DIFFERENTIAL EQUATIONS - Exercises

Nicolae Cotfas, version 02 Mar 2023 (for future updates see https://unibuc.ro/user/nicolae.cotfas/)

A Exercises

1 $\boxed{y''-5y'+6y=0}$

Solution : $\lambda^2-5\lambda+6=0 \Rightarrow \begin{array}{l}\lambda_1=2\\ \lambda_2=3\end{array} \Rightarrow y(x)=C_1\,e^{2x}+C_2\,e^{3x}$.

2 $\boxed{y''-4y'+4y=0}$

Solution : $\lambda^2-4\lambda+4=0 \Rightarrow \lambda_1=\lambda_2=2 \Rightarrow y(x)=C_1\,e^{2x}+C_2\,x\,e^{2x}$.

3 $\boxed{y''-y'+y=0}$

Solution : $\lambda^2-\lambda+1=0 \Rightarrow \lambda_{1,2}=\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

$\Rightarrow y(x)=C_1\,e^{\frac{1}{2}x}\cos\frac{\sqrt{3}}{2}x+C_2\,e^{\frac{1}{2}x}sin\,\frac{\sqrt{3}}{2}x$.

4 $\boxed{y'''-y=0}$ *Solution* : $\lambda^3-1=0 \Rightarrow \lambda_1=1,\quad \lambda_{2,3}=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

$y(x)=C_1\,e^{x}+C_2\,e^{-\frac{1}{2}x}\cos\frac{\sqrt{3}}{2}x+C_3\,e^{-\frac{1}{2}x}\sin\frac{\sqrt{3}}{2}x$,

5 $\boxed{y'=x^3\,y^2}$ *Solution* : $y(x)=0$ is a solution of the equation.

For $y\neq 0$: $\frac{y'}{y^2}=x^3 \Rightarrow \left(-\frac{1}{y}\right)'=x^3 \Rightarrow \frac{1}{y}=-\frac{x^4}{4}+C \Rightarrow y(x)=\frac{4}{4C-x^4}$.

6 $\boxed{y'=-\frac{1}{x}y-1}$ *Solution* : $y'=-\frac{1}{x}y \Rightarrow y(x)=\frac{C}{x}$. We use the method of the variation of parameter.

$y_p(x)=\frac{C(x)}{x}$ satisfies $y'=-\frac{1}{x}y-1$ if $C'(x)=-x$.

The general solution of $y'=-\frac{1}{x}y-1$ is $y(x)=\frac{C}{x}+y_p(x)=\frac{C}{x}-\frac{x}{2}$.

7 $\boxed{\begin{cases} y_1'=y_2\\ y_2'=y_3\\ y_3'=y_1\end{cases}}$ *Solution* : After two substitutions, we get $y_1'''-y_1=0$ with the solution

$y_1(x)=C_1\,e^{x}+C_2\,e^{-\frac{1}{2}x}\cos\frac{\sqrt{3}}{2}x+C_3\,e^{-\frac{1}{2}x}\sin\frac{\sqrt{3}}{2}x$,

and $y_2(x)=y_1'(x),\quad y_3(x)=y_1''(x)$.

B Exercises

1 $\boxed{y'=\frac{y}{x}+e^{\frac{y}{x}}}$ *Solution*: After the change of function $z(x)=\frac{y(x)}{x}$ the equation becomes $e^{-z}z'=\frac{1}{x}$ with solution $z(x)=-\ln(-\ln(Cx))$. Solution of $y'=\frac{y}{x}+e^{\frac{y}{x}}$ is $y(x)=-x\ln(-\ln(Cx))$.

2 $\boxed{\begin{cases} y_1'=y_2+y_3\\ y_2'=y_1+y_3\\ y_3'=y_1+y_2\end{cases}}$ *Solution* :

$$\begin{vmatrix}-\lambda & 1 & 1\\ 1 & -\lambda & 1\\ 1 & 1 & -\lambda\end{vmatrix}=0 \Rightarrow \begin{array}{l}\lambda_1=2\\ \lambda_2=\lambda_3=-1\end{array}$$

By choosing linearly independent eigenvectors, we get

$$\begin{pmatrix}y_1(x)\\ y_2(x)\\ y_3(x)\end{pmatrix}=C_1\begin{pmatrix}1\\1\\1\end{pmatrix}e^{2x}+C_2\begin{pmatrix}1\\0\\-1\end{pmatrix}e^{-x}+C_3\begin{pmatrix}0\\1\\-1\end{pmatrix}e^{-x}.$$

3 $\boxed{y''-4y'+4y=x^2e^{2x}}$ (method of the variation of parameters).

Solution : $y''-4y'+4y=0 \Rightarrow y(x)=C_1\,e^{2x}+C_2\,xe^{2x}$.

$y_p(x)=C_1(x)e^{2x}+C_2(x)xe^{2x}$ satisfies $y''-4y'+4y=x^2e^{2x}$ if $\begin{cases}C_1'(x)\,e^{2x}+C_2'(x)\,xe^{2x}=0\\ C_1'(x)(e^{2x})'+C_2'(x)(xe^{2x})'=x^2e^{2x}\end{cases}$

General solution of $y''-4y'+4y=x^2e^{2x}$ is $y(x)=C_1\,e^{2x}+C_2\,xe^{2x}+y_p(x)=(C_1+C_2x+\frac{x^4}{12})e^{2x}$.

C Exercises

1 $\boxed{(x+y)dx+xdy=0}$ *Solution 1* : $\frac{\partial(x+y)}{\partial y}=\frac{\partial x}{\partial x} \Rightarrow$ eq. is exact.

By integrating $(x+y)dx+xdy$ along the union of paths $\gamma_1:[0,1]\to\mathbb{R}^2,\ \gamma_1(t)=(tx,0)$, $\gamma_2:[0,1]\to\mathbb{R}^2,\ \gamma_2(t)=(x,ty)$,

we get $F(x,y)=\int_{(0,0)}^{(x,y)}(x+y)dx+xdy=\frac{x^2}{2}+xy$. The equation can be written as $d(\frac{x^2}{2}+xy)=0 \Rightarrow$ solution is defined by $\frac{x^2}{2}+xy=C$.

Solution 2 : Written as $y'=-\frac{1}{x}y-1$, the equation coincides to A6.

2 $\boxed{(1+\frac{y}{x})dx+dy=0}$ *Solution* : The equation can be reduced to the previous one by using the integrating factor $\mu(x,y)=x$.

3 $\boxed{\frac{dy}{dx}=\frac{2y^4}{x(x^2+2y^3)}}$ *Hint* : By changing the role of variables, we get the Bernoulli's equation $\frac{dx}{dy}=\frac{1}{y}x+\frac{1}{2y^4}x^3$.

4 $\boxed{y'=\frac{y^2}{x^3}+\frac{y}{x}-\frac{1}{x}}$ *Hint* : This Riccati equation admits the solution $y(x)=x$. The change of function $y(x)=x+\frac{1}{z(x)}$ reduces it to $z'=-(\frac{1}{x}+\frac{2}{x^2})z-\frac{1}{x^3}$.

5 $\boxed{\begin{cases} y_1'=y_1+y_2\\ y_2'=-y_1+y_2\end{cases}}$ that is $\boxed{\frac{d}{dx}\begin{pmatrix}y_1\\y_2\end{pmatrix}=\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}}$

Solution : $\begin{vmatrix}1-\lambda & 1\\ -1 & 1-\lambda\end{vmatrix}=0 \Rightarrow \lambda_{1,2}=1\pm i$.

A solution of $\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}w_1\\w_2\end{pmatrix}=(1+i)\begin{pmatrix}w_1\\w_2\end{pmatrix}$ is $\begin{pmatrix}w_1\\w_2\end{pmatrix}=\begin{pmatrix}1\\i\end{pmatrix}$.

General solution of the system

$$\begin{pmatrix}y_1(x)\\y_2(x)\end{pmatrix}=C_1\,\mathfrak{Re}\left\{\begin{pmatrix}1\\i\end{pmatrix}e^{(1+i)x}\right\}+C_2\,\mathfrak{Im}\left\{\begin{pmatrix}1\\i\end{pmatrix}e^{(1+i)x}\right\}=C_1\begin{pmatrix}\cos x\\ -\sin x\end{pmatrix}e^{x}+C_2\begin{pmatrix}\sin x\\ \cos x\end{pmatrix}e^{x}.$$

6 $\boxed{\begin{cases} y_1'=-y_1+y_2\\ y_2'=-y_2+4y_3\\ y_3'=\ y_1-4y_3\end{cases}}$ *Solution* : The eigenvalues are

$\lambda_1=0$ with eigenspace $\{\alpha(4,4,1)\mid\alpha\in\mathbb{R}\}$

$\lambda_{2,3}=-3$ with eigenspace $\{\alpha(1,-2,1)\mid\alpha\in\mathbb{R}\}$

For $\lambda_{2,3}=-3$, by looking for solutions of the form $\begin{pmatrix}\alpha_1x+\beta_1\\ \alpha_2x+\beta_2\\ \alpha_3x+\beta_3\end{pmatrix}e^{-3x}$,

we get $$\begin{pmatrix}y_1(x)\\y_2(x)\\y_3(x)\end{pmatrix}=C_1\begin{pmatrix}4\\4\\1\end{pmatrix}+C_2\begin{pmatrix}1\\-2\\1\end{pmatrix}e^{-3x}+C_3\begin{pmatrix}x\\-2x+1\\x-1\end{pmatrix}e^{-3x}.$$

7 $\boxed{\begin{cases} y_1'=-y_1-2y_2+\cos x+\sin x+e^{-x}\\ y_2'=2y_1-y_2-\cos x+\sin x\end{cases}}$ $\qquad Y'=AY+F$

Solution :

$Y'=AY \Rightarrow Y(x)=C_1\begin{pmatrix}\cos 2x\\ -\sin 2x\end{pmatrix}e^{-x}+C_2\begin{pmatrix}\sin 2x\\ \cos 2x\end{pmatrix}e^{-x}=W(x)\,C$,

where $W(x)=\begin{pmatrix}e^{-x}\cos 2x & e^{-x}\sin 2x\\ -e^{-x}\sin 2x & e^{-x}\cos 2x\end{pmatrix}$, $\quad C=\begin{pmatrix}C_1\\C_2\end{pmatrix}$.

$Y_p(x)==W(x)\,C(x)$ satisfies $Y'=AY+F$ if $C'(x)=W(x)^{-1}F$.

General solution of $Y'=AY+F$ is $Y(x)=W(x)\,C+Y_p(x)$

$$=C_1\begin{pmatrix}\cos 2x\\ -\sin 2x\end{pmatrix}e^{-x}+C_2\begin{pmatrix}\sin 2x\\ \cos 2x\end{pmatrix}e^{-x}+\frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}e^{-x}-\frac{1}{3}\begin{pmatrix}\cos x-\sin x\\ \cos x+\sin x\end{pmatrix}.$$

- *a* $\boxed{y''-3y'+2y=0}$
- *b* $\boxed{y''+2y'+y=0}$
- *c* $\boxed{y''-y'+2y=0}$
- *d* $\boxed{y'=\frac{x^2+y^2}{xy}}$
- *e* $\boxed{y'+2xy=x^3}$
- *f* $\boxed{y'=\frac{x^2-y^2}{xy}}$
- *g* $\boxed{y'''-y''-y'+y=0}$
- *h* $\boxed{y''=x+\sin x}$
- *k* $\boxed{y'-\frac{2}{x}y=x^3}$
- *l* $\boxed{y'=\frac{2x-2y-1}{x-y+1}}$
- *m* $\boxed{1+y^2+xyy'=0}$
- *n* $\boxed{(1+e^x)yy'=e^x}$
- *o* $\boxed{2xydx+dy=0}$
- *p* $\boxed{y'''+y''=x}$
- *q* $\boxed{xy''=y'}$
- *r* $\boxed{ydx+(x+y)dy=0}$
- *s* $\boxed{\begin{cases} y_1'=y_1+4y_2\\ y_2'=y_1+y_2\end{cases}}$
- *t* $\boxed{y'''-8y=x^2}$
- *u* $\boxed{y''+y=xe^{-x}}$
- *v* $\boxed{y'+y^2=1+x^2}$
- *w* $\boxed{(x-y)ydx-x^2dy=0}$
- *x* $\boxed{x^2y''-2xy'+2y=0}$
- *y* $\boxed{x^2y''+4xy'+2y=\cos x}$
- *z* $\boxed{\begin{cases} y_1'=y_2\\ y_2'=-4y_1+4y_2\\ y_3'=-2y_1+y_2+2y_3\end{cases}}$
- $ $\boxed{\begin{cases} y_1'=y_1-3y_2+3y_3\\ y_2'=-2y_1-6y_2+13y_3\\ y_3'=-y_1-4y_2+8y_3\end{cases}}$
- # $\boxed{\begin{cases} y_1'=-y_2+x^2+6x+1\\ y_2'=y_1-3x^2+3x+1\end{cases}}$