GAUSSIAN FUNCTIONS OF TWO DISCRETE VARIABLES

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Definition

For $a,b,c\in\mathbb{R}$ with $\sigma=\left(\begin{array}{cc} a & b\\ b & c\end{array}\right)>0$, beside the standard Gaussian function of two discrete variables

$$\boxed{\mathbf{g}_\sigma(n_1,n_2)=\sum_{\alpha_1,\alpha_2=-\infty}^{\infty}\mathrm{e}^{-\frac{\pi}{d}(n_1+\alpha_1 d\ \ n_2+\alpha_2 d)\,\sigma\left(\begin{array}{c} n_1+\alpha_1 d\\ n_2+\alpha_2 d\end{array}\right)}=\sum_{\alpha_1,\alpha_2=-\infty}^{\infty}\mathrm{e}^{-\frac{a\pi}{d}(n_1+\alpha_1 d)^2}\,\mathrm{e}^{-\frac{2b\pi}{d}(n_1+\alpha_1 d)(n_2+\alpha_2 d)}\,\mathrm{e}^{-\frac{c\pi}{d}(n_2+\alpha_2 d)^2}}$$

we define other three complementary Gaussian functions of two discrete variables

$$\mathbf{g}_\sigma^{+0}(n_1,n_2)=\sum_{\alpha_1,\alpha_2=-\infty}^{\infty}\mathrm{e}^{-\frac{\pi}{d}(n_1+(\alpha_1+\frac{1}{2})d\ \ n_2+\alpha_2 d)\,\sigma\left(\begin{array}{c} n_1+(\alpha_1+\frac{1}{2})d\\ n_2+\alpha_2 d\end{array}\right)}=\sum_{\alpha_1,\alpha_2=-\infty}^{\infty}\mathrm{e}^{-\frac{a\pi}{d}(n_1+(\alpha_1+\frac{1}{2})d)^2}\,\mathrm{e}^{-\frac{2b\pi}{d}(n_1+(\alpha_1+\frac{1}{2})d)(n_2+\alpha_2 d)}\,\mathrm{e}^{-\frac{c\pi}{d}(n_2+\alpha_2 d)^2}.$$

$$\mathbf{g}_\sigma^{0+}(n_1,n_2)=\sum_{\alpha_1,\alpha_2=-\infty}^{\infty}\mathrm{e}^{-\frac{\pi}{d}(n_1+\alpha_1 d\ \ n_2+(\alpha_2+\frac{1}{2})d)\,\sigma\left(\begin{array}{c} n_1+\alpha_1 d\\ n_2+(\alpha_2+\frac{1}{2})d\end{array}\right)}=\sum_{\alpha_1,\alpha_2=-\infty}^{\infty}\mathrm{e}^{-\frac{a\pi}{d}(n_1+\alpha_1 d)^2}\,\mathrm{e}^{-\frac{2b\pi}{d}(n_1+\alpha_1 d)(n_2+(\alpha_2+\frac{1}{2})d)}\,\mathrm{e}^{-\frac{c\pi}{d}(n_2+(\alpha_2+\frac{1}{2})d)^2}.$$

$$\mathbf{g}_\sigma^{++}(n_1,n_2)=\sum_{\alpha_1,\alpha_2=-\infty}^{\infty}\mathrm{e}^{-\frac{\pi}{d}(n_1+(\alpha_1+\frac{1}{2})d\ \ n_2+(\alpha_2+\frac{1}{2})d)\,\sigma\left(\begin{array}{c} n_1+(\alpha_1+\frac{1}{2})d\\ n_2+(\alpha_2+\frac{1}{2})d\end{array}\right)}=\sum_{\alpha_1,\alpha_2=-\infty}^{\infty}\mathrm{e}^{-\frac{a\pi}{d}(n_1+(\alpha_1+\frac{1}{2})d)^2}\,\mathrm{e}^{-\frac{2b\pi}{d}(n_1+(\alpha_1+\frac{1}{2})d)(n_2+(\alpha_2+\frac{1}{2})d)}\,\mathrm{e}^{-\frac{c\pi}{d}(n_2+(\alpha_2+\frac{1}{2})d)^2}.$$

Fourier transform

LEMMA 1.

$$\bullet 1.\quad \mathbf{F}[\mathbf{g}_\sigma](k_1,k_2)=\frac{1}{\sqrt{\det\sigma}}\sum_{\beta_1,\beta_2=-\infty}^{\infty} g_{\sigma^{-1}}\left((k_1+\beta_1 d)\sqrt{\tfrac{2\pi}{d}},(k_2+\beta_2 d)\sqrt{\tfrac{2\pi}{d}}\right).$$

$$\bullet 2.\quad \mathbf{F}[\mathbf{g}_\sigma^{+0}](k_1,k_2)=\frac{(-1)^{k_1}}{\sqrt{\det\sigma}}\sum_{\beta_1,\beta_2=-\infty}^{\infty}(-1)^{\beta_1}\ g_{\sigma^{-1}}\left((k_1+\beta_1 d)\sqrt{\tfrac{2\pi}{d}},(k_2+\beta_2 d)\sqrt{\tfrac{2\pi}{d}}\right).$$

$$\bullet 3.\quad \mathbf{F}[\mathbf{g}_\sigma^{0+}](k_1,k_2)=\frac{(-1)^{k_2}}{\sqrt{\det\sigma}}\sum_{\beta_1,\beta_2=-\infty}^{\infty}(-1)^{\beta_2}\ g_{\sigma^{-1}}\left((k_1+\beta_1 d)\sqrt{\tfrac{2\pi}{d}},(k_2+\beta_2 d)\sqrt{\tfrac{2\pi}{d}}\right).$$

$$\bullet 4.\quad \mathbf{F}[\mathbf{g}_\sigma^{++}](k_1,k_2)=\frac{(-1)^{k_1+k_2}}{\sqrt{\det\sigma}}\sum_{\beta_1,\beta_2=-\infty}^{\infty}(-1)^{\beta_1+\beta_2}\ g_{\sigma^{-1}}\left((k_1+\beta_1 d)\sqrt{\tfrac{2\pi}{d}},(k_2+\beta_2 d)\sqrt{\tfrac{2\pi}{d}}\right).$$

Proof.

•1 The periodic function of two continuous variables

$$\mathcal{G}_\sigma(q_1,q_2)=\sum_{\alpha_1,\alpha_2=-\infty}^{\infty}\mathrm{e}^{-\frac{\pi}{d}[a(q_1+\alpha_1 d)^2+2b(q_1+\alpha_1 d)(q_2+\alpha_2 d)+c(q_2+\alpha_2 d)^2]}$$

can be expended into a Fourier series

$$\mathcal{G}_\sigma(q_1,q_2)=\sum_{m_1,m_2=-\infty}^{\infty}a_{m_1m_2}\ \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(m_1q_1+m_2q_2)},$$

where

$$a_{m_1m_2}=\frac{1}{d^2}\int_0^d\!\!\int_0^d \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}(m_1q_1+m_2q_2)}\sum_{\alpha_1,\alpha_2=-\infty}^{\infty}\mathrm{e}^{-\frac{\pi}{d}[a(q_1+\alpha_1 d)^2+2b(q_1+\alpha_1 d)(q_2+\alpha_2 d)+c(q_2+\alpha_2 d)^2]}dq_1dq_2.$$

By using the formula

$$\mathcal{F}\left[\mathrm{e}^{-\frac{1}{2}(q_1\ \ q_2)\,\sigma\left(\begin{array}{c} q_1\\ q_2\end{array}\right)}\right](p_1,p_2)=\frac{1}{\sqrt{\det\sigma}}\ \mathrm{e}^{-\frac{1}{2}(p_1\ \ p_2)\,\sigma^{-1}\left(\begin{array}{c} p_1\\ p_2\end{array}\right)}$$

and the change of variables $q_1=y_1\sqrt{\frac{d}{2\pi}}-\alpha_1 d,\ \ q_2=y_2\sqrt{\frac{d}{2\pi}}-\alpha_2 d$, we get

$$a_{m_1m_2}=\frac{1}{2\pi d}\sum_{\alpha_1,\alpha_2=-\infty}^{\infty}\int_{\alpha_1\sqrt{2\pi d}}^{(\alpha_1+1)\sqrt{2\pi d}}\int_{\alpha_2\sqrt{2\pi d}}^{(\alpha_2+1)\sqrt{2\pi d}}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}(m_1(y_1\sqrt{\frac{d}{2\pi}}-\alpha_1 d)+m_2(y_2\sqrt{\frac{d}{2\pi}}-\alpha_2 d))}\ \mathrm{e}^{-\frac{1}{2}[ay_1^2+2by_1y_2+cy_2^2]}dy_1dy_2$$

$$=\frac{1}{2\pi d}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}(m_1y_1\sqrt{\frac{d}{2\pi}}+m_2y_2\sqrt{\frac{d}{2\pi}})}\ \mathrm{e}^{-\frac{1}{2}[ay_1^2+2by_1y_2+cy_2^2]}dy_1dy_2=\frac{1}{d\sqrt{\det\sigma}}\mathrm{e}^{-\frac{\pi}{d}(m_1\ \ m_2)\,\sigma^{-1}\left(\begin{array}{c} m_1\\ m_2\end{array}\right)}.$$

Consequently, we have

$$\mathcal{G}_\sigma(q_1,q_2)=\frac{1}{d\sqrt{\det\sigma}}\sum_{m_1,m_2=-\infty}^{\infty}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(m_1q_1+m_2q_2)}\ \mathrm{e}^{-\frac{\pi}{d}(m_1\ \ m_2)\,\sigma^{-1}\left(\begin{array}{c} m_1\\ m_2\end{array}\right)}.$$

and the relation

$$\begin{aligned}
\mathbf{g}_\sigma(n_1,n_2) &= \frac{1}{d\sqrt{\det\sigma}} \sum_{m_1,m_2=-\infty}^{\infty} \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(m_1n_1+m_2n_2)}\, \mathrm{e}^{-\frac{\pi}{d}(m_1\ \ m_2)\,\sigma^{-1}\begin{pmatrix} m_1\\ m_2\end{pmatrix}}\\
&= \frac{1}{d\sqrt{\det\sigma}} \sum_{k_1,k_2=-j}^{j} \sum_{\beta_1,\beta_2=-\infty}^{\infty} \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}((k_1+\beta_1d)n_1+(k_2+\beta_2d)n_2)}\, \mathrm{e}^{-\frac{\pi}{d}(k_1+\beta_1d\ \ k_2+\beta_2d)\,\sigma^{-1}\begin{pmatrix} k_1+\beta_1d\\ k_2+\beta_2d\end{pmatrix}}\\
&= \frac{1}{d\sqrt{\det\sigma}} \sum_{k_1,k_2=-j}^{j} \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(k_1n_1+k_2n_2)}\, \mathbf{g}_{\sigma^{-1}}(k_1,k_2) = \frac{1}{\sqrt{\det\sigma}}\mathbf{F}^{-1}[\mathbf{g}_{\sigma^{-1}}](n_1,n_2),
\end{aligned}$$

equivalent to $\mathbf{F}[\mathbf{g}_\sigma]=\frac{1}{\sqrt{\det\sigma}}\,\mathbf{g}_{\sigma^{-1}}$.

•2 The periodic function of two continuous variables

$$\mathcal{G}_\sigma^{+0}(q_1,q_2)=\sum_{\alpha_1,\alpha_2=-\infty}^{\infty} \mathrm{e}^{-\frac{\pi}{d}[a(q_1+(\alpha_1+\frac{1}{2})d)^2+2b(q_1+(\alpha_1+\frac{1}{2})d))(q_2+\alpha_2d)+c(q_2+\alpha_2d)^2]}$$

can be expended into a Fourier series

$$\mathcal{G}_\sigma^{+0}(q_1,q_2)=\sum_{m_1,m_2=-\infty}^{\infty} a_{m_1m_2}\, \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(m_1q_1+m_2q_2)}$$

where

$$a_{m_1m_2}=\frac{1}{d^2}\int_0^d\!\!\int_0^d \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}(m_1q_1+m_2q_2)} \sum_{\alpha_1,\alpha_2=-\infty}^{\infty} \mathrm{e}^{-\frac{\pi}{d}[a(q_1+(\alpha_1+\frac{1}{2})d)^2+2b(q_1+(\alpha_1+\frac{1}{2})d))(q_2+\alpha_2d)+c(q_2+\alpha_2d)^2]}dq_1dq_2.$$

By using the formula

$$\mathcal{F}\left[\mathrm{e}^{-\frac{1}{2}(q_1\ \ q_2)\,\sigma\begin{pmatrix} q_1\\ q_2\end{pmatrix}}\right](p_1,p_2)=\frac{1}{\sqrt{\det\sigma}}\mathrm{e}^{-\frac{1}{2}(p_1\ \ p_2)\,\sigma^{-1}\begin{pmatrix} p_1\\ p_2\end{pmatrix}}$$

and the change of variables $q_1=y_1\sqrt{\frac{d}{2\pi}}-\alpha_1d-\frac{d}{2},\ \ q_2=y_2\sqrt{\frac{d}{2\pi}}-\alpha_2d$, we get

$$\begin{aligned}
a_{m_1m_2} &= \frac{1}{2\pi d}\sum_{\alpha_1,\alpha_2=-\infty}^{\infty} \int_{(\alpha_1+\frac{1}{2})\sqrt{2\pi d}}^{(\alpha_1+\frac{3}{2})\sqrt{2\pi d}} \int_{\alpha_2\sqrt{2\pi d}}^{(\alpha_2+1)\sqrt{2\pi d}} \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}(m_1(y_1\sqrt{\frac{d}{2\pi}}-\alpha_1d-\frac{d}{2})+m_2(y_2\sqrt{\frac{d}{2\pi}}-\alpha_2d))}\mathrm{e}^{-\frac{1}{2}[ay_1^2+2by_1y_2+cy_2^2]}dy_1dy_2\\
&= \frac{(-1)^{m_1}}{2\pi d}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{\frac{-2\pi\mathrm{i}}{d}(m_1y_1\sqrt{\frac{d}{2\pi}}+m_2y_2\sqrt{\frac{d}{2\pi}})}\mathrm{e}^{-\frac{1}{2}[ay_1^2+2by_1y_2+cy_2^2]}dy_1dy_2=\frac{(-1)^{m_1}}{d\sqrt{\det\sigma}}\mathrm{e}^{-\frac{\pi}{d}(m_1\ \ m_2)\,\sigma^{-1}\begin{pmatrix} m_1\\ m_2\end{pmatrix}}.
\end{aligned}$$

Consequently, we have

$$\mathcal{G}_\sigma^{+0}(q_1,q_2)=\frac{1}{d\sqrt{\det\sigma}}\sum_{m_1,m_2=-\infty}^{\infty} \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(m_1q_1+m_2q_2)}\,(-1)^{m_1}\,\mathrm{e}^{-\frac{\pi}{d}(m_1\ \ m_2)\,\sigma^{-1}\begin{pmatrix} m_1\\ m_2\end{pmatrix}}.$$

and the relation

$$\begin{aligned}
\mathbf{g}_\sigma^{+0}(n_1,n_2) &= \frac{1}{d\sqrt{\det\sigma}} \sum_{m_1,m_2=-\infty}^{\infty} \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(m_1n_1+m_2n_2)}\,(-1)^{m_1}\, \mathrm{e}^{-\frac{\pi}{d}(m_1\ \ m_2)\,\sigma^{-1}\begin{pmatrix} m_1\\ m_2\end{pmatrix}}\\
&= \frac{1}{d\sqrt{\det\sigma}} \sum_{k_1,k_2=-j}^{j} \sum_{\beta_1,\beta_2=-\infty}^{\infty} \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}((k_1+\beta_1d)n_1+(k_2+\beta_2d)n_2)}\,(-1)^{k_1+\beta_1d}\, \mathrm{e}^{-\frac{\pi}{d}(k_1+\beta_1d\ \ k_2+\beta_2d)\,\sigma^{-1}\begin{pmatrix} k_1+\beta_1d\\ k_2+\beta_2d\end{pmatrix}}\\
&= \frac{1}{d\sqrt{\det\sigma}} \sum_{k_1,k_2=-j}^{j} (-1)^{k_1}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(k_1n_1+k_2n_2)} \sum_{\beta_1,\beta_2=-\infty}^{\infty} (-1)^{\beta_1}\mathrm{e}^{-\frac{\pi}{d}(k_1+\beta_1d\ \ k_2+\beta_2d)\,\sigma^{-1}\begin{pmatrix} k_1+\beta_1d\\ k_2+\beta_2d\end{pmatrix}},
\end{aligned}$$

equivalent to $\mathbf{F}[\mathbf{g}_\sigma^{+0}](k_1,k_2)=\frac{(-1)^{k_1}}{\sqrt{\det\sigma}}\sum_{\beta_1,\beta_2=-\infty}^{\infty}(-1)^{\beta_1}\ g_{\sigma^{-1}}\left((k_1+\beta_1d)\sqrt{\frac{2\pi}{d}},(k_2+\beta_2d)\sqrt{\frac{2\pi}{d}}\right)$.

•3 The periodic function of two continuous variables

$$\mathcal{G}_\sigma^{0+}(q_1,q_2)=\sum_{\alpha_1,\alpha_2=-\infty}^{\infty} \mathrm{e}^{-\frac{\pi}{d}[a(q_1+\alpha_1d)^2+2b(q_1+\alpha_1d)(q_2+(\alpha_2+\frac{1}{2})d)+c(q_2+(\alpha_2+\frac{1}{2})d)^2]}$$

can be expended into a Fourier series

$$\mathcal{G}_\sigma^{0+}(q_1,q_2)=\sum_{m_1,m_2=-\infty}^{\infty} a_{m_1m_2}\, \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(m_1q_1+m_2q_2)},$$

where

$$a_{m_1m_2}=\frac{1}{d^2}\int_0^d\!\!\int_0^d \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}(m_1q_1+m_2q_2)} \sum_{\alpha_1,\alpha_2=-\infty}^{\infty} \mathrm{e}^{-\frac{\pi}{d}[a(q_1+\alpha_1d)^2+2b(q_1+\alpha_1d)(q_2+(\alpha_2+\frac{1}{2})d)d)+c(q_2+(\alpha_2+\frac{1}{2})d)d)^2]}dq_1dq_2.$$

By using the formula

$$\mathcal{F}\left[\mathrm{e}^{-\frac{1}{2}(q_1\ \ q_2)\,\sigma\begin{pmatrix}q_1\\ q_2\end{pmatrix}}\right](p_1,p_2)=\frac{1}{\sqrt{\det\sigma}}\ \mathrm{e}^{-\frac{1}{2}(p_1\ \ p_2)\,\sigma^{-1}\begin{pmatrix}p_1\\ p_2\end{pmatrix}}$$

and the change of variables $q_1=y_1\sqrt{\frac{d}{2\pi}}-\alpha_1 d,\ \ q_2=y_2\sqrt{\frac{d}{2\pi}}-\alpha_2 d-\frac{d}{2}$, we get

$$\begin{aligned}a_{m_1m_2}&=\frac{1}{2\pi d}\sum_{\alpha_1,\alpha_2=-\infty}^{\infty}\int_{\alpha_1\sqrt{2\pi d}}^{(\alpha_1+1)\sqrt{2\pi d}}\int_{(\alpha_2+\frac{1}{2})\sqrt{2\pi d}}^{(\alpha_2+\frac{3}{2})\sqrt{2\pi d}}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}(m_1(y_1\sqrt{\frac{d}{2\pi}}-\alpha_1 d)+m_2(y_2\sqrt{\frac{d}{2\pi}}-\alpha_2 d-\frac{d}{2}))}\ \mathrm{e}^{-\frac{1}{2}[ay_1^2+2by_1y_2+cy_2^2]}dy_1dy_2\\ &=\frac{(-1)^{m_2}}{2\pi d}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}(m_1y_1\sqrt{\frac{d}{2\pi}}+m_2y_2\sqrt{\frac{d}{2\pi}})}\ \mathrm{e}^{-\frac{1}{2}[ay_1^2+2by_1y_2+cy_2^2]}dy_1dy_2=\frac{(-1)^{m_2}}{d\sqrt{\det\sigma}}\mathrm{e}^{-\frac{\pi}{d}(m_1\ \ m_2)\,\sigma^{-1}\begin{pmatrix}m_1\\ m_2\end{pmatrix}}.\end{aligned}$$

Consequently, we have

$$\mathcal{G}_\sigma^{0+}(q_1,q_2)=\frac{1}{d\sqrt{\det\sigma}}\sum_{m_1,m_2=-\infty}^{\infty}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(m_1q_1+m_2q_2)}\,(-1)^{m_2}\,\mathrm{e}^{-\frac{\pi}{d}(m_1\ \ m_2)\,\sigma^{-1}\begin{pmatrix}m_1\\ m_2\end{pmatrix}}.$$

and the relation

$$\begin{aligned}\mathbf{g}_\sigma^{0+}(n_1,n_2)&=\frac{1}{d\sqrt{\det\sigma}}\sum_{m_1,m_2=-\infty}^{\infty}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(m_1n_1+m_2n_2)}\,(-1)^{m_2}\,\mathrm{e}^{-\frac{\pi}{d}(m_1\ \ m_2)\,\sigma^{-1}\begin{pmatrix}m_1\\ m_2\end{pmatrix}}\\ &=\frac{1}{d\sqrt{\det\sigma}}\sum_{k_1,k_2=-j}^{j}\sum_{\beta_1,\beta_2=-\infty}^{\infty}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}((k_1+\beta_1d)n_1+(k_2+\beta_2d)n_2)}\,(-1)^{k_2+\beta_2d}\,\mathrm{e}^{-\frac{\pi}{d}(k_1+\beta_1d\ \ \ k_2+\beta_2d)\,\sigma^{-1}\begin{pmatrix}k_1+\beta_1d\\ k_2+\beta_2d\end{pmatrix}}\\ &=\frac{1}{d\sqrt{\det\sigma}}\sum_{k_1,k_2=-j}^{j}(-1)^{k_2}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(k_1n_1+k_2n_2)}\sum_{\beta_1,\beta_2=-\infty}^{\infty}(-1)^{\beta_2}\mathrm{e}^{-\frac{\pi}{d}(k_1+\beta_1d\ \ \ k_2+\beta_2d)\,\sigma^{-1}\begin{pmatrix}k_1+\beta_1d\\ k_2+\beta_2d\end{pmatrix}},\end{aligned}$$

equivalent to $\ \mathbf{F}[\mathbf{g}_\sigma^{0+}](k_1,k_2)=\frac{(-1)^{k_2}}{\sqrt{\det\sigma}}\sum_{\beta_1,\beta_2=-\infty}^{\infty}(-1)^{\beta_2}\ g_{\sigma^{-1}}\left((k_1+\beta_1d)\sqrt{\frac{2\pi}{d}},(k_2+\beta_2d)\sqrt{\frac{2\pi}{d}}\right).$

•4 The periodic function of two continuous variables

$$\mathcal{G}_\sigma^{++}(q_1,q_2)=\sum_{\alpha_1,\alpha_2=-\infty}^{\infty}\mathrm{e}^{-\frac{\pi}{d}[a(q_1+(\alpha_1+\frac{1}{2})d)^2+2b(q_1+(\alpha_1+\frac{1}{2})d))(q_2+(\alpha_2+\frac{1}{2})d)+c(q_2+(\alpha_2+\frac{1}{2})d)^2]}$$

can be expended into a Fourier series

$$\mathcal{G}_\sigma^{++}(q_1,q_2)=\sum_{m_1,m_2=-\infty}^{\infty}a_{m_1m_2}\ \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(m_1q_1+m_2q_2)}$$

where

$$a_{m_1m_2}=\frac{1}{d^2}\int_0^d\!\!\int_0^d\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}(m_1q_1+m_2q_2)}\sum_{\alpha_1,\alpha_2=-\infty}^{\infty}\mathrm{e}^{-\frac{\pi}{d}[a(q_1+(\alpha_1+\frac{1}{2})d)^2+2b(q_1+(\alpha_1+\frac{1}{2})d))(q_2+(\alpha_2+\frac{1}{2})d)+c(q_2+(\alpha_2+\frac{1}{2})d)^2]}.$$

By using the formula

$$\mathcal{F}\left[\mathrm{e}^{-\frac{1}{2}(q_1\ \ q_2)\,\sigma\begin{pmatrix}q_1\\ q_2\end{pmatrix}}\right](p_1,p_2)=\frac{1}{\sqrt{\det\sigma}}\mathrm{e}^{-\frac{1}{2}(p_1\ \ p_2)\,\sigma^{-1}\begin{pmatrix}p_1\\ p_2\end{pmatrix}}$$

and the change of variables $q_1=y_1\sqrt{\frac{d}{2\pi}}-\alpha_1 d-\frac{d}{2},\ \ q_2=y_2\sqrt{\frac{d}{2\pi}}-\alpha_2 d-\frac{d}{2}$, we get

$$\begin{aligned}a_{m_1m_2}&=\frac{1}{2\pi d}\sum_{\alpha_1,\alpha_2=-\infty}^{\infty}\int_{(\alpha_1+\frac{1}{2})\sqrt{2\pi d}}^{(\alpha_1+\frac{3}{2})\sqrt{2\pi d}}\int_{\alpha_2\sqrt{2\pi d}}^{(\alpha_2+1)\sqrt{2\pi d}}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}(m_1(y_1\sqrt{\frac{d}{2\pi}}-\alpha_1 d-\frac{d}{2})+m_2(y_2\sqrt{\frac{d}{2\pi}}-\alpha_2 d-\frac{d}{2}))}\mathrm{e}^{-\frac{1}{2}[ay_1^2+2by_1y_2+cy_2^2]}dy_1dy_2\\ &=\frac{(-1)^{m_1+m_2}}{2\pi d}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}(m_1y_1\sqrt{\frac{d}{2\pi}}+m_2y_2\sqrt{\frac{d}{2\pi}})}\mathrm{e}^{-\frac{1}{2}[ay_1^2+2by_1y_2+cy_2^2]}dy_1dy_2=\frac{(-1)^{m_1+m_2}}{d\sqrt{\det\sigma}}\mathrm{e}^{-\frac{\pi}{d}(m_1\ \ m_2)\,\sigma^{-1}\begin{pmatrix}m_1\\ m_2\end{pmatrix}}.\end{aligned}$$

Consequently, we have

$$\mathcal{G}_\sigma^{++}(q_1,q_2)=\frac{1}{d\sqrt{\det\sigma}}\sum_{m_1,m_2=-\infty}^{\infty}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(m_1q_1+m_2q_2)}\,(-1)^{m_1+m_2}\,\mathrm{e}^{-\frac{\pi}{d}(m_1\ \ m_2)\,\sigma^{-1}\begin{pmatrix}m_1\\ m_2\end{pmatrix}}.$$

and the relation

$$\begin{aligned}\mathbf{g}_\sigma^{++}(n_1,n_2)&=\frac{1}{d\sqrt{\det\sigma}}\sum_{m_1,m_2=-\infty}^{\infty}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(m_1n_1+m_2n_2)}\,(-1)^{m_1+m_2}\,\mathrm{e}^{-\frac{\pi}{d}(m_1\ \ m_2)\,\sigma^{-1}\begin{pmatrix}m_1\\ m_2\end{pmatrix}}\\ &=\frac{1}{d\sqrt{\det\sigma}}\sum_{k_1,k_2=-j}^{j}\sum_{\beta_1,\beta_2=-\infty}^{\infty}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}((k_1+\beta_1d)n_1+(k_2+\beta_2d)n_2)}\,(-1)^{k_1+\beta_1d}\,(-1)^{k_2+\beta_2d}\,\mathrm{e}^{-\frac{\pi}{d}(k_1+\beta_1d\ \ \ k_2+\beta_2d)\,\sigma^{-1}\begin{pmatrix}k_1+\beta_1d\\ k_2+\beta_2d\end{pmatrix}}\\ &=\frac{1}{d\sqrt{\det\sigma}}\sum_{k_1,k_2=-j}^{j}(-1)^{k_1+k_2}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(k_1n_1+k_2n_2)}\sum_{\beta_1,\beta_2=-\infty}^{\infty}(-1)^{\beta_1+\beta_2}\mathrm{e}^{-\frac{\pi}{d}(k_1+\beta_1d\ \ \ k_2+\beta_2d)\,\sigma^{-1}\begin{pmatrix}k_1+\beta_1d\\ k_2+\beta_2d\end{pmatrix}},\end{aligned}$$

equivalent to $\mathbf{F}[\mathbf{g}_\sigma^{++}](k_1,k_2)=\frac{(-1)^{k_1+k_2}}{\sqrt{\det\sigma}}\sum\limits_{\beta_1,\beta_2=-\infty}^{\infty}(-1)^{\beta_1+\beta_2}\ g_{\sigma^{-1}}\left((k_1+\beta_1 d)\sqrt{\frac{2\pi}{d}},(k_2+\beta_2 d)\sqrt{\frac{2\pi}{d}}\right).$

THEOREM 1. *The discrete Fourier transform of* $\mathbf{g}_\sigma$ *satisfies the relation*

$$\mathbf{F}[\mathbf{g}_\sigma]=\frac{1}{\sqrt{\det\sigma}}\,\mathbf{g}_{\sigma^{-1}}.$$

Proof. This is just the first relation from Lemma 1, written in a different way.

LEMMA 2.

- •1. $\mathbf{F}[\mathbf{g}_{2\sigma}](2k_1,2k_2)=\frac{1}{2\sqrt{\det\sigma}}\left(\mathbf{g}_{2\sigma^{-1}}(k_1,k_2)+\mathbf{g}_{2\sigma^{-1}}^{+0}(k_1,k_2)+\mathbf{g}_{2\sigma^{-1}}^{0+}(k_1,k_2)+\mathbf{g}_{2\sigma^{-1}}^{++}(k_1,k_2)\right).$
- •2. $\mathbf{F}[\mathbf{g}_{2\sigma}^{+0}](2k_1,2k_2)=\frac{1}{2\sqrt{\det\sigma}}\left(\mathbf{g}_{2\sigma^{-1}}(k_1,k_2)-\mathbf{g}_{2\sigma^{-1}}^{+0}(k_1,k_2)+\mathbf{g}_{2\sigma^{-1}}^{0+}(k_1,k_2)-\mathbf{g}_{2\sigma^{-1}}^{++}(k_1,k_2)\right).$
- •3. $\mathbf{F}[\mathbf{g}_{2\sigma}^{0+}](2k_1,2k_2)=\frac{1}{2\sqrt{\det\sigma}}\left(\mathbf{g}_{2\sigma^{-1}}(k_1,k_2)+\mathbf{g}_{2\sigma^{-1}}^{+0}(k_1,k_2)-\mathbf{g}_{2\sigma^{-1}}^{0+}(k_1,k_2)-\mathbf{g}_{2\sigma^{-1}}^{++}(k_1,k_2)\right).$
- •4. $\mathbf{F}[\mathbf{g}_{2\sigma}^{++}](2k_1,2k_2)=\frac{1}{2\sqrt{\det\sigma}}\left(\mathbf{g}_{2\sigma^{-1}}(k_1,k_2)-\mathbf{g}_{2\sigma^{-1}}^{+0}(k_1,k_2)-\mathbf{g}_{2\sigma^{-1}}^{0+}(k_1,k_2)+\mathbf{g}_{2\sigma^{-1}}^{++}(k_1,k_2)\right).$

Proof. The relations

- •1. $\mathbf{F}[\mathbf{g}_{2\sigma}](2k_1,2k_2)=\frac{1}{2\sqrt{\det\sigma}}\sum\limits_{\beta_1,\beta_2=-\infty}^{\infty}g_{(2\sigma)^{-1}}\left((2k_1+\beta_1 d)\sqrt{\frac{2\pi}{d}},(2k_2+\beta_2 d)\sqrt{\frac{2\pi}{d}}\right)$
- •2. $\mathbf{F}[\mathbf{g}_{2\sigma}^{+0}](2k_1,2k_2)=\frac{1}{2\sqrt{\det\sigma}}\sum\limits_{\beta_1,\beta_2=-\infty}^{\infty}(-1)^{\beta_1}\ g_{(2\sigma)^{-1}}\left((2k_1+\beta_1 d)\sqrt{\frac{2\pi}{d}},(2k_2+\beta_2 d)\sqrt{\frac{2\pi}{d}}\right)$
- •3. $\mathbf{F}[\mathbf{g}_{2\sigma}^{0+}](2k_1,2k_2)=\frac{1}{2\sqrt{\det\sigma}}\sum\limits_{\beta_1,\beta_2=-\infty}^{\infty}(-1)^{\beta_2}\ g_{(2\sigma)^{-1}}\left((2k_1+\beta_1 d)\sqrt{\frac{2\pi}{d}},(2k_2+\beta_2 d)\sqrt{\frac{2\pi}{d}}\right)$
- •4. $\mathbf{F}[\mathbf{g}_{2\sigma}^{++}](2k_1,2k_2)=\frac{1}{2\sqrt{\det\sigma}}\sum\limits_{\beta_1,\beta_2=-\infty}^{\infty}(-1)^{\beta_1+\beta_2}\ g_{(2\sigma)^{-1}}\left((2k_1+\beta_1 d)\sqrt{\frac{2\pi}{d}},(2k_2+\beta_2 d)\sqrt{\frac{2\pi}{d}}\right)$

can be written as

- •1. $\mathbf{F}[\mathbf{g}_{2\sigma}](2k_1,2k_2)=\frac{1}{\sqrt{\det\sigma}}\sum\limits_{\beta_1,\beta_2=-\infty}^{\infty}g_{2\sigma^{-1}}\left((k_1+\beta_1\tfrac{d}{2})\sqrt{\frac{2\pi}{d}},(k_2+\beta_2\tfrac{d}{2})\sqrt{\frac{2\pi}{d}}\right)$
- •2. $\mathbf{F}[\mathbf{g}_{2\sigma}^{+0}](2k_1,2k_2)=\frac{1}{\sqrt{\det\sigma}}\sum\limits_{\beta_1,\beta_2=-\infty}^{\infty}(-1)^{\beta_1}\ g_{2\sigma^{-1}}\left((k_1+\beta_1\tfrac{d}{2})\sqrt{\frac{2\pi}{d}},(k_2+\beta_2\tfrac{d}{2})\sqrt{\frac{2\pi}{d}}\right)$
- •3. $\mathbf{F}[\mathbf{g}_{2\sigma}^{0+}](2k_1,2k_2)=\frac{1}{\sqrt{\det\sigma}}\sum\limits_{\beta_1,\beta_2=-\infty}^{\infty}(-1)^{\beta_2}\ g_{2\sigma^{-1}}\left((k_1+\beta_1\tfrac{d}{2})\sqrt{\frac{2\pi}{d}},(k_2+\beta_2\tfrac{d}{2})\sqrt{\frac{2\pi}{d}}\right)$
- •4. $\mathbf{F}[\mathbf{g}_{2\sigma}^{++}](2k_1,2k_2)=\frac{1}{\sqrt{\det\sigma}}\sum\limits_{\beta_1,\beta_2=-\infty}^{\infty}(-1)^{\beta_1+\beta_2}\ g_{2\sigma^{-1}}\left((k_1+\beta_1\tfrac{d}{2})\sqrt{\frac{2\pi}{d}},(k_2+\beta_2\tfrac{d}{2})\sqrt{\frac{2\pi}{d}}\right).$

By separating the even case from the odd case for β_1 and β_2, we get the relations from lemma 2.

Wigner function

LEMMA 3. *If f is such that the series are convergent, then:*

$$\sum_{\alpha,\beta=-\infty}^{\infty}f(\alpha,\beta)=\sum_{\mu,\eta=-\infty}^{\infty}f(\mu+\eta,\mu-\eta)+\sum_{\mu,\eta=-\infty}^{\infty}f(\mu+\eta+1,\mu-\eta).$$

Proof. After separating the sum as

$$\sum_{\alpha,\beta=-\infty}^{\infty}f(\alpha,\beta)=\sum_{\substack{\alpha,\beta\\ \text{both even}\\ \text{or}\\ \text{both odd}}}f(\alpha,\beta)+\sum_{\substack{\alpha,\beta\\ \text{one even}\\ \text{and}\\ \text{other odd}}}f(\alpha,\beta)$$

we use the substitutions $(\alpha,\beta)=(\mu+\eta,\mu-\eta)$ and $(\alpha,\beta)=(\mu+\eta+1,\mu-\eta)$.

THEOREM 2 *The discrete Wigner function of $\mathbf{g}_\sigma$ is an algebraic sum of 16 products of Gaussian like functions:*

$$\begin{aligned}
\mathbf{W}_{\mathbf{g}_\sigma}(n_1,n_2,k_1,k_2) = {} & \tfrac{1}{2d\sqrt{\det\sigma}}\ \mathbf{g}_{2\sigma}(n_1,n_2)\left[\mathbf{g}_{2\sigma^{-1}}(k_1,k_2)+\mathbf{g}^{+0}_{2\sigma^{-1}}(k_1,k_2)+\mathbf{g}^{0+}_{2\sigma^{-1}}(k_1,k_2)+\mathbf{g}^{++}_{2\sigma^{-1}}(k_1,k_2)\right]\\
& +\tfrac{1}{2d\sqrt{\det\sigma}}\ \mathbf{g}^{+0}_{2\sigma}(n_1,n_2)\left[\mathbf{g}_{2\sigma^{-1}}(k_1,k_2)-\mathbf{g}^{+0}_{2\sigma^{-1}}(k_1,k_2)+\mathbf{g}^{0+}_{2\sigma^{-1}}(k_1,k_2)-\mathbf{g}^{++}_{2\sigma^{-1}}(k_1,k_2)\right]\\
& +\tfrac{1}{2d\sqrt{\det\sigma}}\ \mathbf{g}^{0+}_{2\sigma}(n_1,n_2)\left[\mathbf{g}_{2\sigma^{-1}}(k_1,k_2)+\mathbf{g}^{+0}_{2\sigma^{-1}}(k_1,k_2)-\mathbf{g}^{0+}_{2\sigma^{-1}}(k_1,k_2)-\mathbf{g}^{++}_{2\sigma^{-1}}(k_1,k_2)\right]\\
& +\tfrac{1}{2d\sqrt{\det\sigma}}\mathbf{g}^{++}_{2\sigma}(n_1,n_2)\left[\mathbf{g}_{2\sigma^{-1}}(k_1,k_2)-\mathbf{g}^{+0}_{2\sigma^{-1}}(k_1,k_2)-\mathbf{g}^{0+}_{2\sigma^{-1}}(k_1,k_2)+\mathbf{g}^{++}_{2\sigma^{-1}}(k_1,k_2)\right]
\end{aligned}$$

Proof. By using the definition

$$\mathbf{W}_\psi(n_1,n_2,k_1,k_2)\overset{\text{def}}{=}\frac{1}{d^2}\sum_{m_1=-j}^{j}\sum_{m_2=-j}^{j}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}(k_1m_1+k_2m_2)}\ \overline{\psi(n_1-m_1,n_2-m_2)}\,\psi(n_1+m_1,n_2+m_2)$$

and Lemma 3, we get

$$\begin{aligned}
\mathbf{W}_{\mathbf{g}_\sigma}(n_1,n_2,k_1,k_2) = {} & \tfrac{1}{d^2}\sum_{m_1,m_2=-j}^{j}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}(k_1m_1+k_2m_2)}\sum_{\alpha_1,\beta_1=-\infty}^{\infty}\mathrm{e}^{-\frac{a\pi}{d}(n_1-m_1+\alpha_1d)^2}\,\mathrm{e}^{-\frac{2b\pi}{d}(n_1-m_1+\alpha_1d)(n_2-m_2+\beta_1d)}\,\mathrm{e}^{-\frac{c\pi}{d}(n_2-m_2+\beta_1d)^2}\\
& \sum_{\alpha_2,\beta_2=-\infty}^{\infty}\mathrm{e}^{-\frac{a\pi}{d}(n_1+m_1+\alpha_2d)^2}\,\mathrm{e}^{-\frac{2b\pi}{d}(n_1+m_1+\alpha_2d)(n_2+m_2+\beta_2d)}\,\mathrm{e}^{-\frac{c\pi}{d}(n_2+m_2+\beta_2d)^2}\\
= {} & \tfrac{1}{d^2}\sum_{m_1,m_2=-j}^{j}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}(k_1m_1+k_2m_2)}\sum_{\alpha_1,\alpha_2=-\infty}^{\infty}\ \sum_{\beta_1,\beta_2=-\infty}^{\infty}\mathrm{e}^{-\frac{a\pi}{d}(n_1-m_1+\alpha_1d)^2}\,\mathrm{e}^{-\frac{a\pi}{d}(n_1+m_1+\alpha_2d)^2}\\
& \mathrm{e}^{-\frac{2b\pi}{d}(n_1-m_1+\alpha_1d)(n_2-m_2+\beta_1d)}\,\mathrm{e}^{-\frac{2b\pi}{d}(n_1+m_1+\alpha_2d)(n_2+m_2+\beta_2d)}\\
& \mathrm{e}^{-\frac{c\pi}{d}(n_2-m_2+\beta_1d)^2}\,\mathrm{e}^{-\frac{c\pi}{d}(n_2+m_2+\beta_2d)^2}
\end{aligned}$$

$$\begin{aligned}
= {} & \tfrac{1}{d^2}\sum_{m_1,m_2=-j}^{j}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}(k_1m_1+k_2m_2)}\sum_{\mu_1,\eta_1=-\infty}^{\infty}\ \sum_{\mu_2,\eta_2=-\infty}^{\infty}\mathrm{e}^{-\frac{a\pi}{d}(n_1-m_1+(\mu_1+\eta_1)d)^2}\,\mathrm{e}^{-\frac{a\pi}{d}(n_1+m_1+(\mu_1-\eta_1)d)^2}\\
& \mathrm{e}^{-\frac{2b\pi}{d}(n_1-m_1+(\mu_1+\eta_1)d)(n_2-m_2+(\mu_2+\eta_2)d)}\,\mathrm{e}^{-\frac{2b\pi}{d}(n_1+m_1+(\mu_1-\eta_1)d)(n_2+m_2+(\mu_2-\eta_2)d)}\\
& \mathrm{e}^{-\frac{c\pi}{d}(n_2-m_2+(\mu_2+\eta_2)d)^2}\,\mathrm{e}^{-\frac{c\pi}{d}(n_2+m_2+(\mu_2-\eta_2)d)^2}\\
& +\tfrac{1}{d^2}\sum_{m_1,m_2=-j}^{j}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}(k_1m_1+k_2m_2)}\sum_{\mu_1,\eta_1=-\infty}^{\infty}\ \sum_{\mu_2,\eta_2=-\infty}^{\infty}\mathrm{e}^{-\frac{a\pi}{d}(n_1-m_1+(\mu_1+\eta_1+1)d)^2}\,\mathrm{e}^{-\frac{a\pi}{d}(n_1+m_1+(\mu_1-\eta_1)d)^2}\\
& \mathrm{e}^{-\frac{2b\pi}{d}(n_1-m_1+(\mu_1+\eta_1+1)d)(n_2-m_2+(\mu_2+\eta_2)d)}\,\mathrm{e}^{-\frac{2b\pi}{d}(n_1+m_1+(\mu_1-\eta_1)d)(n_2+m_2+(\mu_2-\eta_2)d)}\\
& \mathrm{e}^{-\frac{c\pi}{d}(n_2-m_2+(\mu_2+\eta_2)d)^2}\,\mathrm{e}^{-\frac{c\pi}{d}(n_2+m_2+(\mu_2-\eta_2)d)^2}\\
& +\tfrac{1}{d^2}\sum_{m_1,m_2=-j}^{j}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}(k_1m_1+k_2m_2)}\sum_{\mu_1,\eta_1=-\infty}^{\infty}\ \sum_{\mu_2,\eta_2=-\infty}^{\infty}\mathrm{e}^{-\frac{a\pi}{d}(n_1-m_1+(\mu_1+\eta_1)d)^2}\,\mathrm{e}^{-\frac{a\pi}{d}(n_1+m_1+(\mu_1-\eta_1)d)^2}\\
& \mathrm{e}^{-\frac{2b\pi}{d}(n_1-m_1+(\mu_1+\eta_1)d)(n_2-m_2+(\mu_2+\eta_2+1)d)}\,\mathrm{e}^{-\frac{2b\pi}{d}(n_1+m_1+(\mu_1-\eta_1)d)(n_2+m_2+(\mu_2-\eta_2)d)}\\
& \mathrm{e}^{-\frac{c\pi}{d}(n_2-m_2+(\mu_2+\eta_2+1)d)^2}\,\mathrm{e}^{-\frac{c\pi}{d}(n_2+m_2+(\mu_2-\eta_2)d)^2}\\
& +\tfrac{1}{d^2}\sum_{m_1,m_2=-j}^{j}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}(k_1m_1+k_2m_2)}\sum_{\mu_1,\eta_1=-\infty}^{\infty}\ \sum_{\mu_2,\eta_2=-\infty}^{\infty}\mathrm{e}^{-\frac{a\pi}{d}(n_1-m_1+(\mu_1+\eta_1+1)d)^2}\,\mathrm{e}^{-\frac{a\pi}{d}(n_1+m_1+(\mu_1-\eta_1)d)^2}\\
& \mathrm{e}^{-\frac{2b\pi}{d}(n_1-m_1+(\mu_1+\eta_1+1)d)(n_2-m_2+(\mu_2+\eta_2+1)d)}\,\mathrm{e}^{-\frac{2b\pi}{d}(n_1+m_1+(\mu_1-\eta_1)d)(n_2+m_2+(\mu_2-\eta_2)d)}\\
& \mathrm{e}^{-\frac{c\pi}{d}(n_2-m_2+(\mu_2+\eta_2+1)d)^2}\,\mathrm{e}^{-\frac{c\pi}{d}(n_2+m_2+(\mu_2-\eta_2)d)^2}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{d^2}\sum_{m_1,m_2=-j}^{j} e^{-\frac{4\pi i}{d}(k_1m_1+k_2m_2)}\sum_{\mu_1,\eta_1=-\infty}^{\infty}\ \sum_{\mu_2,\eta_2=-\infty}^{\infty} e^{-\frac{2a\pi}{d}(n_1+\mu_1 d)^2}\, e^{-\frac{2a\pi}{d}(m_1-\eta_1 d)^2}\\
&\qquad e^{-\frac{4b\pi}{d}(n_1+\mu_1 d)(n_2+\mu_2 d)}\, e^{-\frac{4b\pi}{d}(m_1-\eta_1 d)(m_2-\eta_2 d)}\\
&\qquad e^{-\frac{2c\pi}{d}(n_2+\mu_2 d)^2}\, e^{-\frac{2c\pi}{d}(m_2-\eta_2 d)^2}\\
&+ \frac{1}{d^2}\sum_{m_1,m_2=-j}^{j} e^{-\frac{4\pi i}{d}(k_1m_1+k_2m_2)}\sum_{\mu_1,\eta_1=-\infty}^{\infty}\ \sum_{\mu_2,\eta_2=-\infty}^{\infty} e^{-\frac{2a\pi}{d}(n_1+(\mu_1+\frac{1}{2})d)^2}\, e^{-\frac{2a\pi}{d}(m_1-(\eta_1+\frac{1}{2})d)^2}\\
&\qquad e^{-\frac{4b\pi}{d}(n_1+(\mu_1+\frac{1}{2})d)(n_2+\mu_2 d)}\, e^{-\frac{4b\pi}{d}(m_1-(\eta_1+\frac{1}{2})d)(m_2-\eta_2 d)}\\
&\qquad e^{-\frac{2c\pi}{d}(n_2+\mu_2 d)^2}\, e^{-\frac{2c\pi}{d}(m_2-\eta_2 d)^2}\\
&+ \frac{1}{d^2}\sum_{m_1,m_2=-j}^{j} e^{-\frac{4\pi i}{d}(k_1m_1+k_2m_2)}\sum_{\mu_1,\eta_1=-\infty}^{\infty}\ \sum_{\mu_2,\eta_2=-\infty}^{\infty} e^{-\frac{2a\pi}{d}(n_1+\mu_1 d)^2}\, e^{-\frac{2a\pi}{d}(m_1-\eta_1 d)^2}\\
&\qquad e^{-\frac{4b\pi}{d}(n_1+\mu_1 d)(n_2+(\mu_2+\frac{1}{2})d)}\, e^{-\frac{4b\pi}{d}(m_1-\eta_1 d)(m_2-(\eta_2+\frac{1}{2})d)}\\
&\qquad e^{-\frac{2c\pi}{d}(n_2+(\mu_2+\frac{1}{2})d)^2}\, e^{-\frac{2c\pi}{d}(m_2-(\eta_2+\frac{1}{2})d)^2}\\
&+ \frac{1}{d^2}\sum_{m_1,m_2=-j}^{j} e^{-\frac{4\pi i}{d}(k_1m_1+k_2m_2)}\sum_{\mu_1,\eta_1=-\infty}^{\infty}\ \sum_{\mu_2,\eta_2=-\infty}^{\infty} e^{-\frac{2a\pi}{d}(n_1+(\mu_1+\frac{1}{2})d)^2}\, e^{-\frac{2a\pi}{d}(m_1-(\eta_1+\frac{1}{2})d)^2}\\
&\qquad e^{-\frac{4b\pi}{d}(n_1+(\mu_1+\frac{1}{2})d)(n_2+(\mu_2+\frac{1}{2})d)}\, e^{-\frac{4b\pi}{d}(m_1-(\eta_1+\frac{1}{2})d)(m_2-(\eta_2+\frac{1}{2})d)}\\
&\qquad e^{-\frac{2c\pi}{d}(n_2+(\mu_2+\frac{1}{2})d)^2}\, e^{-\frac{2c\pi}{d}(m_2-(\eta_2+\frac{1}{2})d)^2}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{d^2}\sum_{\mu_1,\mu_2=-\infty}^{\infty} e^{-\frac{2a\pi}{d}(n_1+\mu_1 d)^2}\, e^{-\frac{4b\pi}{d}(n_1+\mu_1 d)(n_2+\mu_2 d)}\, e^{-\frac{2c\pi}{d}(n_2+\mu_2 d)^2}\\
&\qquad \sum_{m_1,m_2=-j}^{j} e^{-\frac{4\pi i}{d}(k_1m_1+k_2m_2)}\sum_{\eta_1,\eta_2=-\infty}^{\infty} e^{-\frac{2a\pi}{d}(m_1-\eta_1 d)^2}\, e^{-\frac{4b\pi}{d}(m_1-\eta_1 d)(m_2-\eta_2 d)}\, e^{-\frac{2c\pi}{d}(m_2-\eta_2 d)^2}\\
&+ \frac{1}{d^2}\sum_{\mu_1,\mu_2=-\infty}^{\infty} e^{-\frac{2a\pi}{d}(n_1+(\mu_1+\frac{1}{2})d)^2}\, e^{-\frac{4b\pi}{d}(n_1+(\mu_1+\frac{1}{2})d)(n_2+\mu_2 d)}\, e^{-\frac{2c\pi}{d}(n_2+\mu_2 d)^2}\\
&\qquad \sum_{m_1,m_2=-j}^{j} e^{-\frac{4\pi i}{d}(k_1m_1+k_2m_2)}\sum_{\eta_1,\eta_2=-\infty}^{\infty} e^{-\frac{2a\pi}{d}(m_1-(\eta_1+\frac{1}{2})d)^2}\, e^{-\frac{4b\pi}{d}(m_1-(\eta_1+\frac{1}{2})d)(m_2-\eta_2 d)}\, e^{-\frac{2c\pi}{d}(m_2-\eta_2 d)^2}\\
&+ \frac{1}{d^2}\sum_{\mu_1,\mu_2=-\infty}^{\infty} e^{-\frac{2a\pi}{d}(n_1+\mu_1 d)^2}\, e^{-\frac{4b\pi}{d}(n_1+\mu_1 d)(n_2+(\mu_2+\frac{1}{2})d)}\, e^{-\frac{2c\pi}{d}(n_2+(\mu_2+\frac{1}{2})d)^2}\\
&\qquad \sum_{m_1,m_2=-j}^{j} e^{-\frac{4\pi i}{d}(k_1m_1+k_2m_2)}\sum_{\eta_1,\eta_2=-\infty}^{\infty} e^{-\frac{2a\pi}{d}(m_1-\eta_1 d)^2}\, e^{-\frac{4b\pi}{d}(m_1-\eta_1 d)(m_2-(\eta_2+\frac{1}{2})d)}\, e^{-\frac{2c\pi}{d}(m_2-(\eta_2+\frac{1}{2})d)^2}\\
&+ \frac{1}{d^2}\sum_{\mu_1,\mu_2=-\infty}^{\infty} e^{-\frac{2a\pi}{d}(n_1+(\mu_1+\frac{1}{2})d)^2}\, e^{-\frac{4b\pi}{d}(n_1+(\mu_1+\frac{1}{2})d)(n_2+(\mu_2+\frac{1}{2})d)}\, e^{-\frac{2c\pi}{d}(n_2+(\mu_2+\frac{1}{2})d)^2}\\
&\qquad \sum_{m_1,m_2=-j}^{j} e^{-\frac{4\pi i}{d}(k_1m_1+k_2m_2)}\sum_{\eta_1,\eta_2=-\infty}^{\infty} e^{-\frac{2a\pi}{d}(m_1-(\eta_1+\frac{1}{2})d)^2}\, e^{-\frac{4b\pi}{d}(m_1-(\eta_1+\frac{1}{2})d)(m_2-(\eta_2+\frac{1}{2})d)}\, e^{-\frac{2c\pi}{d}(m_2-(\eta_2+\frac{1}{2})d)^2}.\\
&= \tfrac{1}{d}\,\mathbf{g}_{2\sigma}(n_1,n_2)\ \mathbf{F}[\mathbf{g}_{2\sigma}](2k_1,2k_2) + \tfrac{1}{d}\,\mathbf{g}_{2\sigma}^{+0}(n_1,n_2)\ \mathbf{F}[\mathbf{g}_{2\sigma}^{+0}](2k_1,2k_2)\\
&\qquad + \tfrac{1}{d}\,\mathbf{g}_{2\sigma}^{0+}(n_1,n_2)\ \mathbf{F}[\mathbf{g}_{2\sigma}^{0+}](2k_1,2k_2) + \tfrac{1}{d}\,\mathbf{g}_{2\sigma}^{++}(n_1,n_2)\ \mathbf{F}[\mathbf{g}_{2\sigma}^{++}](2k_1,2k_2).
\end{aligned}
$$

Lemma 2 allows us to transform this relation into the formula from our theorem.

■ Discrete-continuous correspondence

The definition of $\mathbf{g}_\sigma$, the relation from Theorem 1 and the relation from Theorem 2 can be written as:

- $\mathbf{g}_\sigma(n_1,n_2) = \sum_{\alpha_1,\alpha_2=-\infty}^{\infty} g_\sigma\left((n_1+\alpha_1 d)\sqrt{\frac{2\pi}{d}},\,(n_2+\alpha_2 d)\sqrt{\frac{2\pi}{d}}\right).$
- $\mathbf{F}[\mathbf{g}_\sigma](k_1,k_2) = \sum_{\beta_1,\beta_2=-\infty}^{\infty} \mathcal{F}[g_\sigma]\left((k_1+\beta_1 d)\sqrt{\frac{2\pi}{d}},\,(k_2+\beta_2 d)\sqrt{\frac{2\pi}{d}}\right).$
- $\mathbf{W}_{\mathbf{g}_\sigma}(n_1,n_2,k_1,k_2) = C_\sigma \sum_{\alpha_1,\alpha_2=-\infty}^{\infty}\ \sum_{\beta_1,\beta_2=-\infty}^{\infty} (-1)^{\alpha_1\beta_1+\alpha_2\beta_2}$
  $\mathcal{W}_{g_\sigma}\left((n_1+\alpha_1\tfrac{d}{2})\sqrt{\tfrac{2\pi}{d}},\,(n_2+\alpha_2\tfrac{d}{2})\sqrt{\tfrac{2\pi}{d}},\,(k_1+\beta_1\tfrac{d}{2})\sqrt{\tfrac{2\pi}{d}},\,(k_2+\beta_2\tfrac{d}{2})\sqrt{\tfrac{2\pi}{d}}\right).$

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