e}^0 = \mathbb{I}.$$

$\boxed{\#10}$

$$\begin{aligned} (F^\dagger F)[\varphi](k) &= \tfrac{1}{n}\sum_{j=0}^{n-1}\mathrm{e}^{\frac{2\pi\mathrm{i}}{n}kj}\sum_{m=0}^{n-1}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{n}jm}\varphi(m)\\ &= \tfrac{1}{n}\sum_{m=0}^{n-1}\sum_{j=0}^{n-1}\mathrm{e}^{\frac{2\pi\mathrm{i}}{n}(k-m)j}\varphi(m)\\ &= \tfrac{1}{n}\sum_{m=0}^{n-1}\sum_{j=0}^{n-1}\left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{n}(k-m)}\right)^j\;\varphi(m)\\ &= \tfrac{1}{n}\sum_{m=0}^{n-1}n\,\delta_{km}\,\varphi(m) = \varphi(k). \end{aligned}$$

$\boxed{\#11}$

$$\begin{matrix}\varphi\otimes\psi:\mathcal{H}_2\to\mathcal{H}_1,\\ (\varphi\otimes\psi)(\eta)=\langle\eta|\psi\rangle\varphi\end{matrix}\quad\text{is anti-linear, for any}\quad\begin{matrix}\varphi\in\mathcal{H}_1,\\ \psi\in\mathcal{H}_2.\end{matrix}$$

These particular operators span the considered space,

$$\begin{aligned} &(\varphi_1+\varphi_2)\otimes\psi=\varphi_1\otimes\psi+\varphi_2\otimes\psi\\ &\varphi\otimes(\psi_1+\psi_2)=\varphi\otimes\psi_1+\varphi\otimes\psi_2\\ &\lambda(\varphi\otimes\psi)=(\lambda\varphi)\otimes\psi=\varphi\otimes(\lambda\psi). \end{aligned}$$

$\boxed{P25}$

$$\begin{aligned} {}_2\langle\varphi|\mathrm{tr}_1\varrho|\varphi\rangle_2 &= {}_2\langle\varphi|\sum_j {}_1\langle j|\varrho|j\rangle_1|\varphi\rangle_2\\ &= \sum_j {}_2\langle\varphi|_1\langle j|\varrho|j\rangle_1|\varphi\rangle_2\\ &= \sum_j\langle j\varphi|\varrho|j\varphi\rangle\geq 0.\\ \mathrm{tr}(\mathrm{tr}_1\varrho) &= \sum_k {}_2\langle k|\mathrm{tr}_1\varrho|k\rangle_2\\ &= \sum_k {}_2\langle k|\sum_j {}_1\langle j|\varrho|j\rangle_1|k\rangle_2\\ &= \sum_{j,k}\langle jk|\varrho|jk\rangle\\ &= \mathrm{tr}\,\varrho = 1. \end{aligned}$$

$\boxed{P26}$

$$(\mathrm{tr}_1\varrho)^k_\ell = \langle k|\mathrm{tr}_1\varrho|\ell\rangle = \sum_{j=0}^{1}\varrho^{jk}_{j\ell} = \varrho^{0k}_{0\ell}+\varrho^{1k}_{1\ell},$$
$$(\mathrm{tr}_2\varrho)^i_j = \langle i|\mathrm{tr}_2\varrho|j\rangle = \sum_{k=0}^{1}\varrho^{ik}_{jk} = \varrho^{i0}_{j0}+\varrho^{i1}_{j1}.$$

$\boxed{P30}$

See the proof of $\boxed{H17}$ at page 19.

$\boxed{P33}$

$$|\Phi\rangle = \sum_k\sqrt{p_k}\,|\varphi_k\rangle\otimes|\psi_k\rangle = \sum_k\sqrt{p_k}\,|\varphi_k\psi_k\rangle$$
$$\Downarrow$$
$$|\Phi\rangle\langle\Phi| = \sum_{kj}\sqrt{p_kp_j}\,|\varphi_k\psi_k\rangle\langle\varphi_j\psi_j|$$
$$\Downarrow$$
$$\mathrm{tr}_2|\Phi\rangle\langle\Phi| = \sum_{kj}\sqrt{p_kp_j}\;\mathrm{tr}_2(|\varphi_k\psi_k\rangle\langle\varphi_j\psi_j|)$$
$$\Downarrow\ \boxed{\mathfrak{D}12}$$
$$\mathrm{tr}_2|\Phi\rangle\langle\Phi| = \sum_{kj}\sqrt{p_kp_j}\;\langle\psi_j|\psi_k\rangle\,|\varphi_k\rangle\langle\varphi_j|$$
$$\Downarrow$$
$$\mathrm{tr}_2|\Phi\rangle\langle\Phi| = \sum_{kj}\sqrt{p_kp_j}\;\delta_{jk}\,|\varphi_k\rangle\langle\varphi_j|$$
$$\Downarrow$$
$$\mathrm{tr}_2|\Phi\rangle\langle\Phi| = \sum_k p_k\,|\varphi_k\rangle\langle\varphi_k|.$$

$\boxed{\#12}$

Since $|u_iv_j\rangle\langle u_iv_j| = |u_i\rangle\langle u_i| \otimes |v_j\rangle\langle v_j|$, we have

$$\left.\begin{array}{l}\sum_{i=1}^{k}|u_i\rangle\langle u_i| = \mathbb{I}_{\mathcal{H}_1}\\ \sum_{j=1}^{\ell}|v_j\rangle\langle v_j| = \mathbb{I}_{\mathcal{H}_2}\end{array}\right. \Rightarrow \quad \begin{aligned}&\sum_{i=1}^{k}\sum_{j=1}^{\ell}|u_iv_j\rangle\langle u_iv_j| =\\ &\quad = \sum_{i=1}^{k}|u_i\rangle\langle u_i| \otimes \sum_{j=1}^{\ell}|v_j\rangle\langle v_j|\\ &\quad = \mathbb{I}_{\mathcal{H}_1}\otimes\mathbb{I}_{\mathcal{H}_2} = \mathbb{I}_{\mathcal{H}_1\otimes\mathcal{H}_2}.\end{aligned}$$

$\boxed{1}$

The superposition (liniar combination) of states

$$|\psi\rangle = \sum_{k=1}^{n} c_k|\psi_k\rangle$$

is a pure state with the density operator

$$\varrho = |\psi\rangle\langle\psi| = \sum_{k=1}^{n}|c_k|^2|\psi_k\rangle\langle\psi_k| + \sum_{k\neq j} c_k\,\bar{c}_j\,|\psi_k\rangle\langle\psi_j|.$$

The second sum contains the *interference terms.*

$\boxed{2}$

Example. Representations as a probabilistic ensemble of pure states of the maximally mixed state of a qubit:

$$\varrho = \begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix} \Rightarrow$$

- $\varrho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$,
- $\varrho = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -|$,
- $\varrho = \frac{1}{2}|\uparrow_y\rangle\langle\uparrow_y| + \frac{1}{2}|\downarrow_y\rangle\langle\downarrow_y|$,
- $\varrho = \frac{1}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| + \frac{1}{4}|+\rangle\langle +| + \frac{1}{4}|-\rangle\langle -|$,
- etc.

....

$\boxed{\#15}$

$\varrho^\dagger = \varrho \;\Rightarrow\;$ ϱ is diagonalizable, that is, there exists an orthonormal basis $\{|\psi_k\rangle\}$ such that

$$\varrho = \sum_{k=1}^{n} p_k|\psi_k\rangle\langle\psi_k|.$$

$\varrho \geq 0 \quad\Rightarrow\quad p_k = \langle\psi_k|\varrho|\psi_k\rangle \geq 0.$

$\mathrm{tr}\,\varrho = 1 \quad\Rightarrow\quad \sum_{k=1}^{n} p_k = \mathrm{tr}\,\varrho = 1.$

$\boxed{\#16}$

$$\mathrm{i}\,\frac{\partial|\psi\rangle}{\partial t} = H\,|\psi\rangle \quad\Rightarrow\quad -\mathrm{i}\,\frac{\partial\langle\psi|}{\partial t} = \langle\psi|H$$

$$\Rightarrow\; \mathrm{i}\,\frac{\partial}{\partial t}\varrho = \sum_{k=0}^{d-1} p_k\mathrm{i}\,\frac{\partial|\psi_k\rangle}{\partial t}\langle\psi_k| - \sum_{k=0}^{d-1} p_k|\psi_k\rangle\mathrm{i}\,\frac{\partial\langle\psi_k|}{\partial t},$$

$$\mathrm{i}\,\frac{\partial}{\partial t}\varrho = \sum_{k=0}^{d-1} p_k\,H|\psi_k\rangle\langle\psi_k| - \sum_{k=0}^{d-1} p_k|\psi_k\rangle\langle\psi_k|H,$$

$$\mathrm{i}\,\frac{\partial}{\partial t}\varrho = H\varrho - \varrho H.$$

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