An overview on LINEAR DIFFERENTIAL EQUATIONS

Nicolae Cotfas, version 04 Feb 2020 (for future updates see https://unibuc.ro/user/nicolae.cotfas/)

FIRST-ORDER DIFFERENTIAL EQUATIONS

- A *solution* of $\boxed{F(x,y,y')=0}$ is any function $\varphi:(a,b)\to\mathbb{R}$ satisfying $F(x,\varphi(x),\varphi'(x))=0$, for any $x\in(a,b)$.
- A *solution* of $\boxed{y'=f(x,y)}$ is any function $\varphi:(a,b)\to\mathbb{R}$ satisfying $\varphi'(x)=f(x,\varphi(x))$, for any $x\in(a,b)$.
- The *primitives* of a continuous function $f:(a,b)\to\mathbb{R}$ *are* all the functions $F:(a,b)\to\mathbb{R}$, $\boxed{F(x)=\int_{x_0}^{x} f(t)\,dt}$, $x_0\in(a,b)$ is fixed. We have $F'(x)=f(x)$, that is $\boxed{\frac{d}{dx}\left(\int_{x_0}^{x} f(t)\,dt\right)=f(x)}$ for any $x\in(a,b)$.

Separable equations. The solution $y(x)$ of $\boxed{y'=f(x)\,g(y)}$ is defined by $\boxed{\int_{y_0}^{y}\frac{1}{g(u)}du=\int_{x_0}^{x} f(t)\,dt+C}$, x_0, y_0 constants, $g(y_0)\neq 0$.

Proof. $\int_{y_0}^{y}\frac{1}{g(u)}du=\int_{x_0}^{x}f(t)\,dt+C \overset{\frac{d}{dx}}{\Longrightarrow} \frac{d}{dy}\left(\int_{y_0}^{y}\frac{1}{g(u)}du\right)\frac{dy}{dx}=f(x) \Rightarrow \frac{1}{g(y)}y'=f(x)$.

Homogeneous equations. $\boxed{y'=f(\frac{y}{x})}$ $\xrightarrow[y(x)\,=\,x\,z(x)]{\text{change of function}}$ separable equation.

Proof. $y(x)=x\,z(x) \overset{\frac{d}{dx}}{\Longrightarrow} y'(x)=z(x)+x\,z'(x) \Rightarrow z'=\frac{1}{x}(f(z)-z)$.

Linear equation. $\boxed{y'=f(x)y}$ has the general solution $\boxed{y(x)=C\,e^{\int_{x_0}^{x} f(t)dt}}$

Proof. $\frac{y'}{y}=f(x) \mapsto (\ln y)'=f(x) \mapsto \ln y(x)=\int_{x_0}^{x} f(t)\,dt+\ln C$.

Method of the variation of parameter. $\boxed{y'=f(x)y+g(x)}$ admits a particular solution of the form $y_p(x)=C(x)\,e^{\int_{x_0}^{x} f(t)dt}$.

Proof. $y_p'=f(x)y_p+g(x) \Rightarrow C'(x)=g(x)\,e^{-\int_{x_0}^{x} f(t)dt} \Rightarrow C(x)=\ldots$

Linear non-homogeneous equation.

$\boxed{\text{general solution of } y'=f(x)y+g(x)}=\boxed{\text{general solution of } y'=f(x)y}+\boxed{\text{a particular solution of } y'=f(x)y+g(x)}$

Proof. $\left.\begin{array}{l} y'=f(x)y \\ y_p'=f(x)y_p+g(x)\end{array}\right\} \Rightarrow (y+y_p)'=f(x)(y+y_p)+g(x)$.

Bernoulli's equation. $\boxed{y'=f(x)y+g(x)y^{\alpha}}$ $\xrightarrow[z(x)=(y(x))^{1-\alpha}]{\text{change of function}}$ linear equation.

Proof. $y^{-\alpha}y'=f(x)y^{1-\alpha}+g(x) \Rightarrow \frac{1}{1-\alpha}(y^{1-\alpha})'=f(x)y^{1-\alpha}+g(x)$.

Riccati's equation. If we know a particular solution y_p, then $\boxed{y'=f(x)y^2+g(x)y+h(x)}$ $\xrightarrow[y(x)=y_p(x)+\frac{1}{z(x)}]{\text{change of function}}$ linear equation.

Proof. $y'=f(x)y^2+g(x)y+h(x) \Rightarrow z'=-(2f(x)\,y_p(x)-g(x))z-f(x)$.

- $\boxed{y'=f(x)}$ can also be written as $\boxed{\frac{dy}{dx}=f(x)}$ or $\boxed{f(x)dx-dy=0.}$

Symmetric equations.

Definition: $\begin{array}{l}x=\varphi(t)\\ y=\psi(t)\end{array}$ is a solution of $\boxed{P(x,y)dx+Q(x,y)dy=0}$ if $P(\varphi(t),\psi(t))\varphi'(t)+Q(\varphi(t),\psi(t))\psi'(t)=0$, for any t.

Exact equations. In a suitable domain of definition,

$\boxed{\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}} \Rightarrow \boxed{\begin{array}{c}F(x,y)=\int_{(x_0,y_0)}^{(x,y)} Pdx+Qdy \text{ satisfies}\\ P(x,y)dx+Q(x,y)dy=dF\end{array}}$ Solution of $\boxed{dF=0}$ is $F(x,y)=const.$

Method of integrating factor $\mu(x,y)$.

For $\boxed{P(x,y)dx+Q(x,y)dy=0}$ we look for $\mu(x,y)$ such that $\boxed{\frac{\partial(\mu Q)}{\partial x}=\frac{\partial(\mu P)}{\partial y}}$

Method of power series. We look for solutions having the form of a convergent power series $\boxed{y(x)=\sum_{n=0}^{\infty} a_n\,x^n}$

LINEAR DIFFERENTIAL EQUATIONS

- For any $a_0\neq 0, a_1,\ldots,a_n\in\mathbb{R}$, by denoting $D^k=\frac{d^k}{dx^k}$ we get

$$\boxed{\begin{array}{l}Ly=a_0y^{(n)}+\cdots+a_{n-1}y'+a_ny\\ \quad=(a_0D^n+\cdots+a_{n-1}D+a_n)y=P(D)y\end{array}}$$

$P(\lambda)=a_0\lambda^n+\cdots+a_n$ is the characteristic polynomial

Linear equation with constant coefficients

Theorem: Solutions $y:\mathbb{R}\to\mathbb{R}$ of $\boxed{Ly=0}$ form a vector space of dimension n.

General solution of $Ly=0$.

$y=c_1\,y_1+\ldots+c_n\,y_n$ is the *general solution* $\Leftrightarrow$ $\{y_1,\ldots,y_n\}$ is a basis in the vector space of solutions

Particular solutions of $Ly=0$.

$\boxed{y(x)=e^{\lambda x} \text{ is a solution of } Ly=0} \Leftrightarrow \boxed{P(\lambda)=0}$

- In the case $\lambda=\alpha+\beta i$, $\boxed{e^{(\alpha+\beta i)x}\overset{\text{def}}{=}e^{\alpha x}\cos\beta x+i\,e^{\alpha x}\sin\beta x}$
- $P(\alpha+\beta i)=0 \Rightarrow \left\{\begin{array}{l}y_1(x)=e^{\alpha x}\cos\beta x\\ y_2(x)=e^{\alpha x}\sin\beta x\end{array}\right.$ linearly independent solutions of $Ly=0$.
- λ root of P of multiplicity k $\Rightarrow \left\{\begin{array}{l}y_1(x)=e^{\lambda x}\\ y_2(x)=x\,e^{\lambda x}\\ \ldots\ldots\ldots\ldots\\ y_k(x)=x^{k-1}\,e^{\lambda x}\end{array}\right.$ are linearly independent solutions of $Ly=0$.

Method of the variation of parameters.

$\boxed{\begin{array}{c}y=c_1\,y_1+\ldots+c_n\,y_n\\ \text{general solution}\\ \text{of } Ly=0\end{array}} \Rightarrow \boxed{\begin{array}{c}Ly=f \text{ admits a particular}\\ \text{solution of the form}\\ y(x)=c_1(x)\,y_1(x)+\ldots+c_n(x)\,y_n(x)\end{array}}$

The functions $c_1(x),\ldots,c_n(x)$ can be obtained by solving the system

$$\left\{\begin{array}{l}c_1'(x)\,y_1(x)+\ldots+c_n'(x)\,y_n(x)=0\\ c_1'(x)\,y_1'(x)+\ldots+c_n'(x)\,y_n'(x)=0\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ c_1'(x)\,y_1^{(n-2)}(x)+\ldots+c_n'(x)\,y_n^{(n-2)}(x)=0\\ c_1'(x)\,y_1^{(n-1)}(x)+\ldots+c_n'(x)\,y_n^{(n-1)}(x)=\frac{f(x)}{a_0}.\end{array}\right.$$

Linear non-homogeneous equation.

$\boxed{\text{general solution of } Ly=f}=\boxed{\text{general solution of } Ly=0}+\boxed{\text{a particular solution of } Ly=f.}$

Euler's equation.

$\boxed{a_0x^ny^{(n)}+\ldots+a_{n-1}xy'+a_ny=0}$ $\xrightarrow[x=e^t]{\text{change}}$ linear equation with constant coefficients

SYSTEMS OF LINEAR EQUATIONS

Systems of linear equations with constant coefficients.

$$\boxed{Y'=AY+F} \quad Y=\begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix},\quad F=\begin{pmatrix}f_1\\ \vdots\\ f_n\end{pmatrix},\quad A=\begin{pmatrix}a_{11}&\ldots&a_{1n}\\ \vdots&&\vdots\\ a_{n1}&\ldots&a_{nn}\end{pmatrix}.$$

- *Theorem:* Solutions $Y:\mathbb{R}\to\mathbb{R}^n$ of $\boxed{Y'=AY}$ form a vector space of dimension n.

Wronski matrix

$Y_1=\begin{pmatrix}y_{11}\\ \vdots\\ y_{n1}\end{pmatrix},\ldots,Y_n=\begin{pmatrix}y_{1n}\\ \vdots\\ y_{nn}\end{pmatrix}$ form a basis $\Leftrightarrow W=\begin{pmatrix}y_{11}&\ldots&y_{1n}\\ \vdots&&\vdots\\ y_{n1}&\ldots&y_{nn}\end{pmatrix}$ has $\det W\neq 0$

Particular solutions of $Y'=AY$.

$Y=we^{\lambda x}=\begin{pmatrix}w_1\\ \vdots\\ w_n\end{pmatrix}e^{\lambda x}$ is a solution of $Y'=AY$ $\Leftrightarrow$ $\boxed{A\begin{pmatrix}w_1\\ \vdots\\ w_n\end{pmatrix}=\lambda\begin{pmatrix}w_1\\ \vdots\\ w_n\end{pmatrix}}$

- $\lambda=\alpha+\beta i \Rightarrow Y_1(x)=\mathfrak{Re}(we^{\lambda x}),\ Y_2(x)=\mathfrak{Im}(we^{\lambda x})$ solutions.

General solution of $Y'=AY$ can be written as

$$Y=c_1Y_1+\cdots+c_nY_n=W\cdot C,\quad \text{where}\quad C=\begin{pmatrix}c_1\\ \vdots\\ c_n\end{pmatrix}.$$

Method of the variation of parameters.

$\boxed{\begin{array}{c}Y=W\cdot C \text{ general}\\ \text{solution of } Y'=AY\end{array}} \Rightarrow \boxed{\begin{array}{c}Y'=AY+F \text{ admits a}\\ \text{solution } Y(x)=W\cdot C(x)\end{array}}$ with $C'=W^{-1}F$