DIFFERENTIAL EQUATIONS - Solved Exercises 3

Nicolae Cotfas, version 8 April 2026 (for future updates see https://unibuc.ro/user/nicolae.cotfas/)

EXERCISES

■ Exercise 1

Find the general solution of the equations:

$\boxed{1}$ $y''+4y'+3y=0.$

$\boxed{2}$ $y''+4y'+4y=0.$

$\boxed{3}$ $y''+4y'+5y=0.$

$\boxed{4}$ $y'''-y''+y'-y=0.$

■ Exercise 2

Solve the equations

$\boxed{5}$ $y'=\frac{1}{x}y.$

$\boxed{6}$ $y'=\frac{1}{x}y+x\mathrm{e}^{-x}.$

$\boxed{7}$ $x^2y''+xy'+y=0.$

■ Exercise 3

Solve the equations

$\boxed{8}$ $(2x+y)dx+xdy=0.$

$\boxed{9}$ $(2+\frac{y}{x})dx+dy=0.$

■ Exercise 4

Solve the sytem of differential equations

$\boxed{10}$ $\begin{cases} y_1'=2y_1-y_2 \\ y_2'=y_1+2y_2 \end{cases}$

■ Exercise 5

$\boxed{11}$ Find the eigenfunctions of the momentum operator

$$\hat{p}=-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$$

corresponding to the eigenvalues $\lambda\in\mathbb{R}$, that is, solve the differential equation

$$-\mathrm{i}\frac{\mathrm{d}\psi}{\mathrm{d}x}=\lambda\psi.$$

by looking for complex functions as solutions.

■ Exercise 6

$\boxed{12}$ Find the linear second-order differential equation with variable coefficients satisfied by the function

$$\psi(x)=x\mathrm{e}^{-\frac{1}{2}x^2}.$$

SOME DEFINITIONS, THEOREMS and REMARKS

$\boxed{T1}$ **Theorem** (*Linear equations with constant coefficients*).

The space of all the real solutions of

$$a_0y^{(n)}+a_1y^{(n-1)}+\cdots+a_{n-1}y'+a_ny=0 \qquad (1)$$

where $a_0,\ldots,a_n\in\mathbb{R}$, is a real vector space of dimension n.

$\boxed{D1}$ **Definition**. The polynomial

$$P(\lambda)=a_0\lambda^n+a_1\lambda^{n-1}+\cdots+a_{n-1}\lambda+a_n$$

is called the *characteristic polynomial* of the equation (1).

$\boxed{T2}$ **Theorem** (Particular solutions).

$$\boxed{y(x)=\mathrm{e}^{\lambda x}\text{ is a solution of (1)}} \Leftrightarrow \boxed{P(\lambda)=0}$$

$\boxed{D2}$ **Definition** (*Complex exponential*).

$$\boxed{\mathrm{e}^{(\alpha+\beta\mathrm{i})x}=\mathrm{e}^{\alpha x}\cos\beta x+\mathrm{i}\,\mathrm{e}^{\alpha x}\sin\beta x}$$

$\boxed{T3}$ **Theorem**. General solution of $a_0y''+a_1y'+a_2y=0.$

$P(\lambda)=a_0\lambda^2+a_1\lambda+a_2$ has the roots $\lambda_{1,2}=\frac{-a_1\pm\sqrt{a_1^2-4a_0\,a_2}}{2a_0}.$

General solution:

- $\lambda_1\neq\lambda_2\in\mathbb{R} \quad\Rightarrow\quad y(x)=C_1\,\mathrm{e}^{\lambda_1x}+C_2\,\mathrm{e}^{\lambda_2x}.$
- $\lambda_1=\lambda_2=\lambda \quad\Rightarrow\quad y(x)=C_1\,\mathrm{e}^{\lambda x}+C_2\,x\,\mathrm{e}^{\lambda x}.$
- $\lambda_{1,2}=\alpha\pm\beta\mathrm{i} \quad\Rightarrow\quad y(x)=C_1\,\mathrm{e}^{\alpha x}\,\cos\beta x+C_2\,\mathrm{e}^{\alpha x}\sin\beta x.$

$\boxed{T4}$ **Theorem** (Euler's equation).

$$\boxed{a_0x^ny^{(n)}+\ldots+a_{n-1}xy'+a_ny=0}\ \xrightarrow[x=\mathrm{e}^t]{\text{change}}\ \text{linear equation with constant coefficients}$$

$\boxed{T5}$ **Theorem** (*Primitives* of a continuous function)

$$\boxed{\begin{array}{c} f:(a,b)\to\mathbb{R} \\ \text{continuous} \end{array} \Rightarrow \begin{array}{c} \text{Primitives of } f \text{ are:} \\ F:(a,b)\to\mathbb{R}, \\ \boxed{F(x)=\int_{x_0}^{x} f(t)\,dt+C} \\ x_0\in(a,b)\text{ is fixed} \end{array}}$$

We have $F'(x)=f(x)$, that is

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{x_0}^{x}f(t)\,dt\right)=f(x)}$$

for any $x\in(a,b)$.

$\boxed{T6}$ **Theorem** (*Separable equations*).

The solution $y(x)$ of $\boxed{y'=f(x)\,g(y)}$ is defined by $\boxed{\int_{y_0}^{y}\frac{1}{g(u)}du=\int_{x_0}^{x}f(t)\,dt+C}$

$x_0,\ y_0$ constants, $g(y_0)\neq0.$

$\boxed{T7}$ **Linear equation**.

$\boxed{y'=f(x)y}$ has the general solution $\boxed{y(x)=C\,\mathrm{e}^{\int_{x_0}^{x}f(t)dt}}$

$\boxed{T8}$ **Method of the variation of parameter**.

$\boxed{y'=f(x)y+g(x)}$ admits a particular solution of the form $y_p(x)=C(x)\,\mathrm{e}^{\int_{x_0}^{x}f(t)dt}.$

$\boxed{T9}$ **Linear non-homogeneous equation**.

$$\boxed{\begin{array}{c}\text{general solution of}\\ y'=f(x)y+g(x)\end{array}}=\boxed{\begin{array}{c}\text{general solution of}\\ y'=f(x)y\end{array}}+\boxed{\begin{array}{c}\text{a particular solution of}\\ y'=f(x)y+g(x)\end{array}}$$

$\boxed{T10}$ **Theorem** (*Exact equations*).

$$\boxed{\begin{array}{c}\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}\\ \text{in a simply}\\ \text{connected}\\ \text{domain } D\end{array}}\Rightarrow\boxed{\begin{array}{l}\text{The function } F:D\to\mathbb{R},\\ \qquad F(x,y)=\int_\gamma Pdx+Qdy,\\ \text{where } \gamma:[a,b]\to D \text{ is an arbitrary path}\\ \text{connecting a fixed point } (x_0,y_0) \text{ with } (x,y),\\ \text{defines a function satisfying the relation}\\ \qquad P(x,y)dx+Q(x,y)dy=dF\end{array}}$$

$\boxed{R3}$ **Remark**.

In D, the equation $P(x,y)dx+Q(x,y)dy=0$ can be written as $dF=0$, and its solution is described implicitly by $F(x,y)=C$.

$\boxed{R4}$ **Remark**. By denoting $Y(x)=\begin{pmatrix}y_1(x)\\ y_2(x)\end{pmatrix}$, $A=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$,

$$\boxed{\begin{cases}y_1'=a_{11}y_1+a_{12}y_2\\ y_2'=a_{21}y_1+a_{22}y_2\end{cases}}\ \text{can be written as}\ \boxed{Y'=AY}$$

$\boxed{T11}$ **Theorem**.

The space of all the real solutions of $Y'=AY$, where $a_{ij}\in\mathbb{R}$, is a real vector space of dimension 2.

$\boxed{D4}$ **Definition**. The polynomial

$$P(\lambda)=\begin{vmatrix}a_{11}-\lambda & a_{12}\\ a_{21} & a_{22}-\lambda\end{vmatrix}=\lambda^2-(a_{11}+a_{22})\lambda+a_{11}a_{22}-a_{12}a_{21}$$

is called the *characteristic polynomial* of $Y'=AY$.

$\boxed{T12}$ **Theorem** (Particular non-null solutions)

$$Y(x)=\begin{pmatrix}p\\ q\end{pmatrix}\mathrm{e}^{\lambda x}\neq\begin{pmatrix}0\\ 0\end{pmatrix}\ \text{satisfies } Y'=AY\ \Leftrightarrow\begin{cases}P(\lambda)=0 \quad\text{and}\\ A\begin{pmatrix}p\\ q\end{pmatrix}=\lambda\begin{pmatrix}p\\ q\end{pmatrix}.\end{cases}$$

$\boxed{T13}$ **Theorem.** If λ_1 and λ_2 are the solutions of $P(\lambda)=0$, then:

$$\left.\begin{array}{l}\bullet\quad \lambda_1,\ \lambda_2\in\mathbb{R}\\ A\begin{pmatrix}p_j\\ q_j\end{pmatrix}=\lambda_j\begin{pmatrix}p_j\\ q_j\end{pmatrix}\\ \text{linearly independent}\end{array}\right\}\Rightarrow\ Y(x)=C_1\begin{pmatrix}p_1\\ q_1\end{pmatrix}\mathrm{e}^{\lambda_1x}+C_2\begin{pmatrix}p_2\\ q_2\end{pmatrix}\mathrm{e}^{\lambda_2x}.$$

$$\left.\begin{array}{l}\bullet\quad \lambda_{1,2}=\alpha\pm\beta\mathrm{i}\notin\mathbb{R}\\ A\begin{pmatrix}p\\ q\end{pmatrix}=(\alpha+\beta\mathrm{i})\begin{pmatrix}p\\ q\end{pmatrix}\end{array}\right\}\Rightarrow\ \begin{array}{l}Y(x)=C_1\,\mathfrak{Re}\left\{\begin{pmatrix}p\\ q\end{pmatrix}\mathrm{e}^{(\alpha+\beta\mathrm{i})x}\right\}\\ \qquad +C_2\,\mathfrak{Im}\left\{\begin{pmatrix}p\\ q\end{pmatrix}\mathrm{e}^{(\alpha+\beta\mathrm{i})x}\right\}.\end{array}$$

SOLUTIONS

Exercise 1.

We use T2 and T3.

$\boxed{1}$ $\lambda^2+4\lambda+3=0 \quad\Rightarrow\quad \lambda_1=-1,\ \lambda_2=-3.$

$T3 \quad\Rightarrow\quad y(x)=C_1\,\mathrm{e}^{-x}+C_2\,\mathrm{e}^{-3x}.$

$\boxed{2}$ $\lambda^2+4\lambda+4=0 \quad\Rightarrow\quad \lambda_1=\lambda_2=-2.$

$T3 \quad\Rightarrow\quad y(x)=C_1\,\mathrm{e}^{-2x}+C_2\,x\,\mathrm{e}^{-2x}.$

$\boxed{3}$ $\lambda^2+4\lambda+5=0 \quad\Rightarrow\quad \lambda_{1,2}=-2\pm\mathrm{i}.$

$T3 \quad\Rightarrow\quad y(x)=C_1\,\mathrm{e}^{-2x}\cos x+C_2\,\mathrm{e}^{-2x}\sin x.$

$\boxed{4}$ $(\lambda-1)(\lambda^2+1)=0 \quad\Rightarrow\quad \lambda_1=1,\ \lambda_{2,3}=\pm\mathrm{i}.$

$T3 \quad\Rightarrow\quad y(x)=C_1\,\mathrm{e}^{x}+C_2\,\cos x+C_3\,\sin x.$

Exercise 2

$\boxed{5}$ We use T7.

The equation can be written successively as follows:

$$y'=\tfrac{1}{x}y,\qquad \tfrac{y'}{y}=\tfrac{1}{x},\qquad (\ln y)'=\tfrac{1}{x},$$
$$\ln y=\ln x+\ln C,\qquad y=Cx.$$

$\boxed{6}$ We use T8 and T9. Looking for a particular solution of the form $y_p(x)=C(x)\,x$ we get successively:

$C'\,x+C=\tfrac{1}{x}\;C\;x+x\mathrm{e}^{-x},\qquad C'=\mathrm{e}^{-x},$

$C(x)=\int \mathrm{e}^{-x}dx=-\mathrm{e}^{-x}$, and consequently $y_p(x)=-x\,\mathrm{e}^{-x}$.

T9 ⇒ the general solution of $y'=\frac{1}{x}y+x\mathrm{e}^{-x}$ is

$$y(x)=C\,x-x\,\mathrm{e}^{-x}.$$

$\boxed{7}$ We use T4, T2 and T3.

We use the change of variables:

$$\begin{matrix} x\mapsto t, \\ y\mapsto z,\end{matrix}\quad \text{satisfying}\quad \begin{matrix} x=\mathrm{e}^t, & y(x)=z(\ln x), \\ t=\ln x, & z(t)=y(\mathrm{e}^t).\end{matrix}$$

Since $y'(x)=z'(\ln x)(\ln x)'=\frac{1}{x}\,z'(\ln x)$,

$$y''(x)=-\tfrac{1}{x^2}\,z'(\ln x)+\tfrac{1}{x^2}\,z''(\ln x),$$

in the new variables, the equation becomes

$$\mathrm{e}^{2t}\left(-\tfrac{1}{\mathrm{e}^{2t}}\,z'+\tfrac{1}{\mathrm{e}^{2t}}\,z''\right)+\mathrm{e}^t\tfrac{1}{\mathrm{e}^t}\,z'+z=0,$$

that is $\quad z''+z=0.$

Because $\lambda^2+1=0 \quad\Rightarrow\quad \lambda_{1,2}=\pm\mathrm{i}$,

and consequently, $\quad T3 \quad\Rightarrow\quad z(t)=C_1\,\cos t+C_2\,\sin t,$

we get $\quad y(x)=C_1\,\cos(\ln x)+C_2\,\sin(\ln x).$

Remark.

The given equation can be written as

$$\left(x^2\tfrac{\mathrm{d}^2}{\mathrm{d}x^2}+x\tfrac{\mathrm{d}}{\mathrm{d}x}+1\right)y=0.$$

and by using the operatorial relation $\frac{\mathrm{d}}{\mathrm{d}x}=\frac{\mathrm{d}t}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}t}=\mathrm{e}^{-t}\frac{\mathrm{d}}{\mathrm{d}t}$,

we get the equation for $z(t)$,

$$\left(\mathrm{e}^{2t}\mathrm{e}^{-t}\tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{-t}\tfrac{\mathrm{d}}{\mathrm{d}t}+\mathrm{e}^{t}\mathrm{e}^{-t}\tfrac{\mathrm{d}}{\mathrm{d}t}+1\right)z=0.$$

namely, $\quad z''+z=0.$

Exercise 3

$\boxed{8}$ The equation is exact in $D=\mathbb{R}^2$,

$$\tfrac{\partial(2x+y)}{\partial y}=1=\tfrac{\partial\,x}{\partial x}.$$

By using T10, R3 and the path

$$\gamma:[0,1]\to\mathbb{R}^2,\quad \gamma(t)=(xt,yt)$$

connecting $(0,0)$ with (x,y), we get

$$F(x,y)=\int_\gamma(2x+y)dx+xdy=\int_0^1[(2xt+yt)x+xt\,y]\,dt$$
$$=(2x^2+2xy)\int_0^1 t\,dt=(2x^2+2xy)\tfrac{t^2}{2}\Big|_{t=0}^{t=1}=x^2+xy.$$

The given equation can be written as (see R3)

$$d(x^2+xy)=0,$$

and its solution is described by

$$x^2+xy=C,$$

namely $\quad y(x)=\frac{C}{x}-x \quad$ for $\ x\neq 0.$

$\boxed{9}$ This equation can be reduced to the exact equation $\boxed{8}$ by multiplying it with the integrating factor $\mu(x)=x$.

The two equations have the same solutions, namely

$$x^2+xy=C.$$

Exercise 4

$\boxed{10}$ The system can be written as $Y'=AY$, where.

$$Y(x)=\begin{pmatrix} y_1(x)\\ y_2(x)\end{pmatrix},\ A=\begin{pmatrix} 2 & -1\\ 1 & 2\end{pmatrix},$$

In this case,

$$\begin{vmatrix} 2-\lambda & -1\\ 1 & 2-\lambda\end{vmatrix}=0 \;\Rightarrow\; \lambda_{1,2}=2\pm\mathrm{i}\quad\text{and}$$

$$\begin{pmatrix} 2 & -1\\ 1 & 2\end{pmatrix}\begin{pmatrix} p\\ q\end{pmatrix}=(2+\mathrm{i})\begin{pmatrix} p\\ q\end{pmatrix}\quad\Rightarrow\quad\begin{pmatrix} p\\ q\end{pmatrix}=\begin{pmatrix} p\\ -p\mathrm{i}\end{pmatrix}.$$

By choosing the eigenvector $\begin{pmatrix} 1\\ -\mathrm{i}\end{pmatrix}$, we get (see D2 and T13)

$$\begin{pmatrix} y_1(x)\\ y_2(x)\end{pmatrix}=C_1\,\mathfrak{Re}\left\{\begin{pmatrix} 1\\ -\mathrm{i}\end{pmatrix}\mathrm{e}^{(2+\mathrm{i})x}\right\}+C_2\,\mathfrak{Im}\left\{\begin{pmatrix} 1\\ -\mathrm{i}\end{pmatrix}\mathrm{e}^{(2+\mathrm{i})x}\right\}$$
$$=C_1\begin{pmatrix} \mathrm{e}^{2x}\cos x\\ \mathrm{e}^{2x}\sin x\end{pmatrix}+C_2\begin{pmatrix} \mathrm{e}^{2x}\sin x\\ -\mathrm{e}^{2x}\cos x\end{pmatrix}.$$

that is

$$y_1(x)=C_1\,\mathrm{e}^{2x}\,\cos x+C_2\,\mathrm{e}^{2x}\,\sin x,$$
$$y_2(x)=C_1\,\mathrm{e}^{2x}\,\sin x-C_2\,\mathrm{e}^{2x}\,\cos x.$$

Exercise 5.

$\boxed{11}$ The equation can be written successively as follows:

$$-\mathrm{i}\,\psi'=\lambda\,\psi,$$
$$\psi'=\mathrm{i}\,\lambda\,\psi,$$
$$\tfrac{\psi'}{\psi}=\mathrm{i}\,\lambda$$
$$\psi(x)=C\,\mathrm{e}^{\mathrm{i}\,\lambda x},$$
$$\psi(x)=C\,(\,\cos(\lambda x)+\mathrm{i}\,\sin(\lambda x)\,).$$

Exercise 6.

$\boxed{12}$ Because

$$\psi(x)=x\mathrm{e}^{-\frac{1}{2}x^2},$$
$$\psi'(x)=\mathrm{e}^{-\frac{1}{2}x^2}-x^2\,\mathrm{e}^{-\frac{1}{2}x^2},$$
$$\psi''(x)=-x\,\mathrm{e}^{-\frac{1}{2}x^2}-2x\,\mathrm{e}^{-\frac{1}{2}x^2}+x^3\,\mathrm{e}^{-\frac{1}{2}x^2},$$
$$\psi''(x)=-3x\,\mathrm{e}^{-\frac{1}{2}x^2}+x^3\,\mathrm{e}^{-\frac{1}{2}x^2},$$

the function ψ satisfies the equation

$$-\psi''+x^2\,\psi=3\psi.$$

Remark.

The last equation written in the form

$$-\tfrac{1}{2}\psi''+\tfrac{1}{2}x^2\,\psi=\tfrac{3}{2}\psi.$$

shows that ψ is an eigenfunction of the Hamiltonian

$$\hat{H}=-\tfrac{1}{2}\tfrac{\mathrm{d}^2}{\mathrm{d}x^2}+\tfrac{1}{2}x^2$$

of the quantum harmonic oscillator

corresponding to the eigenvalue $E_1=\frac{3}{2}$

It describes the first excited state.