LINEAR ALGEBRA - Solved Exercises 1

(Real and complex finite-dimensional vector spaces)

Nicolae Cotfas, version 09 Feb. 2021 (for future updates see https://unibuc.ro/user/nicolae.cotfas/)

A PARTIAL EVALUATION 1

A1 Find the values of $\lambda\in\mathbb{R}$ for which $\mathfrak{B}=\{(1,-1),(2,\lambda)\}$ is a basis of $\mathbb{R}^2$, and the coordinates of $v=(4,-3)$ with respect to $\mathfrak{B}$.

A2 In $\mathbb{R}^3$ consider the subspaces
$\mathcal{V}=\{(x,y,z)\in\mathbb{R}^3 \mid x-2z=0\}$ and
$$\mathcal{W}=\left\{(x,y,z)\in\mathbb{R}^3 \,\middle|\, \begin{array}{l} x-y+z=0,\\ 2x-y-z=0\end{array}\right\}.$$
a) Compute: $\dim\mathcal{V}$, $\dim(\mathcal{V}\cap\mathcal{W})$, $\dim\mathcal{W}$, $\dim(\mathcal{V}+\mathcal{W})$.
b) Is it true that $\mathbb{R}^3=\mathcal{V}\oplus\mathcal{W}$?

A3 Let $A:\mathbb{R}^3\to\mathbb{R}^3$ be the linear map defined by its action on the elements of the canonical basis:
$$Ae_1=e_1-e_2,\quad Ae_2=e_1+2e_2+3e_3,\quad Ae_3=e_1-2e_2-e_3.$$
a) Give the matrix of A in the canonical basis.
b) Compute $\dim(\mathrm{Ker}A)$ and $\dim(\mathrm{Im}A)$.
c) Decide whether $\mathbb{R}^3=\mathrm{Ker}A\oplus\mathrm{Im}A$ or not.

A4 Consider the basis $\mathfrak{B}'=\{(1,\mathrm{i}),(1,-\mathrm{i})\}$ of $\mathbb{C}^2$ and the linear operator
$$A:\mathbb{C}^2\to\mathbb{C}^2,\ A(z,w)=(z-\mathrm{i}w,\mathrm{i}z+w).$$
a) Find the change of basis matrix from the canonical basis $\mathfrak{B}$ of $\mathbb{C}^2$ to $\mathfrak{B}'$.
b) Find the matrix of A with respect to $\mathfrak{B}'$.
c) Determine A^n, for $n\in\mathbb{N}$.

Some definitions, notations and results

In a vector space $\mathcal{V}$ over $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$,

1 $\mathrm{span}\,\{v_1,v_2,...,v_n\}=\{\alpha_1v_1+...+\alpha_nv_n \mid \alpha_k\in\mathbb{K}\}$.

$\mathfrak{B}=\{v_1,v_2,...,v_n\}$ is a *basis* of $\mathcal{V}$ if
- $\mathcal{V}=\mathrm{span}\,\{v_1,v_2,...,v_n\}$ and
- $\alpha_1v_1+\alpha_2v_2+...+\alpha_nv_n=0 \ \Downarrow\ \alpha_1=\alpha_2=...=\alpha_n=0$

2 *Canonical basis* of
- $\mathbb{K}^2\equiv\{x=(x_1,x_2)\mid x_k\in\mathbb{K}\}$ is $\{e_1=(1,0),e_2=(1,0)\}$.
- $\mathbb{K}^2\equiv\left\{x=\begin{pmatrix}x_1\\x_2\end{pmatrix}\,\middle|\,x_k\in\mathbb{K}\right\}$ is $\left\{e_1=\begin{pmatrix}1\\0\end{pmatrix},e_2=\begin{pmatrix}0\\1\end{pmatrix}\right\}$.

3 *Change of basis matrix* in $\mathbb{K}^3$ from the old basis $\mathfrak{B}=\{v_1,v_2,v_3\}$ to the new basis $\mathfrak{B}'=\{v_1',v_2',v_3'\}$ is
$$\mathbf{S}=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\ \alpha_{21}&\alpha_{22}&\alpha_{23}\\ \alpha_{31}&\alpha_{32}&\alpha_{33}\end{pmatrix}\quad\text{where}\quad \begin{array}{l}v_1'=\alpha_{11}v_1+\alpha_{21}v_2+\alpha_{31}v_3\\ v_2'=\alpha_{12}v_1+\alpha_{22}v_2+\alpha_{32}v_3\\ v_3'=\alpha_{13}v_1+\alpha_{23}v_2+\alpha_{33}v_3.\end{array}$$

4 *Matrix* of $A:\mathbb{K}^3\to\mathbb{K}^3$ relative to $\mathfrak{B}=\{e_1,e_2,e_3\}$ is
$$\mathbf{A}=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\ a_{21}&a_{22}&a_{23}\\ a_{31}&a_{32}&a_{33}\end{pmatrix}\quad\text{where}\quad \begin{array}{l}Ae_1=a_{11}e_1+a_{21}e_2+a_{31}e_3\\ Ae_2=a_{12}e_1+a_{22}e_2+a_{32}e_3\\ Ae_3=a_{13}e_1+a_{23}e_2+a_{33}e_3\end{array}$$
Matrix of $A:\mathbb{K}^3\to\mathbb{K}^3$ relative to $\mathfrak{B}'$ is $\mathbf{A}'=\mathbf{S}^{-1}\mathbf{A}\mathbf{S}$.

5 *Sum of subspaces*
- $\mathcal{W}_1+\mathcal{W}_2=\{w_1+w_2 \mid w_1\in\mathcal{W}_1,\ w_2\in\mathcal{W}_2\}$.
- $\mathcal{W}_1+\mathcal{W}_2$ is a direct sum denoted by $\mathcal{W}_1\oplus\mathcal{W}_2$ $\Leftrightarrow$ $\mathcal{W}_1\cap\mathcal{W}_2=\{0\}$.
- $\boxed{\dim(\mathcal{W}_1+\mathcal{W}_2)=\dim\mathcal{W}_1+\dim\mathcal{W}_2-\dim(\mathcal{W}_1\cap\mathcal{W}_2)}$

6 *Kernel of* $A:\mathcal{V}\to\mathcal{W}$ $\ \mathrm{Ker}\,A=\{x\in\mathcal{V}\mid Ax=0\}$.
Image of $A:\mathcal{V}\to\mathcal{W}$ $\ \mathrm{Im}\,A=\{Ax \mid x\in\mathcal{V}\}$.
For $A:\mathcal{V}\to\mathcal{W}$, $\ \boxed{\dim\mathcal{V}=\dim\mathrm{Ker}\,A+\dim\mathrm{Im}\,A}$

7 *Standard inner product*
- in $\mathbb{R}^2$: $\langle(x_1,x_2),(y_1,y_2)\rangle=x_1y_1+x_2y_2$,
- in $\mathbb{C}^2$: $\langle(x_1,x_2),(y_1,y_2)\rangle=\bar{x}_1y_1+\bar{x}_2y_2$.

B PARTIAL EVALUATION 2

B1 a) Find an orthonormal basis of
$\mathcal{K}=\{x=(x_1,x_2,x_3,x_4)\in\mathbb{R}^4 \mid x_2+x_3+x_4=0\}$.
b) Describe explicitly the orthogonal projector $P:\mathbb{R}^4\to\mathbb{R}^4$ corresponding to $\mathcal{K}\subset\mathbb{R}^4$.
c) Find an orthonormal basis of
$$\mathcal{H}=\left\{A=\begin{pmatrix}x_1&x_2\\x_3&x_4\end{pmatrix}\in\mathcal{M}_{2\times2}(\mathbb{R})\,\middle|\,A^T=A\right\}.$$

B2 a) Find the eigenvalues and eigenspaces of
$A:\mathbb{R}^2\to\mathbb{R}^2$, $A(x_1,x_2)=(\pi x_1+x_2,\pi x_2)$,
$B:\mathbb{C}^2\to\mathbb{C}^2$, $B(z_1,z_2)=(\mathrm{i}z_1+z_2,z_2)$.
b) Investigate the existence of the diagonal form.

B3 Let
$$\mathbf{A}=\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}.$$
a) Find an orthogonal matrix $\mathbf{S}$ such that $\mathbf{S}^T\mathbf{A}\mathbf{S}$ is a diagonal matrix.
b) Find an explicit expression of $\mathbf{A}^n$ for $n\in\mathbb{N}$.

B4 a) Find the adjoint $U^\dagger$ of the operator
$U:\mathbb{C}^2\to\mathbb{C}^2$, $U(z_1,z_2)=\frac{1}{\sqrt{2}}(\mathrm{i}z_1+z_2,-z_1-\mathrm{i}z_2)$.
b) Prove that U is a unitary operator.
c) Diagonalize U.

B5 Let $t\in(0,1)$ and $u_1,u_2\in\mathbb{R}^3$ be two orthogonal unit vectors, $\langle u_1,u_2\rangle=0$, $||u_1||=||u_2||=1$.
a) Prove that the operator $A:\mathbb{R}^3\to\mathbb{R}^3$,
$$Ax=(1-t)\langle u_1,x\rangle u_1+t\langle u_2,x\rangle u_2.$$
is a self-adjoint operator.
b) Find the eigenvalues of A.
c) Prove that $\langle x,Ax\rangle\geq0$, for any $x\in\mathbb{R}^3$.

B6 a) Assume that A is a self-adjoint (Hermitian) operator on a Hilbert space $\mathcal{H}$. Prove that $\mathrm{Ker}\,A\perp\mathrm{Im}\,A$ and $\mathcal{H}=\mathrm{Ker}\,A\oplus\mathrm{Im}\,A$.
b) Solve the Schrödinger equation
$$\mathrm{i}\frac{d\psi}{dt}=H\psi$$
describing the time evolution of a qubit with the Hamiltonian
$$H:\mathbb{C}^2\to\mathbb{C}^2,\quad H=-\omega\sigma_2=-\omega\begin{pmatrix}0&-\mathrm{i}\\ \mathrm{i}&0\end{pmatrix},$$
wavefunction
$$\psi(t)=\psi_0(t)|0\rangle+\psi_1(t)|1\rangle\equiv\begin{pmatrix}\psi_0(t)\\ \psi_1(t)\end{pmatrix}$$
and initial state
$$\psi(0)=\psi_0(0)|0\rangle+\psi_1(0)|1\rangle\equiv\frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix}.$$
Prove that $||\psi(t)||=1$, for any $t\in\mathbb{R}$.

8 *Orthogonal projection* of x on $w\neq0$ is $P_wx=\frac{\langle w,x\rangle}{\langle w,w\rangle}w$.

9 *Gram-Schmidt orthogonalization method:*

$\boxed{\{v_1,v_2,...,v_n\}\text{ linearly independent set of vectors}}$

$\Downarrow$

- $\{w_1,w_2,...,w_n\}$, where
$w_1=v_1$,
$w_2=v_2-\frac{\langle w_1,v_2\rangle}{\langle w_1,w_1\rangle}w_1$,
....
$w_n=v_n-\frac{\langle w_1,v_n\rangle}{\langle w_1,w_1\rangle}w_1-\cdots-\frac{\langle w_{n-1},v_n\rangle}{\langle w_{n-1},w_{n-1}\rangle}w_{n-1}$,
is an orthogonal set of vectors such that
- $\mathrm{span}\,\{w_1,w_2,...,w_k\}=\mathrm{span}\,\{v_1,v_2,...,v_k\}$, for any $k\in\{1,2,...,n\}$.

10 Basis $\mathfrak{B}=\{v_1,...,v_n\}$ is *orthonormal* if $\langle v_j,v_k\rangle=\delta_{jk}$.

11 *Dirac's "bra-ket" notation* (in the case of $\mathbb{C}^2$)

$$\boxed{|x\rangle\equiv\begin{pmatrix}x_1\\x_2\end{pmatrix},\quad \langle x|\equiv(\bar{x}_1\ \ \bar{x}_2)}\quad \langle x|y\rangle=(\bar{x}_1\ \ \bar{x}_2)\begin{pmatrix}y_1\\y_2\end{pmatrix}=\bar{x}_1y_1+\bar{x}_2y_2,\quad |x\rangle\langle y|=\begin{pmatrix}x_1\\x_2\end{pmatrix}(\bar{y}_1\ \ \bar{y}_2)=\begin{pmatrix}x_1\bar{y}_1 & x_1\bar{y}_2\\x_2\bar{y}_1 & x_2\bar{y}_2\end{pmatrix}.$$

12 $||u||=1\ \Rightarrow\ P_u=|u\rangle\langle u|$.

13 $\{u_1,...,u_m\}$ orthonormal basis in a subspace $\mathcal{K}\subset\mathcal{H}$ $\Rightarrow$ Orthogonal projector on $\mathcal{K}$ is $P:\mathcal{H}\to\mathcal{H}$, $P=\sum|u_k\rangle\langle u_k|$.

14 *Eigenvalues* of $A:\mathbb{C}^2\to\mathbb{C}^2$, $\quad A\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$

are the roots λ of the equation $\begin{vmatrix}a_{11}-\lambda & a_{12}\\ a_{21} & a_{22}-\lambda\end{vmatrix}=0$.

Eigenspace corresponding to λ is $\{\,x\mid Ax=\lambda x\,\}$.

15 The *adjoint* of $A:\mathcal{H}\to\mathcal{H}$ is the operator $A^\dagger$ satisfying $\boxed{\langle x,Ay\rangle=\langle A^\dagger x,y\rangle,\ \text{for any } x,y\in\mathcal{H}.}$

16 With respect to an orthonormal basis, the matrix $\mathbf{A}^\dagger$ of the adjoint of A is $\mathbf{A}^\dagger=\bar{\mathbf{A}}^T$.

17 $A:\mathcal{H}\to\mathcal{H}$ is Hermitian (self-adjoint) $\Leftrightarrow$ $\boxed{\langle x,Ay\rangle=\langle Ax,y\rangle,\ \text{for any } x,y\in\mathcal{H}.}$ $\Leftrightarrow$ $\boxed{A^\dagger=A}$

18 $\mathbf{A}\in\mathcal{M}_{n\times n}(\mathbb{R})$ is a *symmetric* matrix if $\mathbf{A}^T=\mathbf{A}$.
$\mathbf{A}\in\mathcal{M}_{n\times n}(\mathbb{R})$ is an *orthogonal* matrix if $\mathbf{A}^T\mathbf{A}=\mathbf{I}$.
$\mathbf{A}\in\mathcal{M}_{n\times n}(\mathbb{C})$ is a *unitary* matrix if $\mathbf{A}^\dagger\mathbf{A}=\mathbf{I}$.

SOLUTIONS for **PARTIAL EVALUATION 1**

A1 $\alpha_1(1,-1)+\alpha_2(2,\lambda)=(0,0)\ \Downarrow\ \alpha_1=\alpha_2=0$ only for $\lambda\neq-2$.

$(4,-3)=\alpha_1(1,-1)+\alpha_2(2,\lambda)\ \Rightarrow\ \alpha_1=\frac{6+4\lambda}{2+\lambda},\ \alpha_2=\frac{1}{2+\lambda}$.

A2 *a)* $\mathcal{V}=\{(2z,y,z)\mid y,z\in\mathbb{R}\}$
$=\{y(0,1,0)+z(2,0,1)\mid y,z\in\mathbb{R}\}$
$\mathcal{W}=\{(2z,3z,z)\mid z\in\mathbb{R}\}=\{z(2,3,1)\mid z\in\mathbb{R}\}=\mathcal{V}\cap\mathcal{W}$.
$\{(0,1,0),(2,0,1)\}$ basis in $\mathcal{V}$ $\Rightarrow\dim\mathcal{V}=2$.
$\{(2,3,1)\}$ basis in $\mathcal{W}$ $\Rightarrow\dim\mathcal{W}=\dim(\mathcal{V}\cap\mathcal{W})=1$.
$\dim(\mathcal{V}+\mathcal{W})=\dim\mathcal{V}+\dim\mathcal{W}-\dim(\mathcal{V}\cap\mathcal{W})=2$.
b) $\dim(\mathcal{V}+\mathcal{W})=2\ \Rightarrow\ \mathbb{R}^3\neq\mathcal{V}\oplus\mathcal{W}$.

A3 *a)*
$$\begin{matrix}Ae_1=e_1-e_2+0\,e_3,\\ Ae_2=e_1+2e_2+3e_3,\\ Ae_3=e_1-2e_2-e_3.\end{matrix}\ \Rightarrow\mathbf{A}=\begin{pmatrix}1&1&1\\-1&2&-2\\0&3&-1\end{pmatrix}\Rightarrow$$
$$A\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}1&1&1\\-1&2&-2\\0&3&-1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}x_1+x_2+x_3\\-x_1+2x_2-2x_3\\3x_2-x_3\end{pmatrix},$$
$A(x_1,x_2,x_3)=(x_1+x_2+x_3,\ -x_1+2x_2-2x_3,\ 3x_2-x_3)$.

b) $$\operatorname{Ker}A=\left\{(x_1,x_2,x_3)\ \middle|\ \begin{matrix}x_1+x_2+x_3=0\\-x_1+2x_2-2x_3=0\\3x_2-x_3=0\end{matrix}\right\}$$
$=\{\alpha(-4,1,3)\mid\alpha\in\mathbb{R}\}\ \Rightarrow\ \dim\operatorname{Ker}A=1$.
$\dim\operatorname{Im}A=\dim\mathbb{R}^3-\dim\operatorname{Ker}A=3-1=2\ \Rightarrow$
$\operatorname{Im}A=\operatorname{span}\{Ae_1,Ae_2\}=\{\alpha(1,-1,0)+\beta(1,2,3)\mid\alpha,\beta\in\mathbb{R}\}$
c) $(-4,1,3)\notin\operatorname{Im}A\ \Rightarrow\ \operatorname{Ker}A\cap\operatorname{Im}A=\{(0,0,0)\}$
$\Rightarrow\ \mathbb{R}^3=\operatorname{Ker}A\oplus\operatorname{Im}A$.

A4 Change of basis matrix from $\mathfrak{B}$ to $\mathfrak{B}'$ is $\mathbf{S}=\begin{pmatrix}1&1\\\mathrm{i}&-\mathrm{i}\end{pmatrix}$.

b) Matrix of A relative to $\mathfrak{B}$ is $\mathbf{A}=\begin{pmatrix}1&-\mathrm{i}\\\mathrm{i}&1\end{pmatrix}$.

Matrix of A relative to $\mathfrak{B}'$ is $\mathbf{A}'=\mathbf{S}^{-1}\mathbf{A}\mathbf{S}=\begin{pmatrix}2&0\\0&0\end{pmatrix}$.

c)
$A^2(z,w)=A(z-\mathrm{i}w,\mathrm{i}z+w)=2(z-\mathrm{i}w,\mathrm{i}z+w)=2A(z,w)$,
$\Rightarrow A^2=2A,\ A^3=2^2A,\dots,\ A^n=AA^{n-1}=2^{n-2}A^2=2^{n-1}A$.

SOLUTIONS for **PARTIAL EVALUATION 2**

B1 *a)* $\mathcal{K}=\{(x_1,x_2,x_3,-x_2-x_3)\mid x_1,x_2,x_3\in\mathbb{R}\}$
$=\{x_1(1,0,0,0)+x_2(0,1,0,-1)+x_3(0,0,1,-1))\mid x_k\in\mathbb{R}\}$
$=\operatorname{span}\{v_1=(1,0,0,0),\ v_2=(0,1,0,-1),\ v_3=(0,0,1,-1))\}$

Gram-Schmidt $$\Rightarrow\begin{cases}w_1=v_1=(1,0,0,0),\\ w_2=v_2-\frac{\langle w_1,v_2\rangle}{\langle w_1,w_1\rangle}w_1=(0,1,0,-1),\\ w_3=v_3-\frac{\langle w_1,v_3\rangle}{\langle w_1,w_1\rangle}w_1-\frac{\langle w_2,v_3\rangle}{\langle w_2,w_2\rangle}w_2=(0,-\frac12,1,-\frac12)\end{cases}$$
is an orthogonal basis of $\mathcal{K}$ $\Rightarrow$
$\{u_1=(1,0,0,0),\ u_2=(0,\frac{1}{\sqrt2},0,-\frac{1}{\sqrt2}),\ u_3=(0,-\frac{1}{\sqrt6},\frac{2}{\sqrt6},-\frac{1}{\sqrt6})\}$
is an orthonormal basis of $\mathcal{K}$.

b) $$P=|u_1\rangle\langle u_1|+|u_2\rangle\langle u_2|+|u_3\rangle\langle u_3|=\frac13\begin{pmatrix}3&0&0&0\\0&2&-1&-1\\0&-1&2&-1\\0&-1&-1&2\end{pmatrix}.$$

c) $\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$ is an orthonormal basis of $\mathcal{H}$.

B2 *a, b)*
$\mathbf{A}=\begin{pmatrix}\pi&1\\0&\pi\end{pmatrix},\quad \begin{vmatrix}\pi-\lambda&1\\0&\pi-\lambda\end{vmatrix}=0\ \Rightarrow\ \lambda_1=\lambda_2=\pi$.
$A(x_1,x_2)=\pi(x_1,x_2)\ \Rightarrow\ (x_1,x_2)\in\{\alpha(1,0)\mid\alpha\in\mathbb{R}\}$.
A is not diagonalizable (there does not exist a basis containing only eigenvectors of A).

$\mathbf{B}=\begin{pmatrix}\mathrm{i}&1\\0&1\end{pmatrix},\quad \begin{vmatrix}\mathrm{i}-\lambda&1\\0&1-\lambda\end{vmatrix}=0\ \Rightarrow\ \lambda_1=\mathrm{i},\quad\lambda_2=1$.
$B(z_1,z_2)=\mathrm{i}(z_1,z_2)\ \Rightarrow\ (z_1,z_2)\in\{\alpha(1,0)\mid\alpha\in\mathbb{C}\}$.
$B(z_1,z_2)=1(z_1,z_2)\ \Rightarrow\ (z_1,z_2)\in\{\alpha(1,1-\mathrm{i})\mid\alpha\in\mathbb{C}\}$.
Matrix of B relative to $\{(1,0),(1,1-\mathrm{i})\}$ is diagonal.

B3 *a)* $\begin{vmatrix}-\lambda&1&0\\1&-\lambda&1\\0&1&-\lambda\end{vmatrix}=0\ \Rightarrow\ \lambda_1=0,\quad\lambda_{2,3}=\pm\sqrt2$.

$Ax=0\ \Rightarrow\ x=\frac{\alpha}{\sqrt2}\begin{pmatrix}1\\0\\-1\end{pmatrix},\quad Ax=\pm\sqrt2x\ \Rightarrow\ x=\frac{\alpha}{2}\begin{pmatrix}1\\\pm\sqrt2\\1\end{pmatrix}$.

$\mathbf{S}=\frac12\begin{pmatrix}\sqrt2&1&1\\0&\sqrt2&-\sqrt2\\-\sqrt2&1&1\end{pmatrix}$ satisfies $\mathbf{S}^T\mathbf{A}\mathbf{S}=\begin{pmatrix}0&0&0\\0&\sqrt2&0\\0&0&-\sqrt2\end{pmatrix}$
and is orthogonal, that is, $\mathbf{S}^T\mathbf{S}=\mathbf{I}$.

b)
$$\mathbf{A}^n=\mathbf{S}\begin{pmatrix}0&0&0\\0&\sqrt2&0\\0&0&-\sqrt2\end{pmatrix}^n\mathbf{S}^T=\mathbf{S}\begin{pmatrix}0&0&0\\0&(\sqrt2)^n&0\\0&0&(-\sqrt2)^n\end{pmatrix}\mathbf{S}^T=\dots$$

B4 $\mathbf{U}=\frac{1}{\sqrt2}\begin{pmatrix}\mathrm{i}&1\\-1&-\mathrm{i}\end{pmatrix}\ \Rightarrow\ \mathbf{U}^\dagger=\frac{1}{\sqrt2}\begin{pmatrix}-\mathrm{i}&-1\\1&\mathrm{i}\end{pmatrix}\ \Rightarrow\ \mathbf{U}\mathbf{U}^\dagger=\mathbf{I}$.

c)
$\begin{vmatrix}\frac{\mathrm{i}}{\sqrt2}-\lambda&\frac{1}{\sqrt2}\\\frac{-1}{\sqrt2}&\frac{-\mathrm{i}}{\sqrt2}-\lambda\end{vmatrix}=0\ \Rightarrow\ \lambda_{1,2}=\pm\mathrm{i}$. Matrix of U relative to $\left\{\frac{(1,(\sqrt2-1)\mathrm{i})}{\sqrt{4-2\sqrt2}},\frac{(1,-(\sqrt2+1)\mathrm{i})}{\sqrt{4+2\sqrt2}}\right\}$ is a diagonal matrix.

B5 *a)* We extend $\{u_1,u_2\}$ up to an orthonormal basis $\mathfrak{B}=\{u_1,u_2,u_3\}$ of $\mathbb{R}^3$. Matrix of A relative to $\mathfrak{B}$ is

$\mathbf{A}=\begin{pmatrix}1-t&0&0\\0&t&0\\0&0&0\end{pmatrix}=\mathbf{A}^\dagger$. *b)* Eigenvalues of A are $1-t$, t and 0.

c) $\langle x,Ax\rangle=(1-t)|\langle u_1,x\rangle|^2+t|\langle u_2,x\rangle|^2\geq0$ for any $x\in\mathbb{R}^3$.

B6 $\begin{matrix}x\in\operatorname{Ker}A,\\Ay\in\operatorname{Im}A\end{matrix}\ \Rightarrow\langle x,Ay\rangle=\langle Ax,y\rangle=0\Rightarrow\operatorname{Ker}A\perp\operatorname{Im}A\ \Rightarrow$
$\operatorname{Ker}A\cap\operatorname{Im}A=\{0\}$. But, $\dim\operatorname{Ker}A+\dim\operatorname{Im}A=\dim\mathcal{H}$.

$$\mathrm{i}\frac{d\psi}{dt}=H\psi\ \Rightarrow\ \mathrm{i}\begin{pmatrix}\psi_0'\\\psi_1'\end{pmatrix}=-\omega\begin{pmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{pmatrix}\begin{pmatrix}\psi_0\\\psi_1\end{pmatrix}\Rightarrow\begin{cases}\psi_0'=\omega\psi_1\\\psi_1'=-\omega\psi_0\end{cases}$$
$$\Rightarrow\begin{cases}\psi_0''+\omega^2\psi_0=0\\\psi_0'=\omega\psi_1\end{cases}\ \Rightarrow\ \begin{pmatrix}\psi_0(t)\\\psi_1(t)\end{pmatrix}=\begin{pmatrix}C_1\sin\omega t+C_2\cos\omega t\\C_1\cos\omega t-C_2\sin\omega t\end{pmatrix}$$
$$\psi(0)=\frac{1}{\sqrt5}\begin{pmatrix}1\\2\end{pmatrix}\Rightarrow\psi(t)=\begin{pmatrix}\psi_0(t)\\\psi_1(t)\end{pmatrix}=\begin{pmatrix}\frac{2}{\sqrt5}\sin\omega t+\frac{1}{\sqrt5}\cos\omega t\\\frac{2}{\sqrt5}\cos\omega t-\frac{1}{\sqrt5}\sin\omega t\end{pmatrix}.$$
$$||\psi(t)||^2=\left(\frac{2}{\sqrt5}\sin\omega t+\frac{1}{\sqrt5}\cos\omega t\right)^2+\left(\frac{2}{\sqrt5}\cos\omega t-\frac{1}{\sqrt5}\sin\omega t\right)^2=1.$$