On the CONTINUOUS – DISCRETE quantum system correspondence

$$L^2(\mathbb{R}) \quad \longleftrightarrow \quad \mathbb{C}^{2s+1} \equiv \{\, \psi : \{-s, -s+1, ..., s\} \to \mathbb{C} \,\} \equiv \ell^2(\mathbb{Z}_{2s+1})$$

"Quantum" version

■ **Definition.** For $d = 2s+1 \in \{1, 3, 5, ...\}$, the well-known correspondence

■ Fourier transform	$D0$	$F[\psi](p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} pq} \psi(q) dq$	$\longleftrightarrow$	$D0'$	$F[\psi](k) = \frac{1}{\sqrt{d}} \sum_{n=-s}^{s} \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d} kn}\, \psi(n)$

suggests for continuous-discrete correspondence the following choice:

■ Position operator	$D1$	$\hat{q}\psi(q) = q\, \psi(q)$	$\longleftrightarrow$	$D1'$	$Q\psi(n) = \tilde{n}\, \psi(n)$ $-s \le \tilde{n} \le s$, $\tilde{n} \equiv n (mod\, d)$
■ Momentum operator	$D2$	$\hat{p} = F^\dagger \hat{q} F$	$\longleftrightarrow$	$D2'$	$P = F^\dagger Q F$
■ Translation operator	$D3$	$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar} q\hat{p}} \psi(q') = \psi(q'-q)$	$\longleftrightarrow$	$D3'$	$\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d} nP} \psi(n') = \psi(n'-n)$
■ Translation operator	$D4$	$\mathrm{e}^{\frac{\mathrm{i}}{\hbar} p\hat{q}} \psi(q) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar} pq}\, \psi(q)$	$\longleftrightarrow$	$D4'$	$\mathrm{e}^{\frac{2\pi\mathrm{i}}{d} kQ} \psi(n) = \mathrm{e}^{\frac{2\pi\mathrm{i}}{d} kn}\, \psi(n)$
■ Displacement operators	$D5$	$D(q,p) = \mathrm{e}^{-\frac{\mathrm{i}}{2\hbar} pq}\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar} p\hat{q}}\, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} q\hat{p}}$	$\longleftrightarrow$	**$D5'$**	$\boxed{D(n,k) = \mathrm{e}^{-\frac{\pi\mathrm{i}}{d} nk}\, \mathrm{e}^{\frac{2\pi\mathrm{i}}{d} kQ}\, \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d} nP}}$
■ Parity operators	$D6$	$\Pi(q,p) = D(q,p)\, \Pi\, D(q,p)^\dagger$	$\longleftrightarrow$	$D6'$	$\Pi(n,k) = D(n,k)\, \Pi\, D(n,k)^\dagger$ $\Pi\psi(x) = \psi(-x)$
■ Wigner function	$D7$	$\mathcal{W}_A(q,p) = \frac{1}{\pi\hbar} \mathrm{tr}(\Pi(q,p)\, A)$	$\longleftrightarrow$	$D7'$	$\mathcal{W}_A(n,k) = \frac{1}{d} \mathrm{tr}(\Pi(n,k)\, A)$
■ Ambiguity function	$D8$	$\chi_A(q,p) = \frac{1}{\pi\hbar} \mathrm{tr}(D(q,p)^\dagger\, A)$	$\longleftrightarrow$	**$D8'$**	$\chi_A(n,k) = \frac{1}{d} \mathrm{tr}(D(n,k)^\dagger\, A)$

■ **Similarity of (formal) properties**

$P1$	$\mathrm{e}^{\frac{\mathrm{i}}{\hbar} p\hat{q}} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} q\hat{p}} = \mathrm{e}^{\frac{\mathrm{i}}{\hbar} pq}\, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} q\hat{p}}\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar} p\hat{q}}$	$\longleftrightarrow$	$P1'$	$\mathrm{e}^{\frac{2\pi\mathrm{i}}{d} kQ}\, \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d} nP} = \mathrm{e}^{\frac{2\pi\mathrm{i}}{d} nk}\, \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d} nP}\, \mathrm{e}^{\frac{2\pi\mathrm{i}}{d} kQ}$
$P2$	$D(q,p)\psi(q') = \mathrm{e}^{-\frac{\mathrm{i}}{2\hbar} pq}\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar} pq'}\, \psi(q'-q)$ $\Pi(q,p)\psi(q') = \mathrm{e}^{\frac{2\mathrm{i}}{\hbar} p(q'-q)}\, \psi(2q-q')$	$\longleftrightarrow$	**$P2'$**	$D(n,k)\psi(n') = \mathrm{e}^{-\frac{\pi\mathrm{i}}{d} nk}\, \mathrm{e}^{\frac{2\pi\mathrm{i}}{d} kn'}\, \psi(n'-n)$ $\Pi(n,k)\psi(n') = \mathrm{e}^{\frac{4\pi\mathrm{i}}{d} k(n'-n)}\, \psi(2n-n')$
$P3$	$\langle q'\|D(q,p)\|q''\rangle = \mathrm{e}^{\frac{\mathrm{i}}{2\hbar} p(q'+q'')}\, \delta_{q'}(q''+q)$ $\langle q'\|\Pi(q,p)\|q''\rangle = \mathrm{e}^{\frac{\mathrm{i}}{\hbar} p(q'-q'')}\, \delta_{2q}(q'+q'')$	$\longleftrightarrow$	**$P3'$**	$\langle n'\|D(n,k)\|n''\rangle = \mathrm{e}^{\frac{\pi\mathrm{i}}{d} k(n'+n'')}\, \delta_{n'}(n''+n)$ $\langle n'\|\Pi(n,k)\|n''\rangle = \mathrm{e}^{\frac{2\pi\mathrm{i}}{d} k(n'-n'')}\, \delta_{2n}(n'+n'')$
$P4$	$D(q,p)\, D(q',p') = \mathrm{e}^{\frac{\mathrm{i}}{2\hbar}(pq'-p'q)} D(q+q', p+p')$	$\longleftrightarrow$	**$P4'$**	$D(n,k)\, D(n',k') = \mathrm{e}^{\frac{\pi\mathrm{i}}{d}(kn'-k'n)} D(\widetilde{n+n'}, \widetilde{k+k'})$
$P5$	$D(q,p)\, D(q',p') = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(pq'-p'q)} D(q',p')\, D(q,p)$	$\longleftrightarrow$	**$P5'$**	$D(n,k)\, D(n',k') = \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(kn'-k'n)} D(n,k)\, D(n',k')$
$P6$	$D(q,p)^\dagger = D(-q,-p) = D^{-1}(q,p)$	$\longleftrightarrow$	**$P6'$**	$D(n,k)^\dagger = D(-n,-k) = D^{-1}(n,k)$
$P7$	$A = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{W}_A(q,p)\, \Pi(q,p) dq dp$	$\longleftrightarrow$	$P7'$	$A = \sum_{n=-s}^{s} \sum_{k=-s}^{s} \mathcal{W}_A(n,k)\, \Pi(n,k)$
$P8$	$\mathcal{W}_A(q,p) = \frac{1}{\pi\hbar} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{2\mathrm{i}}{\hbar} pq'}\, \langle q+q'\|A\|q-q'\rangle dq'$	$\longleftrightarrow$	$P8'$	$\mathcal{W}_A(n,k) = \frac{1}{d} \sum_{n'\in\mathbb{Z}_d} \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d} kn'}\, \langle n+n'\|A\|n-n'\rangle$
$P9$	$\mathcal{W}_\psi(q,p) = \frac{1}{\pi\hbar} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{2\mathrm{i}}{\hbar} pq'}\, \psi(q+q')\, \overline{\psi(q-q')} dq'$	$\longleftrightarrow$	$P9'$	$\mathcal{W}_\psi(n,k) = \frac{1}{d} \sum_{n'\in\mathbb{Z}_d} \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d} kn'}\, \psi(n+n')\overline{\psi(n-n')}$
$P10$	$\int_{-\infty}^{\infty} \mathcal{W}_\psi(q,p) dp = \|\psi(q)\|^2$ $\int_{-\infty}^{\infty} \mathcal{W}_\psi(q,p) dq = \|F[\psi](p)\|^2$	$\longleftrightarrow$	$P10'$	$\sum_{k=-s}^{s} \mathcal{W}_\psi(n,k) = \|\psi(n)\|^2$ $\sum_{n=-s}^{s} \mathcal{W}_\psi(n,k) = \|F[\psi](k)\|^2$
$P11$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{W}_A(q,p) dq dp = \mathrm{tr}\, A$	$\longleftrightarrow$	$P11'$	$\sum_{n=-s}^{s} \sum_{k=-s}^{s} \mathcal{W}_A(n,k) = \mathrm{tr}\, A$
$A12$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{W}_\psi^2(q,p) dq dp = \frac{\|\|\psi\|\|^4}{2\pi\hbar}$	$\longleftrightarrow$	$A12'$	$\sum_{n=-s}^{s} \sum_{k=-s}^{s} \mathcal{W}_\psi^2(n,k) = \frac{\|\|\psi\|\|^4}{d}$
$P13$	$\|\langle\psi\|\varphi\rangle\|^2 = 2\pi\hbar \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{W}_\psi(q,p) \mathcal{W}_\varphi(q,p) dq dp$	$\longleftrightarrow$	$P13'$	$\|\langle\psi\|\varphi\rangle\|^2 = d \sum_{n=-s}^{s} \sum_{k=-s}^{s} \mathcal{W}_\psi(n,k) \mathcal{W}_\varphi(n,k)$

■ **Remarks:** The constant d corresponds to $2\pi\hbar$ in D1-D6, and to $\pi\hbar$ in D7-D8 (!?).

$D(n,k)$ are unitary, but their definition is not modulo d invariant.
$\Pi(n,k)$ are self-adjoint, and their definition is modulo d invariant. $\Rightarrow$

$P15'$ $\Pi(n,k) \neq \frac{1}{d} \sum_{n'=-s}^{s} \sum_{k'=-s}^{s} \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(kn'-k'n)}\, D(n',k')$,

$P16'$ $\mathcal{W}_A(n,k) \neq \frac{1}{d} \sum_{n'=-s}^{s} \sum_{k'=-s}^{s} \mathrm{e}^{\pm\frac{2\pi\mathrm{i}}{d}(kn'-k'n)}\, \chi_A(n',k')$.

"Signal" version

■ **Definition.** For $d=2s+1\in\{1,3,5,...\}$, the well-known correspondence

■ Fourier transform	*D0*	$F[\psi](p)=\frac{1}{\sqrt{2\pi\hbar}}\int\limits_{-\infty}^{\infty}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}pq}\psi(q)dq$	$\longleftrightarrow$	*D0′*	$F[\psi](k)=\frac{1}{\sqrt{d}}\sum\limits_{n=-s}^{s}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}kn}\,\psi(n)$

suggests for continuous-discrete correspondence the following choice:

■ Position operator	*D1*	$\hat{q}\psi(q)=q\,\psi(q)$	$\longleftrightarrow$	*D1′*	$Q\psi(n)=\tilde{n}\,\psi(n)$	$-s\leq\tilde{n}\leq s$, $\tilde{n}\equiv n(mod\,d)$
■ Momentum operator	*D2*	$\hat{p}=F^{\dagger}\hat{q}F$	$\longleftrightarrow$	*D2′*	$P=F^{\dagger}QF$	
■ Translation operator	*D3*	$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}q\hat{p}}\psi(q')=\psi(q'-q)$	$\longleftrightarrow$	*D3′*	$\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}\psi(n')=\psi(n'-n)$	
■ Translation operator	*D4*	$\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p\hat{q}}\psi(q)=\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq}\,\psi(q)$	$\longleftrightarrow$	*D4′*	$\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}\psi(n)=\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kn}\,\psi(n)$	
■ Displacement operators	*D5*	$D(q,p)=\mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}pq}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p\hat{q}}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}q\hat{p}}$	$\longleftrightarrow$	**D5″**	$\mathbf{D}(n,k)=\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}$	
■ Parity operators	*D6*	$\Pi(q,p)=D(q,p)\,\Pi\,D(q,p)^{\dagger}$	$\longleftrightarrow$	*D6′*	$\Pi(n,k)=\mathbf{D}(n,k)\,\Pi\,\mathbf{D}(n,k)^{\dagger}$	$\Pi\psi(x)=\psi(-x)$
■ Wigner function	*D7*	$\mathcal{W}_A(q,p)=\frac{1}{\pi\hbar}\mathrm{tr}(\Pi(q,p)\,A)$	$\longleftrightarrow$	*D7′*	$\mathcal{W}_A(n,k)=\frac{1}{d}\mathrm{tr}(\Pi(n,k)\,A)$	
■ Ambiguity function	*D8*	$\chi_A(q,p)=\frac{1}{\pi\hbar}\mathrm{tr}(D(q,p)^{\dagger}\,A)$	$\longleftrightarrow$	**D8″**	$\chi_{\mathbf{A}}(n,k)=\frac{1}{d}\mathrm{tr}(\mathbf{D}(n,k)^{\dagger}\,A)$	

■ **Similarity of (formal) properties**

P1	$\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p\hat{q}}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}q\hat{p}}=\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}q\hat{p}}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p\hat{q}}$	$\longleftrightarrow$	*P1′*	$\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}=\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}nk}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}$
P2	$D(q,p)\psi(q')=\mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}pq}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq'}\,\psi(q'-q)$ $\Pi(q,p)\psi(q')=\mathrm{e}^{\frac{2\mathrm{i}}{\hbar}p(q'-q)}\,\psi(2q-q')$	$\longleftrightarrow$	**P2″**	$\mathbf{D}(n,k)\psi(n')=\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kn'}\,\psi(n'-n)$ $\Pi(n,k)\psi(n')=\mathrm{e}^{\frac{4\pi\mathrm{i}}{d}k(n'-n)}\,\psi(2n-n')$
P3	$\langle q'\vert D(q,p)\vert q''\rangle=\mathrm{e}^{\frac{\mathrm{i}}{2\hbar}p(q'+q'')}\,\delta_{q'}(q''+q)$ $\langle q'\vert\Pi(q,p)\vert q''\rangle=\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p(q'-q'')}\,\delta_{2q}(q'+q'')$	$\longleftrightarrow$	**P3″**	$\langle n'\vert\mathbf{D}(n,k)\vert n''\rangle=\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kn'}\,\delta_{n''}(n'-n)$ $\langle n'\vert\Pi(n,k)\vert n''\rangle=\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k(n'-n'')}\,\delta_{2n}(n'+n'')$
P4	$D(q,p)\,D(q',p')=\mathrm{e}^{\frac{\mathrm{i}}{2\hbar}(pq'-p'q)}D(q+q',p+p')$	$\longleftrightarrow$	**P4″**	$\mathbf{D}(n,k)\,\mathbf{D}(n',k')=\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nk'}\mathbf{D}(n+n',k+k')$
P5	$D(q,p)\,D(q',p')=\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(pq'-p'q)}D(q',p')\,D(q,p)$	$\longleftrightarrow$	**P5″**	$\mathbf{D}(n,k)\,\mathbf{D}(n',k')=\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(kn'-k'n)}\mathbf{D}(n,k)\,\mathbf{D}(n',k')$
P6	$D^{-1}(q,p)=D(-q,-p)=D(q,p)^{\dagger}$	$\longleftrightarrow$	**P6″**	$\mathbf{D}^{-1}(n,k)=\mathbf{D}(n,k)^{\dagger}=\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nk}\mathbf{D}(-n,-k)$
P7	$A=\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\mathcal{W}_A(q,p)\,\Pi(q,p)dqdp$	$\longleftrightarrow$	*P7′*	$A=\sum\limits_{n=-s}^{s}\sum\limits_{k=-s}^{s}\mathcal{W}_A(n,k)\,\Pi(n,k)$
P8	$\mathcal{W}_A(q,p)=\frac{1}{\pi\hbar}\int\limits_{-\infty}^{\infty}\mathrm{e}^{-\frac{2\mathrm{i}}{\hbar}pq'}\,\langle q+q'\vert A\vert q-q'\rangle dq'$	$\longleftrightarrow$	*P8′*	$\mathcal{W}_A(n,k)=\frac{1}{d}\sum\limits_{n'\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}kn'}\,\langle n+n'\vert A\vert n-n'\rangle$
P9	$\mathcal{W}_{\psi}(q,p)=\frac{1}{\pi\hbar}\int\limits_{-\infty}^{\infty}\mathrm{e}^{-\frac{2\mathrm{i}}{\hbar}pq'}\,\psi(q+q')\,\overline{\psi(q-q')}dq'$	$\longleftrightarrow$	*P9′*	$\mathcal{W}_{\psi}(n,k)=\frac{1}{d}\sum\limits_{n'\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}kn'}\,\psi(n+n')\overline{\psi(n-n')}$
P10	$\int\limits_{-\infty}^{\infty}\mathcal{W}_{\psi}(q,p)dp=\vert\psi(q)\vert^2$	$\longleftrightarrow$	*P10′*	$\sum\limits_{k=-s}^{s}\mathcal{W}_{\psi}(n,k)=\vert\psi(n)\vert^2$
P11	$\int\limits_{-\infty}^{\infty}\mathcal{W}_{\psi}(q,p)dq=\vert F[\psi](p)\vert^2$	$\longleftrightarrow$	*P11′*	$\sum\limits_{n=-s}^{s}\mathcal{W}_{\psi}(n,k)=\vert F[\psi](k)\vert^2$
P12	$\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\mathcal{W}_A(q,p)dqdp=\mathrm{tr}\,A$	$\longleftrightarrow$	*P12′*	$\sum\limits_{n=-s}^{s}\sum\limits_{k=-s}^{s}\mathcal{W}_A(n,k)=\mathrm{tr}\,A$
P13	$\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\mathcal{W}_{\psi}^2(q,p)dqdp=\frac{\vert\vert\psi\vert\vert^4}{2\pi\hbar}$	$\longleftrightarrow$	*A13′*	$\sum\limits_{n=-s}^{s}\sum\limits_{k=-s}^{s}\mathcal{W}_{\psi}^2(n,k)=\frac{\vert\vert\psi\vert\vert^4}{d}$
P14	$\vert\langle\psi\vert\varphi\rangle\vert^2=2\pi\hbar\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\mathcal{W}_{\psi}(q,p)\mathcal{W}_{\varphi}(q,p)dqdp$	$\longleftrightarrow$	*P14′*	$\vert\langle\psi\vert\varphi\rangle\vert^2=d\sum\limits_{n=-s}^{s}\sum\limits_{k=-s}^{s}\mathcal{W}_{\psi}(n,k)\mathcal{W}_{\varphi}(n,k)$

■ **Remarks:** The constant d corresponds to $2\pi\hbar$ in D1-D6, and to $\pi\hbar$ in D7-D8 (!?).

$\mathbf{D}(n,k)$ are unitary, and their definition is modulo d invariant.
$\Pi(n,k)$ are self-adjoint, and their definition is modulo d invariant.

but

P15″ $\quad\Pi(n,k)\neq\frac{1}{d}\sum\limits_{n'=-s}^{s}\sum\limits_{k'=-s}^{s}\mathrm{e}^{\pm\frac{2\pi\mathrm{i}}{d}(kn'-k'n)}\,\mathbf{D}(n',k'),$

P16″ $\quad\mathcal{W}_A(n,k)\neq\frac{1}{d}\sum\limits_{n'=-s}^{s}\sum\limits_{k'=-s}^{s}\mathrm{e}^{\pm\frac{2\pi\mathrm{i}}{d}(kn'-k'n)}\,\chi_{\mathbf{A}}(n',k').$

■ Some details

In the case of $\boxed{P15'}$, the computation

$$\begin{aligned}\frac{1}{d}\sum_{n'=-s}^{s}\sum_{k'=-s}^{s}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(kn'-k'n)}\,D(n',k')\psi(n'') &= \frac{1}{d}\sum_{n'=-s}^{s}\sum_{k'=-s}^{s}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(kn'-k'n)}\,\mathrm{e}^{-\frac{\pi\mathrm{i}}{d}n'k'}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k'n''}\,\psi(n''-n')\\ &= \sum_{n'=-s}^{s}\left(\frac{1}{d}\sum_{k'=-s}^{s}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k'(n''-\frac{n'}{2}-n)}\right)\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kn'}\,\psi(n''-n')\\ &= \sum_{n'=-s}^{s}\delta_{n'}(2n''-2n)\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kn'}\,\psi(n''-n')\\ &= \mathrm{e}^{\frac{4\pi\mathrm{i}}{d}k(n''-n)}\,\psi(n''-2n''+2n)\\ &= \mathrm{e}^{\frac{4\pi\mathrm{i}}{d}k(n''-n)}\,\psi(2n-n'')\\ &= \Pi(n,k)\psi(n'')\end{aligned}$$

is not correct because n' takes even as well odd values, and hence

$$\frac{1}{d}\sum_{k'=-s}^{s}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k'(n''-\frac{n'}{2}-n)}\neq\delta_{n'}(2n''-2n).$$

In the case of $\boxed{P15''}$, instead of the definition $\boxed{\mathbf{D5'}}$ we use $\boxed{\mathbf{D5''}}$, and we get

$$\begin{aligned}\frac{1}{d}\sum_{n'=-s}^{s}\sum_{k'=-s}^{s}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(kn'-k'n)}\,\mathbf{D}(n',k')\psi(n'') &= \frac{1}{d}\sum_{n'=-s}^{s}\sum_{k'=-s}^{s}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(kn'-k'n)}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k'n''}\,\psi(n''-n')\\ &= \sum_{n'=-s}^{s}\left(\frac{1}{d}\sum_{k'=-s}^{s}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k'(n''-n)}\right)\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kn'}\,\psi(n''-n')\\ &= \sum_{n'=-s}^{s}\delta_{n}(n'')\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kn'}\,\psi(n''-n')\\ &= \sum_{n'=-s}^{s}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kn'}\,\psi(n-n')\\ &\neq \Pi(n,k)\psi(n'')\end{aligned}$$