An introduction to MATHEMATICAL DESCRIPTION OF QUANTUM SYSTEMS

Nicolae Cotfas, version 28 Jul 2021 (for future updates see https://unibuc.ro/user/nicolae.cotfas/)

From a mathematical point of view:
Quantum system ≡ a complex Hilbert space $\mathcal{H}$.

A AXIOMS OF QUANTUM MECHANICS

■ **Definition.** *Closed system* = a perfectly isolated system.
Open system = an imperfectly isolated system.

1 A *ray* in $\mathcal{H}$ is a set of the form $\{\ \lambda|\psi\rangle\ \mid\ 0\neq\lambda\in\mathbb{C}\ \}$ described by a representative $|\psi\rangle$ with $||\psi||=\sqrt{\langle\psi|\psi\rangle}=1$.
Normalized vectors $e^{i\vartheta}|\psi\rangle$, with $\vartheta\in\mathbb{R}$, represent the same ray.

■ *Axioms of quantum mechanics* (for closed quantum systems).

- *Axiom 1.* **Pure state** = ray $|\psi\rangle$ in a Hilbert space $\mathcal{H}$
- *Axiom 2.* **Observable** = a self-adjoint operator $A\in\mathcal{A}(\mathcal{H})$.
- *Axiom 3.* **Measurement.** The measurement of an observable $A:\mathcal{H}\to\mathcal{H}$ with the spectral decomposition

2 $$\boxed{A=\sum a_k\, E_k} \qquad \left(E_k=\begin{array}{c}\text{the orthogonal projector}\\ \text{onto the eigenspace of } a_k\end{array}\right)$$

prepares an eigenstate of A, and the observer learns the value of the corresponding eigenvalue.
If the quantum state *just prior to the measurement* is $|\psi\rangle$, then the *outcome* a_k is obtained with *a priori probability*

3 $$\boxed{prob(a_k)=\|\ E_k|\psi\rangle\ \|^2=\langle\psi|E_k|\psi\rangle=\mathrm{tr}(E_k\,|\psi\rangle\langle\psi|)}.$$

If a_k is attained, then the quantum state *just after the measurement* is $\frac{E_k|\psi\rangle}{\|E_k|\psi\rangle\|}$, that is $|\psi\rangle$ collapses to $\frac{E_k|\psi\rangle}{\|E_k|\psi\rangle\|}$.
If many identically prepared systems are measured, each described by the state $|\psi\rangle$, then the expectation value of the outcomes is

4 $$\boxed{\langle A\rangle=\sum a_k\langle\psi|E_k|\psi\rangle=\langle\psi|A|\psi\rangle=\mathrm{tr}(A\,|\psi\rangle\langle\psi|)}.$$

- *Axiom 4.* **Dynamics.** The *time evolution* of a closed system is described by unitary operators $U(t',t):\mathcal{H}\to\mathcal{H}$,
$$|\psi(t')\rangle=U(t',t)\,|\psi(t)\rangle.$$
The infinitesimal time evolution is described by the

5 *Schrödinger equation* $\boxed{\mathrm{i}\,\frac{d}{dt}|\psi(t)\rangle=H(t)\,|\psi(t)\rangle}$

where $H(t)$ is the *Hamiltonian* of the system.

- *Axiom 5.* **Composite systems.**

$\boxed{\begin{array}{l}\mathcal{H}_A=\text{Hilbert space of system A}\\ \mathcal{H}_B=\text{Hilbert space of system B}\end{array}} \Rightarrow \boxed{\begin{array}{c}\text{Hilbert space of the}\\ \text{composite system is}\\ \mathcal{H}_A\otimes\mathcal{H}_B.\end{array}}$

If the system A is prepared in the state $|\psi\rangle_A$ and B in the state $|\psi\rangle_B$, then the composite system state is the product $|\psi\rangle_A\otimes|\psi\rangle_B$.

6 $\langle\psi_1|\psi_2\rangle$ = amplitude probability that ψ_2 is in state ψ_1.

7 *Measurement in the orthonormal eigenbasis of A*
If the eigenvalues of A are distinct, then $A=\sum a_k|\psi_k\rangle\langle\psi_k|$,
$E_k=|\psi_k\rangle\langle\psi_k|$ and $prob(a_k)=|\langle\psi_k|\psi\rangle|^2$.

B DENSITY OPERATORS

■

1 A pure state $|\psi\rangle$ can alternatively be described by using the orthogonal *projector corresponding to* $|\psi\rangle$ $\boxed{\begin{array}{l}E_\psi:\mathcal{H}\to\mathcal{H},\\ E_\psi=|\psi\rangle\langle\psi|\end{array}\ \text{satisfying}\ \begin{array}{r}E_\psi\geq 0,\\ \mathrm{tr}\,E_\psi=1.\end{array}}$

■ **Definition.**

2 A *density operator* is a linear operator $\boxed{\varrho:\mathcal{H}\to\mathcal{H}\quad\text{satisfying}\quad \begin{array}{r}\varrho\geq 0,\\ \mathrm{tr}\,\varrho=1.\end{array}}$

3 *Quantum state* of an open system ≡ a density operator $\varrho:\mathcal{H}\longrightarrow\mathcal{H}$.

$\mathcal{D}(\mathcal{H})=\boxed{\begin{array}{c}\text{Set of all the}\\ \text{quantum states}\end{array}}\equiv\boxed{\begin{array}{c}\text{Set of all the}\\ \text{density operators}\end{array}}.$

4 A *mixed state* ≡ a state which is not a pure state.

5 *Purity* of a state ϱ is $\boxed{\mathrm{tr}\,\varrho^2}$ satisfying $\frac{1}{\dim\mathcal{H}}\leq\mathrm{tr}\,\varrho^2\leq 1$.

6 *Fidelity* of two states is $F(\varrho_1,\varrho_2)=\left(\mathrm{tr}\sqrt{\varrho_1^{\frac{1}{2}}\,\varrho_2\,\varrho_1^{\frac{1}{2}}}\right)^2$.

■ **Theorem.**

7 Two probabilistic representations of a density operator ϱ are related through a unitary transform (see *B*7 at pag. 9)

$$\boxed{\begin{array}{l}\varrho=\sum p_j|\psi_j\rangle\langle\psi_j|\\ \quad=\sum q_k|\varphi_k\rangle\langle\varphi_k|\end{array}}\quad \begin{array}{c}0\leq p_j,q_k\leq 1,\\ \sum p_j=\sum q_k=1.\end{array}$$
$$\Updownarrow$$
$$\boxed{\begin{array}{c}\text{There exists a unitary}\\ \text{matrix } (u_{jk}) \text{ such that}\\ \sqrt{p_j}|\psi_j\rangle=\sum_k u_{jk}\sqrt{q_k}|\varphi_k\rangle\end{array}}$$

8 $\varrho\in\mathcal{D}(\mathcal{H})\quad\Rightarrow\quad 0\leq\varrho\leq\mathbb{I}$.

9 *Expectation value* of an observable A in a state ϱ is $\boxed{\langle A\rangle\equiv\langle A\rangle_\varrho=\mathrm{tr}(A\varrho)=\mathrm{tr}(\varrho A).}$

10 $\boxed{\varrho\text{ is a pure state}}\Leftrightarrow\boxed{\varrho^2=\varrho}\Leftrightarrow\boxed{\mathrm{tr}\,\varrho^2=1}\Leftrightarrow\boxed{||\varrho||=1}.$

11 $\boxed{\varrho\text{ is a mixed state}}\Leftrightarrow\boxed{\varrho^2\neq\varrho}\Leftrightarrow\boxed{\mathrm{tr}\,\varrho^2<1}\Leftrightarrow\boxed{||\varrho||<1}.$

12 $$\left.\begin{array}{r}\varrho\in\mathcal{D}(\mathcal{H})\\ K\in U(\mathcal{H})\end{array}\right\}\ \Rightarrow\quad K^\dagger\varrho K\in\mathcal{D}(\mathcal{H}).$$

13 $\mathcal{D}(\mathcal{H})$ is a *convex set*, that is $\boxed{\left.\begin{array}{c}\varrho_1,\varrho_2\in\mathcal{D}(\mathcal{H})\\ 0\leq\lambda\leq 1\end{array}\right\}\Rightarrow(1-\lambda)\varrho_1+\lambda\varrho_2\in\mathcal{D}(\mathcal{H})}.$

14 $\boxed{\varrho\text{ is a pure state}}\Leftrightarrow\boxed{\begin{array}{l}\varrho\text{ is an } \textit{extremal point} \text{ of } \mathcal{D}(\mathcal{H}):\\ \left.\begin{array}{c}\varrho=(1-\lambda)\varrho_1+\lambda\varrho_2,\\ \varrho_1,\varrho_2\in\mathcal{D}(\mathcal{H}),\\ 0\leq\lambda\leq 1\end{array}\right\}\Rightarrow\varrho_1=\varrho_2=\varrho.\end{array}}$

C QUBITS (quantum bits)

■ **Definition.**
Qubit = quantum system with a two-dimensional Hilbert space $\mathcal{H}$.

■ **Remarks:**
By choosing an orthonormal basis $\{|0\rangle,|1\rangle\}$ in $\mathcal{H}$, $\mathcal{H}=\{\alpha|0\rangle+\beta|1\rangle\ \mid\ \alpha,\beta\in\mathbb{C}\}$ can be identified with

1 $$\boxed{\begin{array}{l}\mathbb{C}^2=\left\{|\psi\rangle=\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\ \middle|\ \alpha,\beta\in\mathbb{C}^2\right\}\equiv\{\psi:\{0,1\}\to\mathbb{C}\},\\ \qquad\langle\psi_1|\psi_2\rangle=\bar{\alpha}_1\beta_1+\bar{\alpha}_2\beta_2=\sum_{k=0}^{1}\overline{\psi_1(k)}\,\psi_2(k).\end{array}}$$

2 Computational state basis is $|0\rangle\equiv|\uparrow\rangle=\begin{pmatrix}1\\0\end{pmatrix},\quad|1\rangle\equiv|\downarrow\rangle=\begin{pmatrix}0\\1\end{pmatrix}.$

3 The “bra - ket” correspondence $\langle\psi|\equiv(\,\bar{\alpha}\quad\bar{\beta}\,)\quad\Leftrightarrow\quad|\psi\rangle\equiv\begin{pmatrix}\alpha\\ \beta\end{pmatrix}.$

■ **Theorem.**

- In the complex Hilbert space of linear operators

4 $\mathcal{L}(\mathbb{C}^2)=\left\{A:\mathbb{C}^2\to\mathbb{C}^2\ \middle|\ \begin{array}{c}A\text{ is linear}\\ \text{operator}\end{array}\right\},\ \langle A,B\rangle=\mathrm{tr}(A^\dagger B),$

the *Pauli operators* (several notations are presented)

5 $$\boxed{\begin{array}{ll}\sigma_0\equiv\mathbb{I}=\begin{pmatrix}1&0\\0&1\end{pmatrix}, & \sigma_1\equiv\sigma_x\equiv X=\begin{pmatrix}0&1\\1&0\end{pmatrix},\\ \sigma_2\equiv\sigma_y\equiv Y=\begin{pmatrix}0&-\mathrm{i}\\ \mathrm{i}&0\end{pmatrix}, & \sigma_3\equiv\sigma_z\equiv Z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\end{array}}$$

form an orthogonal basis: $\langle\sigma_j,\sigma_k\rangle=2\delta_{jk}\mathbb{I}$, and
$\sigma_j^\dagger=\sigma_j,\quad\sigma_1^2=\sigma_2^2=\sigma_3^2=\mathbb{I},\quad\sigma_j\sigma_k+\sigma_k\sigma_j=0$ for $j\neq k$.

6 $$A\in\mathcal{L}(\mathbb{C}^2)\Rightarrow\boxed{\begin{array}{l}\text{there exist } r_0,r_1,r_2,r_3\in\mathbb{C}\text{ such that}\\ A=\frac{1}{2}(r_0\,\sigma_0+r_1\,\sigma_1+r_2\,\sigma_2+r_3\,\sigma_3)\\ \quad=\frac{1}{2}\begin{pmatrix}r_0+r_3 & r_1-\mathrm{i}r_2\\ r_1+\mathrm{i}r_2 & r_0-r_3\end{pmatrix}\end{array}}$$

7 Eigenvalues of $A\in\mathcal{L}(\mathbb{C}^2)$ are $\lambda_{1,2}=\frac{r_0\pm\sqrt{r_1^2+r_2^2+r_3^2}}{2}$.

8 $A\in\mathcal{L}(\mathbb{C}^2)\quad\Rightarrow\quad\mathrm{tr}\,A=r_0$.

• In the real Hilbert space of Hermitian operators

9 $\mathcal{A}(\mathbb{C}^2)=\{A:\mathbb{C}^2\to\mathbb{C}^2 \mid A^\dagger=A\}$, $\langle A,B\rangle=\mathrm{tr}(AB)$, the Pauli operators form also an orthogonal basis.

10 $A\in\mathcal{A}(\mathbb{C}^2)\Rightarrow$ there exist $r_0\in\mathbb{R},\ r=(r_1,r_2,r_3)\in\mathbb{R}^3$ such that $A=\frac{1}{2}(r_0\,\mathbb{I}+r\cdot\sigma) = \frac{1}{2}\begin{pmatrix} r_0+r_3 & r_1-\mathrm{i}r_2\\ r_1+\mathrm{i}r_2 & r_0-r_3\end{pmatrix}$

11 Eigenvalues of $A\in\mathcal{A}(\mathbb{C}^2)$ are $\lambda_{1,2}=\frac{r_0\pm||r||}{2}$.

12 $A\in\mathcal{A}(\mathbb{C}^2) \quad\Rightarrow\quad \boxed{A\geq 0 \ \Leftrightarrow\ ||r||\leq r_0}$.

13 $\varrho(r)=\frac{1}{2}(r_0\,\mathbb{I}+r\cdot\sigma)$ is a density operator $\Leftrightarrow \begin{cases} r_0=1\\ ||r||\leq 1.\end{cases}$

14 The set of all the density operators is $\boxed{\mathcal{D}(\mathbb{C}^2)=\{\,\varrho=\frac{1}{2}(\mathbb{I}+r\cdot\sigma)\ \mid\ ||r||\leq 1\}}$.

15 Eigenvalues of $\varrho\in\mathcal{D}(\mathbb{C}^2)$ are $\lambda_{1,2}=\frac{1\pm||\mathbf{r}||}{2}$.

16 Purity of $\varrho\in\mathcal{D}(\mathbb{C}^2)$ is $\frac{1+||r||^2}{2}\in\left[\frac{1}{2},1\right]$.

17 *Bloch sphere* $\Omega_3=\{r\in\mathbb{R}^3 \mid ||r||\leq 1\}$

• $\Omega_3\to\mathcal{D}(\mathbb{C}^2)$: $r\mapsto\varrho(r)=\frac{1}{2}(\mathbb{I}+r\cdot\sigma)$ is a one-to-one map.

• r corresponds to a pure state $\Leftrightarrow ||r||=1$.

18 $|\psi\rangle$ is a pure state $\Rightarrow$ there exist $\theta\in[0,\pi]$, $\varphi\in[0,2\pi)$ such that $\boxed{|\psi\rangle=\cos\frac{\theta}{2}|0\rangle+\sin\frac{\theta}{2}\mathrm{e}^{\mathrm{i}\varphi}|1\rangle}$ (a phase factor $\mathrm{e}^{\mathrm{i}\eta}$ is ignored)

$\Downarrow$

$\boxed{\varrho=|\psi\rangle\langle\psi|=\frac{1}{2}\begin{pmatrix}1+\cos\theta & \sin\theta\,\mathrm{e}^{-\mathrm{i}\phi}\\ \sin\theta\,\mathrm{e}^{\mathrm{i}\phi} & 1+\cos\theta\end{pmatrix}}$

corresponds to

$r=(\sin\theta\,\cos\phi,\ \sin\theta\,\sin\phi,\ \cos\theta)\in\partial\Omega_3$.

19 Orthogonal pure states correspond to diametrically opposite points of $\partial\Omega_3=\{r\in\mathbb{R}^3 \mid ||r||=1\}$.

20 Spectral decomposition of Pauli operators

$$\sigma_x=|\uparrow_x\rangle\langle\uparrow_x|-|\downarrow_x\rangle\langle\downarrow_x|,\ \text{where}\ \begin{cases}|\uparrow_x\rangle\equiv|+\rangle=\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\\ |\downarrow_x\rangle\equiv|-\rangle=\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\end{cases}$$

$$\sigma_y=|\uparrow_y\rangle\langle\uparrow_y|-|\downarrow_y\rangle\langle\downarrow_y|,\ \text{where}\ \begin{cases}|\uparrow_y\rangle=\frac{1}{\sqrt{2}}(|0\rangle+\mathrm{i}|1\rangle)\\ |\downarrow_y\rangle=\frac{1}{\sqrt{2}}(|0\rangle-\mathrm{i}|1\rangle)\end{cases}$$

$$\sigma_z=|\uparrow_z\rangle\langle\uparrow_z|-|\downarrow_z\rangle\langle\downarrow_z|,\ \text{where}\ \begin{cases}|\uparrow_z\rangle=|0\rangle\\ |\downarrow_z\rangle=|1\rangle\end{cases}$$

21 $A^2=\mathbb{I}\ \Rightarrow\ \boxed{\mathrm{e}^{\mathrm{i}\theta A}=\cos\theta\,\mathbb{I}+\mathrm{i}\,\sin\theta\,A}$, for any $\theta\in\mathbb{R}$.

22 $\sigma_x^2=\mathbb{I}\ \Rightarrow\ \mathrm{e}^{-\mathrm{i}\frac{\theta}{2}\sigma_x}=\begin{pmatrix}\cos\frac{\theta}{2} & -\mathrm{i}\,\sin\frac{\theta}{2}\\ -\mathrm{i}\,\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}$.

23 $\sigma_y^2=\mathbb{I}\ \Rightarrow\ \mathrm{e}^{-\mathrm{i}\frac{\theta}{2}\sigma_y}=\begin{pmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}$.

24 $\sigma_z^2=\mathbb{I}\ \Rightarrow\ \mathrm{e}^{-\mathrm{i}\frac{\theta}{2}\sigma_z}=\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\frac{\theta}{2}} & 0\\ 0 & \mathrm{e}^{\mathrm{i}\frac{\theta}{2}}\end{pmatrix}$.

25 Through $\Omega_3\to\mathcal{D}(\mathbb{C}^2): r\mapsto\varrho(r)=\frac{1}{2}(\mathbb{I}+r\cdot\sigma)$, the transform $\mathcal{D}(\mathbb{C}^2)\to\mathcal{D}(\mathbb{C}^2):\varrho\mapsto\mathrm{e}^{-\mathrm{i}\frac{\theta}{2}\sigma_x}\varrho\,\mathrm{e}^{\mathrm{i}\frac{\theta}{2}\sigma_x}$ corresponds to the rotation of Ω_3 around Ox

$$\Omega_3\to\Omega_3:\begin{pmatrix}r_1\\ r_2\\ r_3\end{pmatrix}\mapsto\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}r_1\\ r_2\\ r_3\end{pmatrix}.$$

26 The transform $\mathcal{D}(\mathbb{C}^2)\to\mathcal{D}(\mathbb{C}^2):\varrho\mapsto\mathrm{e}^{-\mathrm{i}\frac{\theta}{2}\sigma_y}\varrho\,\mathrm{e}^{\mathrm{i}\frac{\theta}{2}\sigma_y}$ corresponds to the rotation of Ω_3 around Oy

$$\Omega_3\to\Omega_3:\begin{pmatrix}r_1\\ r_2\\ r_3\end{pmatrix}\mapsto\begin{pmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{pmatrix}\begin{pmatrix}r_1\\ r_2\\ r_3\end{pmatrix}.$$

27 The transform $\mathcal{D}(\mathbb{C}^2)\to\mathcal{D}(\mathbb{C}^2):\varrho\mapsto\mathrm{e}^{-\mathrm{i}\frac{\theta}{2}\sigma_z}\varrho\,\mathrm{e}^{\mathrm{i}\frac{\theta}{2}\sigma_z}$ corresponds to the rotation of Ω_3 around Oz

$$\Omega_3\to\Omega_3:\begin{pmatrix}r_1\\ r_2\\ r_3\end{pmatrix}\mapsto\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}r_1\\ r_2\\ r_3\end{pmatrix}.$$

28 For any $U\in U(\mathbb{C}^2)$, there exist $\theta,\alpha,\beta,\gamma\in\mathbb{R}$ such that

$$U=\mathrm{e}^{\mathrm{i}\theta}\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}} & 0\\ 0 & \mathrm{e}^{\mathrm{i}\frac{\alpha}{2}}\end{pmatrix}\begin{pmatrix}\cos\frac{\beta}{2} & -\sin\frac{\beta}{2}\\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\frac{\gamma}{2}} & 0\\ 0 & \mathrm{e}^{\mathrm{i}\frac{\gamma}{2}}\end{pmatrix}.$$

29 For any $U\in U(2)\equiv U(\mathbb{C}^2)$, the transform $\mathcal{D}(\mathbb{C}^2)\to\mathcal{D}(\mathbb{C}^2):\varrho\mapsto U\varrho\,U^\dagger$ corresponds to a rotation of the Bloch sphere Ω_3.

■ Bit versus Qubit

Bit 30

1. Indivisible unit of classical information.
2. It describes the state of a two-state classical system.
3. It takes one of the possible values 0 and 1.
4. There exists only one observable.
5. We can measure a bit without disturbing it.
6. We can obtain the entire information it encodes by a single measurement.

Qubit 31

$1'$. indivisible unit of quantum information;

$2'$. It describes the state of a quantum system with a two-dimensional Hilbert space $\mathcal{H}$.

$3'$. It takes any of the values

$$|\psi\rangle=a|0\rangle+b|1\rangle=\cos\tfrac{\theta}{2}\mathrm{e}^{\mathrm{i}\varphi_0}|0\rangle+\sin\tfrac{\theta}{2}\mathrm{e}^{\mathrm{i}\varphi_1}|1\rangle=\mathrm{e}^{\mathrm{i}\varphi_0}\left(\cos\tfrac{\theta}{2}|0\rangle+\sin\tfrac{\theta}{2}\mathrm{e}^{\mathrm{i}\varphi}|1\rangle\right)$$

where $\{|0\rangle,|1\rangle\}$ is an orthonormal basis of $\mathcal{H}$, and $\varphi=\varphi_1-\varphi_0$. The physically irrelevant phase factor $\mathrm{e}^{\mathrm{i}\varphi_0}$ can be ignored.

$4'$. There exist more than one observable.

$5'$. After a measurement, the state before the measurement $|\psi\rangle$ collapses to a known state which, in general, is different from $|\psi\rangle$.

$6'$. If the state $|\psi\rangle=a|0\rangle+b|1\rangle$ of a qubit is unknown, then there is no way to determine a and b with a single measurement.

32 If we measure a qubit in the state $|\psi\rangle=a|0\rangle+b|1\rangle$ in the eigenbasis $\{|0\rangle,|1\rangle\}$ of σ_z, then $|\psi\rangle$ collapses to $|0\rangle$ with the probability $|a|^2=\cos^2\frac{\theta}{2}$, and to $|1\rangle$ with the probability $|b|^2=\sin^2\frac{\theta}{2}$.

33 The state before the measurement $|\psi\rangle=a|0\rangle+b|1\rangle$ is a *coherent superposition* of $|0\rangle$ and $|1\rangle$. It is not a *probabilistic mixture* of $|0\rangle$ with the probability $|a|^2$ and $|1\rangle$ with the probability $|b|^2$ because, the state $|\psi\rangle$ depends not only on $|a|$ and $|b|$, but also on the relative phase φ of a and b.

34 The result of certain measurements depends on the phase factor $\mathrm{e}^{\mathrm{i}\varphi}$ (there exists quantum interference). If we measure $|\psi\rangle$ in the eigenbasis $\{|+\rangle,|-\rangle\}$ of σ_x, then the state before the measurement $|\psi\rangle$ collapses to $|+\rangle$ with probability $|a+b|^2/2=\left|\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\mathrm{e}^{\mathrm{i}\varphi}\right|^2/2$ and $|-\rangle$ with probability $|a-b|^2/2=\left|\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\mathrm{e}^{\mathrm{i}\varphi}\right|^2/2$.

D COMPOSITE QUANTUM SYSTEMS

■ Definitions and notations:

$\mathcal{H}_1 \equiv \mathcal{H}_A$ = Hilbert space of the main quantum system.
$\mathcal{H}_2 \equiv \mathcal{H}_B$ = Hilbert space of a reference quantum system.
$\{|j\rangle_1\} \equiv \{|j\rangle_A\}$ – orthonormal basis of $\mathcal{H}_1 \equiv \mathcal{H}_A$
$\{|k\rangle_2\} \equiv \{|k\rangle_B\}$ – orthonormal basis of $\mathcal{H}_2 \equiv \mathcal{H}_B$

1 $$|\varphi\psi\rangle \equiv |\varphi\rangle_1 \otimes |\psi\rangle_2 \equiv |\varphi\rangle_A \otimes |\psi\rangle_B$$

2 $$\boxed{\langle\varphi_1\psi_1|\varphi_2\psi_2\rangle = \langle\varphi_1|\varphi_2\rangle\,\langle\psi_1|\psi_2\rangle}$$
$$\langle ik|j\ell\rangle = \delta_{ij}\,\delta_{k\ell}$$

Pure state of the composite system $|\Phi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ $\boxed{|\Phi\rangle = \sum_{i,k} \Phi^{ik}|ik\rangle}$ $\langle\Phi|\Phi\rangle = \sum_{i,k} |\Phi^{ik}|^2 = 1.$

Quantum state of the composite system $\varrho : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2$ $\boxed{\varrho = \sum_{i,j,k,\ell} \varrho^{ik}_{j\ell}|ik\rangle\langle j\ell|}$ $\varrho \geq 0,$ $\mathrm{tr}\,\varrho = 1.$

3 $${}_1\langle a| : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_2, \quad {}_2\langle b| : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1 \qquad \boxed{\begin{aligned} {}_1\langle a|\varphi\psi\rangle &= \langle a|\varphi\rangle\,|\psi\rangle \\ {}_2\langle b|\varphi\psi\rangle &= \langle b|\psi\rangle\,|\varphi\rangle \end{aligned}}$$

4 $$\langle ab| : \mathcal{H}_1 \to \mathcal{H}_2^*, \quad \langle ab| : \mathcal{H}_2 \to \mathcal{H}_1^* \qquad \boxed{\begin{aligned} \langle ab|\varphi\rangle_1 &= \langle a|\varphi\rangle\,\langle b| \\ \langle ab|\psi\rangle_2 &= \langle b|\psi\rangle\,\langle a| \end{aligned}}$$

For $\mathcal{H}_1 \otimes \mathcal{H}_2 \xrightarrow{T} \mathcal{H}_1 \otimes \mathcal{H}_2$ we define the operators:

5 $$\boxed{\begin{aligned} {}_1\langle a|T &: \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_2 : \ |\varphi\psi\rangle \mapsto {}_1\langle a|T|\varphi\psi\rangle \\ {}_2\langle a|T &: \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1 : \ |\varphi\psi\rangle \mapsto {}_2\langle a|T|\varphi\psi\rangle \\ T|b\rangle_1 &: \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2 : \ |\psi\rangle \mapsto T|b\psi\rangle \\ T|b\rangle_2 &: \mathcal{H}_1 \to \mathcal{H}_1 \otimes \mathcal{H}_2 : \ |\varphi\rangle \mapsto T|\varphi b\rangle \end{aligned}}$$

6 $$\left.\begin{aligned} A &: \mathcal{H}_1 \to \mathcal{H}_1 \\ B &: \mathcal{H}_2 \to \mathcal{H}_2 \\ &\text{linear operators} \end{aligned}\right\} \Rightarrow \begin{aligned} A \otimes B : \mathcal{H}_1 \otimes \mathcal{H}_2 &\longrightarrow \mathcal{H}_1 \otimes \mathcal{H}_2 \\ \boxed{(A \otimes B)(\varphi \otimes \psi) = (A\varphi) \otimes (B\psi)} \\ \text{defines a linear operator.} \end{aligned}$$

7 For $\mathcal{H}_1 \otimes \mathcal{H}_2 \xrightarrow{T} \mathcal{H}_1 \otimes \mathcal{H}_2$
$$\boxed{\begin{aligned} \mathrm{tr}_1 T &: \mathcal{H}_2 \to \mathcal{H}_2, \\ \mathrm{tr}_2 T &: \mathcal{H}_1 \to \mathcal{H}_1 \end{aligned}} \qquad \boxed{\begin{aligned} \mathrm{tr}\,T &= \sum_{j,k} \langle jk|T|jk\rangle \\ \mathrm{tr}_1 T &= \sum_j {}_1\langle j|T|j\rangle_1, \\ \mathrm{tr}_2 T &= \sum_k {}_2\langle k|T|k\rangle_2, \\ \mathrm{tr}\,T &= \mathrm{tr}_1\mathrm{tr}_2 T = \mathrm{tr}_2\mathrm{tr}_1 T \end{aligned}}$$

■ Fundamental formulas:

8 $$\mathrm{tr}(|\varphi\rangle\langle\psi|) = \langle\psi|\varphi\rangle$$

9 $$\boxed{\begin{aligned} {}_1\langle i|j\ell\rangle &= \delta_{ij}\,|\ell\rangle_2 \\ {}_2\langle k|j\ell\rangle &= \delta_{k\ell}\,|j\rangle_1 \end{aligned}} \qquad \boxed{\begin{aligned} \langle ik|j\rangle_1 &= \delta_{ij}\,{}_2\langle k| \\ \langle ik|\ell\rangle_2 &= \delta_{k\ell}\,{}_1\langle i| \end{aligned}}$$

10 $$\boxed{\sum_j {}_2\langle j|\varphi\psi\rangle \otimes |j\rangle_2 = |\varphi\psi\rangle}$$

11 $$\begin{aligned} {}_1\langle\varphi_1|A \otimes B|\varphi_2\rangle_1 &= \langle\varphi_1|A|\varphi_2\rangle\,B, \\ {}_2\langle\psi_1|A \otimes B|\psi_2\rangle_2 &= \langle\psi_1|B|\psi_2\rangle\,A. \end{aligned}$$

12 $$\left.\begin{aligned} A &: \mathcal{H}_1 \to \mathcal{H}_1 \\ B &: \mathcal{H}_2 \to \mathcal{H}_2 \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} \mathrm{tr}_1(A \otimes B) &= (\mathrm{tr}\,A)\,B, \\ \mathrm{tr}_2(A \otimes B) &= (\mathrm{tr}\,B)\,A. \end{aligned}}$$

13 $$|\varphi_1\psi_1\rangle\langle\varphi_2\psi_2| = |\varphi_1\rangle\langle\varphi_2| \otimes |\psi_1\rangle\langle\psi_2|.$$

14 $$\begin{aligned} \mathrm{tr}_1(|\varphi_1\psi_1\rangle\langle\varphi_2\psi_2|) &= \langle\varphi_2|\varphi_1\rangle\ |\psi_1\rangle\langle\psi_2|, \\ \mathrm{tr}_2(|\varphi_1\psi_1\rangle\langle\varphi_2\psi_2|) &= \langle\psi_2|\psi_1\rangle\ |\varphi_1\rangle\langle\varphi_2|. \end{aligned}$$

15 $$\begin{aligned} \mathcal{H}_1 \otimes \mathcal{H}_2 &\xrightarrow{T} \mathcal{H}_1 \otimes \mathcal{H}_2 \\ \boxed{T^{ik}_{j\ell} = \langle ik|T|j\ell\rangle} \end{aligned} \Rightarrow \boxed{\begin{aligned} (\mathrm{tr}_1 T)^k_\ell &= {}_2\langle k|\mathrm{tr}_1 T|\ell\rangle_2 = \sum_j T^{jk}_{j\ell} \\ (\mathrm{tr}_2 T)^i_j &= {}_1\langle i|\mathrm{tr}_2 T|j\rangle_1 = \sum_k T^{ik}_{jk} \end{aligned}}$$

16 $$\varrho = \sum_{i,j,k,\ell} \varrho^{ik}_{j\ell}|ik\rangle\langle j\ell| \ \Rightarrow \ \boxed{\mathrm{tr}_1\,\varrho = \sum_{j,k,\ell} \varrho^{jk}_{j\ell}|k\rangle\langle\ell|} \quad \boxed{\mathrm{tr}_2\,\varrho = \sum_{i,j,k} \varrho^{ik}_{jk}|i\rangle\langle j|}$$

17 $$\mathcal{H}_1 \otimes \mathcal{H}_2 \xrightarrow{T} \mathcal{H}_1 \otimes \mathcal{H}_2 \ \Rightarrow \boxed{\sum_{k,\ell} {}_2\langle k|T|\ell\rangle_2 \otimes |k\rangle_2\,{}_2\langle\ell| = T}.$$

18 $$\mathcal{H} \xrightarrow{A} \mathcal{H} \quad \mapsto \quad \mathrm{tr}(|\varphi\rangle\langle\psi|\,A) = \langle\psi|A|\varphi\rangle$$

19 $$\begin{aligned} \mathrm{tr}_2((\mathbb{I} \otimes (|b\rangle\langle b|))T) &= {}_2\langle b|T|b\rangle_2 \\ \mathrm{tr}_2(T(\mathbb{I} \otimes (|b\rangle\langle b|))) &= {}_2\langle b|T|b\rangle_2 \end{aligned}$$

20 $$\mathrm{tr}((A \otimes B)T) = \mathrm{tr}(A\ \mathrm{tr}_2((\mathbb{I} \otimes B)T))$$

21 $$\mathrm{tr}((A \otimes B)T) = \mathrm{tr}(B\ \mathrm{tr}_1((A \otimes \mathbb{I})T))$$

22 $$\mathrm{tr}((A \otimes B)T) = \mathrm{tr}((A \otimes \mathbb{I})(\mathbb{I} \otimes B)T)$$

23 $$\boxed{\mathrm{tr}((A \otimes \mathbb{I})\varrho) = \mathrm{tr}(A\ \mathrm{tr}_2\,\varrho)}$$

24 $$\boxed{\mathrm{tr}((\mathbb{I} \otimes B)\varrho) = \mathrm{tr}(B\ \mathrm{tr}_1\,\varrho)}$$

25 $$\begin{aligned} \mathcal{H}_1 \otimes \mathcal{H}_2 &\xrightarrow{\varrho} \mathcal{H}_1 \otimes \mathcal{H}_2 \\ &\text{density operator} \end{aligned} \Rightarrow \boxed{\begin{aligned} \mathrm{tr}_1\,\varrho &: \mathcal{H}_2 \to \mathcal{H}_2, \\ \mathrm{tr}_2\,\varrho &: \mathcal{H}_1 \to \mathcal{H}_1 \\ &\text{density operators} \end{aligned}}$$

26 For $\varrho : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$,
$$\varrho = \begin{pmatrix} \varrho^{00}_{00} & \varrho^{00}_{01} & \varrho^{00}_{10} & \varrho^{00}_{11} \\ \varrho^{01}_{00} & \varrho^{01}_{01} & \varrho^{01}_{10} & \varrho^{01}_{11} \\ \varrho^{10}_{00} & \varrho^{10}_{01} & \varrho^{10}_{10} & \varrho^{10}_{11} \\ \varrho^{11}_{00} & \varrho^{11}_{01} & \varrho^{11}_{10} & \varrho^{11}_{11} \end{pmatrix} \Rightarrow \begin{aligned} \mathrm{tr}_1\varrho &= \begin{pmatrix} \varrho^{00}_{00} & \varrho^{00}_{01} \\ \varrho^{01}_{00} & \varrho^{01}_{01} \end{pmatrix} + \begin{pmatrix} \varrho^{10}_{10} & \varrho^{10}_{11} \\ \varrho^{11}_{10} & \varrho^{11}_{11} \end{pmatrix}, \\ \mathrm{tr}_2\varrho &= \begin{pmatrix} \mathrm{tr}\begin{pmatrix} \varrho^{00}_{00} & \varrho^{00}_{01} \\ \varrho^{01}_{00} & \varrho^{01}_{01} \end{pmatrix} & \mathrm{tr}\begin{pmatrix} \varrho^{00}_{10} & \varrho^{00}_{11} \\ \varrho^{01}_{10} & \varrho^{01}_{11} \end{pmatrix} \\ \mathrm{tr}\begin{pmatrix} \varrho^{10}_{00} & \varrho^{10}_{01} \\ \varrho^{11}_{00} & \varrho^{11}_{01} \end{pmatrix} & \mathrm{tr}\begin{pmatrix} \varrho^{10}_{10} & \varrho^{10}_{11} \\ \varrho^{11}_{10} & \varrho^{11}_{11} \end{pmatrix} \end{pmatrix}. \end{aligned}$$

■ Schmidt form:

27 Given $$\boxed{\begin{aligned} &|\Phi\rangle = \sum_{kj} \Phi^{kj}|\varphi_k\rangle \otimes |j\rangle_2, \\ &\text{with } \{\varphi_k\} \text{ chosen such that} \\ &\mathrm{tr}_2|\Phi\rangle\langle\Phi| = \sum_{k=0}^{r-1} p_k|\varphi_k\rangle\langle\varphi_k| \end{aligned}}$$

28 $$\Downarrow$$ $$\sum_j \Phi^{kj}\,\overline{\Phi^{\ell j}} = p_k\,\delta_{k\ell}.$$

29 By choosing $$\boxed{|\psi_k\rangle = \frac{1}{\sqrt{p_k}} \sum_j \Phi^{kj}|j\rangle_2}$$

30 we get $$\boxed{\boxed{|\Phi\rangle = \sum_{k=0}^{r-1} \sqrt{p_k}|\varphi_k\rangle \otimes |\psi_k\rangle}}$$

■ <u>Purification</u>: **mixed state = tr_2(pure state)**

31 Given $$\boxed{\begin{aligned} &\varrho : \mathbb{C}^n \to \mathbb{C}^n, \ \varrho = \sum_{k=0}^{n-1} p_k|\varphi_k\rangle\langle\varphi_k|, \\ &\{|\varphi_k\rangle\} \text{ orthonormal basis in } \mathbb{C}^n \end{aligned}}$$

32 by choosing $$\boxed{\begin{aligned} &\{|\psi_k\rangle\} \text{ orthonormal basis in } \mathbb{C}^n, \\ &|\Phi\rangle = \sum_{k=0}^{n-1} \sqrt{p_k}\,|\varphi_k\rangle \otimes |\psi_k\rangle \end{aligned}}$$

33 we get $$\boxed{\boxed{\varrho = \mathrm{tr}_2|\Phi\rangle\langle\Phi|}}$$

$\boxed{F}$ **ORTHOGONAL MEASUREMENTS**

■ **Description of a measurement**

• To perform a measurement $\{P_j\}_{j=0}^{n-1}$ of $\mathcal{A}$,

1 $$\boxed{P_j P_k = \delta_{jk} P_j, \quad P_j = P_j^\dagger, \quad \sum_{j=0}^{n-1} P_j = \mathbb{I}}$$

we use a n-dimensional *pointer* $\mathcal{B}$ with fiducial orthonormal basis states $\{|j\rangle\}_{j=0}^{n-1}$.

• The *unitary operator* $U : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_A \otimes \mathcal{H}_B$,

2 $$\boxed{U = \sum_{j,k} P_j \otimes |k+j\rangle\langle k|}$$

transforms an initial state $|\psi\rangle \otimes |0\rangle$ into

3 $$|\Psi'\rangle = U(|\psi\rangle \otimes |0\rangle) = \sum_j P_j|\psi\rangle \otimes |j\rangle,$$

that is, it correlates the states of $\mathcal{A}$ and $\mathcal{B}$.

• If we perform the measurement $\{\mathbb{I} \otimes |j\rangle\langle j|\}$, the outcome j occurs with probability

4 $$\boxed{p_j = \langle \Psi'|(\mathbb{I} \otimes |j\rangle\langle j|)|\Psi'\rangle = \langle \psi|P_j|\psi\rangle},$$

and the post-measurement state of $\mathcal{A}\&\mathcal{B}$ is

5 $$\frac{(\mathbb{I} \otimes |j\rangle\langle j|)|\Psi'\rangle}{\sqrt{p_j}} = \frac{P_j|\psi\rangle \otimes |j\rangle}{||P_j|\psi\rangle||} = \frac{P_j|\psi\rangle}{||P_j|\psi\rangle||} \otimes |j\rangle.$$

The post-measurement state of $\mathcal{A}$ is

6 $$\boxed{\frac{P_j|\psi\rangle}{||P_j|\psi\rangle||}}.$$

• Once the system becomes entangled,the state of $\mathcal{A}$ is described by the operator

$\boxed{E}$ **ENTANGLEMENT**

....

7 $$\sum_j p_j \frac{E_j|\psi\rangle\langle\psi|P_j}{\langle\psi|P_j|\psi\rangle} = \sum_j P_j|\psi\rangle\langle\psi|P_j.$$

• If the initial state of our system is a mixed state ϱ, the measurement modifies it as

8 $$\varrho \mapsto \sum_j P_j \varrho P_j.$$

■ **Example**. *Qubit* $\mathcal{A}$ (system) & *qubit* $\mathcal{B}$ (pointer)

System state: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$,

Orthogonal proj.: $\{P_0 = |0\rangle\langle 0|,\ P_1 = |1\rangle\langle 1|\}$,

Pointer basis: $\{|j\rangle\}_{j=0}^{1} = \{|0\rangle, |1\rangle\}$

Entangled correlated state:

9 $$|\Psi'\rangle = U(|\psi\rangle \otimes |0\rangle) = \alpha|0\rangle \otimes |0\rangle + \beta|1\rangle \otimes |1\rangle.$$

If we measure $\mathcal{B}$ in basis$\{|0\rangle, |1\rangle\}$, that is, the measurement $\{\mathbb{I} \otimes |0\rangle\langle 0|, \mathbb{I} \otimes |1\rangle\langle 1|\}$ of $\mathcal{A}\&\mathcal{B}$:

	Outcome	Probability	Post-measurement state of $\mathcal{A}$
10	0	$\|\alpha\|^2$	$\|0\rangle$
	1	$\|\beta\|^2$	$\|1\rangle$.

■ **Remarks**. Since $P_j^\dagger = P_j = P_j^2$,

11 $\boxed{\sum_{j=0}^{n-1} P_j = \mathbb{I}}$ can be written as $\boxed{\sum_{j=0}^{n-1} P_j^\dagger P_j = \mathbb{I}}$,

12 $\boxed{p_j = \langle\psi|P_j|\psi\rangle}$ can be written as $\boxed{p_j = \mathrm{tr}(P_j|\psi\rangle\langle\psi|P_j^\dagger)}$.

If the state *just prior to the measurement* is ϱ, then

13 the *outcome* j is obtained with *a priori probability* $\boxed{p_j = \mathrm{tr}(P_j \varrho P_j^\dagger)}$.

If the *outcome* j is attained, then

14 the state *just after the measurement* is $\boxed{\frac{P_j \varrho P_j^\dagger}{p_j}}$.

G GENERALIZED MEASUREMENTS

■ Definition.

1 A *generalized measurement* *is* a system of measurement operators $M_k:\mathcal{H}\to\mathcal{H}$ with $\boxed{\sum_{k=0}^{m-1} M_k^\dagger M_k = \mathbb{I}.}$

2 In a state ϱ
- k occurs with the probability $p_k = \mathrm{tr}(M_k\varrho M_k^\dagger)$
- if k occurs, post-measurement state is $\varrho_k = \frac{M_k\varrho M_k^\dagger}{p_k}$.

■ Theorem.

3 In the case of a pure state $|\psi\rangle$
- k occurs with the probability $p_k = ||M_k|\psi\rangle||^2$
- if k occurs, post-measurement state is $\frac{M_k|\psi\rangle}{\sqrt{p_k}}$.

4 Post-measurement states need not be orthogonal.

5 A generalized measurement is repetable $\Leftrightarrow$ it is orthogonal.

■ Theorem (Naimark)

6 • Any generalized measurement in $\mathcal{H}_A$ can be described by a standard observable on a larger Hilbert space $\mathcal{H}_A\otimes\mathcal{H}_B$ ($\mathcal{H}_B$ represents the apparatus).

- $\sum_{k=0}^{m-1} M_k^\dagger M_k = \mathbb{I}$,
- $p_k = \mathrm{tr}(M_k\varrho_A M_k^\dagger)$,
- $\varrho_A \mapsto \varrho_{Ak} = \frac{M_k\varrho_A M_k^\dagger}{p_k}$

$\Downarrow$

There exist:
- $\mathcal{H}_B$ of dimension m (called apparatus, pointer or ancilla),
- an initial state $\varrho_B = |\omega\rangle\langle\omega|$ of $\mathcal{H}_B$
- an entangling unitary transform
$$\mathcal{H}_A\otimes\mathcal{H}_B \xrightarrow{U} \mathcal{H}_A\otimes\mathcal{H}_B$$
which correlates the states of the system and of the apparatus,
- an orthogonal measurement in $\mathcal{H}_B$ $\{P_k = |k\rangle\langle k|\}_{k=0}^{m-1}$ corresponding to the orthogonal measurement $\{\mathbb{I}\otimes P_k\}$ in $\mathcal{H}_A\otimes\mathcal{H}_B$

such that:
- $M_k|\psi\rangle = {}_B\langle k|U|\psi\omega\rangle$
- $p_k = \mathrm{tr}(U\varrho_A\otimes\varrho_B U^\dagger \mathbb{I}\otimes P_k)$
$= \mathrm{tr}_A({}_B\langle k|U\varrho_A\otimes\varrho_B U^\dagger|k\rangle_B)$
- $\varrho_A \mapsto \varrho_{Ak} = \frac{\mathrm{tr}_B(U\varrho_A\otimes\varrho_B U^\dagger \mathbb{I}\otimes P_k)}{p_k}$
$= \frac{{}_B\langle k|U\varrho_A\otimes\varrho_B U^\dagger|k\rangle_B}{p_k}$.

7 • From any such a measurement in $\mathcal{H}_A\otimes\mathcal{H}_B$, a generalized measurement in $\mathcal{H}_A$ is obtained by restricting the attention to $\mathcal{H}_A$ alone (as an indirect measurement).

■ Remark.

We obtain an information about a system by coupling it to an ancilla, let them evolve together and then perform the measurement of an observable on the ancilla.

■ Description of a measurement

- To perform a measurement $\{M_k\}_{k=0}^{m-1}$ of $\mathcal{A}$, we use a m-dimensional *pointer* $\mathcal{B}$ with fiducial orthonormal basis states $\{|k\rangle\}_{k=0}^{m-1}$.
- The *unitary operator* $U:\mathcal{H}_A\otimes|0\rangle\to\mathcal{H}_A\otimes\mathcal{H}_B$,

8 $$\boxed{U = \sum_{j,k} M_k\otimes|j+k\rangle\langle j|}$$

transforms an initial state $|\psi\rangle\otimes|0\rangle$ into

9 $$|\Psi\rangle = U(|\psi\rangle\otimes|0\rangle) = \sum_k M_k|\psi\rangle\otimes|k\rangle,$$

that is, it correlates the states of $\mathcal{A}$ and $\mathcal{B}$.

- If we perform the measurement $\{\mathbb{I}\otimes|j\rangle\langle j|\}$, the outcome k occurs with probability

10 $$\boxed{p_k = \langle\Psi|(\mathbb{I}\otimes|k\rangle\langle k|)|\Psi\rangle = \langle\psi|M_k|\psi\rangle},$$

and the post-measurement state of $\mathcal{A}\&\mathcal{B}$ is

11 $$\frac{(\mathbb{I}\otimes|k\rangle\langle k|)|\Psi'\rangle}{\sqrt{p_k}} = \frac{M_k|\psi\rangle\otimes|k\rangle}{||M_k|\psi\rangle||} = \frac{M_k|\psi\rangle}{||M_k|\psi\rangle||}\otimes|k\rangle.$$

The post-measurement state of $\mathcal{A}$ is

12 $$\boxed{\frac{M_k|\psi\rangle}{||M_k|\psi\rangle||}}.$$

■ Example. *Qubit* $\mathcal{A}$ (system) & *qubit* $\mathcal{B}$ (pointer)

System state: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$,

Orthogonal proj.: $\{E_0 = |0\rangle\langle 0|,\ E_1 = |1\rangle\langle 1|\}$,

Pointer bases: $|0\rangle$, $|1\rangle$ and $|0'\rangle = \frac{1}{\sqrt2}(|0\rangle+|1\rangle)$, $|1'\rangle = \frac{1}{\sqrt2}(|0\rangle-|1\rangle)$

Entangled correlated state:

13 $$U(|\psi\rangle\otimes|0\rangle) = E_0|\psi\rangle\otimes|0\rangle + E_1|\psi\rangle\otimes|1\rangle = M_0|\psi\rangle\otimes|0'\rangle + M_1|\psi\rangle\otimes|1'\rangle,$$

where

14 $$M_0 = \frac{1}{\sqrt2}\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad M_1 = \frac{1}{\sqrt2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

If we measure the pointer in $\{|0'\rangle, |1'\rangle\}$, we get:

15

Outcome	Probability	Post-measurement state of $\mathcal{A}$
$0'$	$p_0 = \langle\psi|M_0^\dagger M_0|\psi\rangle = \frac12$	$\frac{M_0|\psi\rangle}{\sqrt{p_0}} = \alpha|0\rangle+\beta|1\rangle$
$1'$	$p_1 = \langle\psi|M_1^\dagger M_1|\psi\rangle = \frac12$	$\frac{M_1|\psi\rangle}{\sqrt{p_1}} = \alpha|0\rangle-\beta|1\rangle$.

Remarks:
- $|\alpha|\neq|\beta| \Rightarrow$ post-measurement states are not orthogonal.
- The orthogonal measurement $\{\mathbb{I}\otimes|0'\rangle\langle 0'|,\ \mathbb{I}\otimes|1'\rangle\langle 1'|\}$ of $\mathcal{A}\&\mathcal{B}$ realizes a non-orthogonal measurement of $\mathcal{A}$.

H POSITIVE OPERATOR-VALUED MEASURES (POVM)

■ Definition.

1 A *POVM* *is* a system of operators $\Pi_k:\mathcal{H}\to\mathcal{H}$ such that $\boxed{\Pi_k\geq 0,\ \sum_k \Pi_k = \mathbb{I}.}$

2 A probability $\boxed{p_k = \mathrm{tr}(\varrho\,\Pi_k)}$ is attributed to each measurement outcome k.

■ Theorem.

3 $$\{M_k\} = \text{generalized measurement} \Rightarrow \{\Pi_k = M_k^\dagger M_k\} = \text{POVM}.$$

4 $$\{\Pi_k\} = \text{POVM} \Rightarrow \begin{cases}\text{For any } \{U_k\} = \text{unitary operators}\\ \{M_k = U_k\sqrt{\Pi_k}\} = \text{generalized measurement}.\end{cases}$$

■ Remark.

5 The post-measurement state $U_k\left(\frac{\sqrt{\Pi_k}|\psi\rangle}{||\sqrt{\Pi_k}|\psi\rangle||}\right)$ depends on the unitary operators $\{U_k\}$ we choose.

I QUANTUM CHANNELS/OPERATIONS

■ **Definition**. A map $\mathcal{L}(\mathcal{H}_A)\xrightarrow{\mathcal{E}}\mathcal{L}(\mathcal{H}_{A'}):\ \varrho\mapsto\mathcal{E}(\varrho)$

- is *linear* if $\mathcal{E}(\alpha\varrho_1+\beta\varrho_2)=\alpha\mathcal{E}(\varrho_1)+\beta\mathcal{E}(\varrho_2)$;
- *preserves Hermiticity* if $\varrho^\dagger=\varrho\ \Rightarrow\ \mathcal{E}(\varrho)^\dagger=\mathcal{E}(\varrho)$;
- *preserves trace* if $\operatorname{tr}\mathcal{E}(\varrho)=\operatorname{tr}\varrho$;
- is *positive* ($\mathcal{E}\geq 0$) if $\varrho\geq 0\ \Rightarrow\ \mathcal{E}(\varrho)\geq 0$;
- is a *completely positive map* if $\mathcal{E}\otimes\mathbb{I}_{\mathcal{L}(\mathcal{H})}\geq 0$, for any $\mathcal{H}$.

■ **Definition**.

1 A *quantum channel* is a linear map $\mathcal{E}:\mathcal{L}(\mathcal{H}_A)\to\mathcal{L}(\mathcal{H}_{A'})$ such that
- $\operatorname{tr}\mathcal{E}(\varrho)=\operatorname{tr}\varrho$
- $\mathcal{E}\geq 0$
- $\mathcal{E}\otimes\mathbb{I}_{\mathcal{L}(\mathcal{H})}\geq 0,\ \forall\mathcal{H}$

A *quantum operation* is a linear map $\mathcal{E}:\mathcal{L}(\mathcal{H}_A)\to\mathcal{L}(\mathcal{H}_{A'})$ such that
- $0\leq\operatorname{tr}\mathcal{E}(\varrho)\leq\operatorname{tr}\varrho$
- $\mathcal{E}\geq 0$
- $\mathcal{E}\otimes\mathbb{I}_{\mathcal{L}(\mathcal{H})}\geq 0,\ \forall\mathcal{H}$

■ **Lemma.**

2 Any map $\varrho\mapsto\mathcal{E}(\varrho)$ admitting an operator sum representation is a channel

There exist $K_j:\mathcal{H}_A\to\mathcal{H}_{A'}$ linear operators such that
- $\mathcal{E}(\varrho)=\sum_{j=0}^{m-1}K_j\varrho K_j^\dagger$
- $\sum_{j=0}^{m-1}K_j^\dagger K_j=\mathbb{I}$

$\Downarrow$

$\mathcal{L}(\mathcal{H}_A)\xrightarrow{\mathcal{E}}\mathcal{L}(\mathcal{H}_{A'})$, $\varrho\mapsto\mathcal{E}(\varrho)$ is a channel

■ **Theorem** (*Channel-state / Choi-Jamilkowski duality*).

3 Up to normalization, there exists a one-to-one correspondence: channel $\updownarrow$ state of a larger system $\mathcal{H}_{A'}\otimes\mathcal{H}_B$

$\mathcal{L}(\mathcal{H}_A)\xrightarrow{\mathcal{E}}\mathcal{L}(\mathcal{H}_{A'})$ channel

$\updownarrow$

$(\mathcal{E}\otimes\mathbb{I}_B)(|\Phi\rangle\langle\Phi|)=\sum_{j=0}^{m-1}p_j\,|\Psi_j\rangle\langle\Psi_j|$ density operator

$\{|k\rangle_A\}_{k=0}^{n-1}$ basis in $\mathcal{H}_A$

$\{|k\rangle_B\}_{k=0}^{n-1}$ basis in $\mathcal{H}_B$

$|\Phi\rangle=\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}|k\rangle_A\otimes|k\rangle_B$

■ **Theorem** (*Kraus*).

4 A map $\varrho\mapsto\mathcal{E}(\varrho)$ is a channel if and only if it admits an operator sum -representation

$\mathcal{L}(\mathcal{H}_A)\xrightarrow{\mathcal{E}}\mathcal{L}(\mathcal{H}_{A'})$, $\varrho\mapsto\mathcal{E}(\varrho)$ is a channel

$\Updownarrow$

There exist $K_j:\mathcal{H}_A\to\mathcal{H}_{A'}$, $K_j|k\rangle_A=\sqrt{n\,p_j}\,{}_B\langle k|\Psi_j\rangle_{A'B}$

such that $\mathcal{E}(\varrho)=\sum_{j=0}^{m-1}K_j\varrho K_j^\dagger$, $\quad\sum_{j=0}^{m-1}K_j^\dagger K_j=\mathbb{I}$

■ **Theorem** (*Stinespring dilatation*).

5 A map $\varrho\mapsto\mathcal{E}(\varrho)$ is a channel if and only if it is a partial trace of a unitary evolution in a larger Hilbert space with factorized initial condition

$\mathcal{L}(\mathcal{H}_A)\xrightarrow{\mathcal{E}}\mathcal{L}(\mathcal{H}_{A'})$, $\varrho\mapsto\mathcal{E}(\varrho)$ is a channel $\quad\mathcal{E}(\varrho)=\sum_{j=0}^{m-1}K_j\varrho K_j^\dagger$, $\sum_{j=0}^{m-1}K_j^\dagger K_j=\mathbb{I}$

$\Updownarrow$

There exist:
- $\mathcal{H}_B$ of dimension m
- $\{|j\rangle_B\}_{j=0}^{m-1}$ basis in $\mathcal{H}_B$
- a unitary operator $U:\mathcal{H}_A\otimes\mathcal{H}_B\to\mathcal{H}_A\otimes\mathcal{H}_B$ acting as $U(|\psi\rangle\otimes|0\rangle)=\sum_j K_j|\psi\rangle\otimes|j\rangle$

such that:
- $\mathcal{E}(\varrho)=\operatorname{tr}_B(U(\varrho\otimes|0\rangle\langle 0|)U^\dagger)$
- $K_j|\psi\rangle={}_B\langle j|U|\psi\rangle\otimes|0\rangle$

■ **Remark**.

6 Any channel $\varrho\mapsto\mathcal{E}(\varrho)$ is the partial trace of a unitary operator

$\varrho\mapsto\mathcal{E}(\varrho)$ is a channel, $\mathcal{E}(\varrho)=\sum_{j=0}^{m-1}K_j\varrho K_j^\dagger$, $\sum_j K_j^\dagger K_j=\mathbb{I}$

$\Downarrow$

There exist:
- $\mathcal{H}_B$ of dimension m
- $\{|j\rangle_B\}_{j=0}^{m-1}$ basis in $\mathcal{H}_B$
- the unitary operator $U:\mathcal{H}_A\to\mathcal{H}_A\otimes\mathcal{H}_B$, $U|\psi\rangle=\sum_j K_j|\psi\rangle\otimes|j\rangle$

such that:
- $\mathcal{E}(\varrho)=\operatorname{tr}_B(U\varrho U^\dagger)$
- $K_j={}_B\langle j|U$

■ **Theorem**.

7 *If $\varrho\mapsto\mathcal{E}(\varrho)$ is a quantum channel, then $\mathcal{E}$(density operator) = density operator.*

8 *Sum-representation of a channel is not unique.*

9 *Quantum channels form a semigroup*

10 $\boxed{\mathcal{E}=\text{reversible channel}}\Leftrightarrow\boxed{\mathcal{E}=\text{unitary transform}}$

11 *Heisenberg picture*

$$\begin{array}{l}\varrho(t)=U(t)\varrho U(t)^\dagger\\ A(t)=U(t)^\dagger AU(t)\end{array}\Rightarrow\operatorname{tr}(A\varrho(t))=\operatorname{tr}(A(t)\varrho).$$

12 *Dual $\mathcal{E}^*$ of a channel $\mathcal{E}$*

$$\begin{array}{l}\mathcal{E}(\varrho)=\sum K_j\varrho K_j^\dagger\\ \mathcal{E}^*(A)=\sum K_j^\dagger AK_j\end{array}\Rightarrow\operatorname{tr}(A\mathcal{E}(\varrho))=\operatorname{tr}(\mathcal{E}^*(A)\varrho).$$

■ **Example**. *Non-selective measurement* (measurement is performed, but its outcome is not known).

System state: $|\psi\rangle=\alpha|0\rangle+\beta|1\rangle$,

Orthogonal proj.: $\{E_0=|0\rangle\langle 0|,\ E_1=|1\rangle\langle 1|\}$,

Pointer bases: $|0\rangle$, $|1\rangle$ and $|0'\rangle=\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$, $|1'\rangle=\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)$

The unitary entangling transform

13 $$U=\sum_{j,k}E_a\otimes|k+j\rangle\langle k|,$$

transforms the product state $|\psi\rangle\otimes|0\rangle$ as

14 $$\begin{aligned}U(|\psi\rangle\otimes|0\rangle)&=E_0|\psi\rangle\otimes|0\rangle+E_1|\psi\rangle\otimes|1\rangle\\&=\alpha|0\rangle\otimes|0\rangle+\beta|1\rangle\otimes|1\rangle\\&=\tfrac{1}{\sqrt{2}}(\alpha|0\rangle+\beta|1\rangle)\otimes|0'\rangle\\&\quad+\tfrac{1}{\sqrt{2}}(\alpha|0\rangle-\beta|1\rangle)\otimes|1'\rangle\\&=M_0|\psi\rangle\otimes|0'\rangle+M_1|\psi\rangle\otimes|1'\rangle,\end{aligned}$$

where

15 $$M_0=\tfrac{1}{\sqrt{2}}\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad M_1=\tfrac{1}{\sqrt{2}}\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

For A, considered alone, U defines the channel

16 $$\begin{aligned}|\psi\rangle\langle\psi|\mapsto\mathcal{E}(|\psi\rangle\langle\psi|)&=\operatorname{tr}_B(U(|\psi\rangle\otimes|0\rangle)(\langle\psi|\otimes\langle 0|)U^\dagger)\\&=M_0|\psi\rangle\langle\psi|M_0^\dagger+M_1|\psi\rangle\langle\psi|M_1^\dagger,\end{aligned}$$

that is

17 $$|\psi\rangle\langle\psi|\mapsto\mathcal{E}(|\psi\rangle\langle\psi|)=\sum_j M_j|\psi\rangle\langle\psi|M_j^\dagger.$$

For a mixed state ϱ of A

18 $$\boxed{\varrho\mapsto\mathcal{E}(\varrho)=\sum_j M_j\varrho M_j^\dagger.}$$

■ **Example.** *Depolarizing channel* $(\mathcal{H}_A = \mathbb{C}^2)$

• *Description.*

27 $$\mathcal{E}_{A\to A} : |\psi\rangle \mapsto \begin{array}{l} |\psi\rangle \text{ with probability } 1-p \\ \sigma_1|\psi\rangle \text{ with probability } p/3 \\ \sigma_2|\psi\rangle \text{ with probability } p/3 \\ \sigma_3|\psi\rangle \text{ with probability } p/3 \end{array}$$

where

28 $$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

• *Unitary representation* $U_{A\to AE} : \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^4$

By using a 4-dimensional environment E, we define the scalar product preserving map

29 $$|\psi\rangle_A \mapsto \sqrt{1-p}\,|\psi\rangle_A \otimes |0\rangle_E + \sqrt{\tfrac{p}{3}}\,\sigma_1|\psi\rangle_A \otimes |1\rangle_E + \sqrt{\tfrac{p}{3}}\,\sigma_2|\psi\rangle_A \otimes |2\rangle_E + \sqrt{\tfrac{p}{3}}\,\sigma_3|\psi\rangle_A \otimes |4\rangle_E.$$

• *Operator-sum representation.*

The Kraus operators are: $M_a = {}_E\langle a|U$.

30 $$\varrho \mapsto \mathcal{E}(\varrho) = \sum_{a=0}^{3} M_a \varrho M_a^\dagger \quad \boxed{\begin{array}{l} M_0 = \sqrt{1-p}\,\mathbb{I} \\ M_1 = \sqrt{p/3}\,\sigma_1 \\ M_2 = \sqrt{p/3}\,\sigma_2 \\ M_3 = \sqrt{p/3}\,\sigma_3 \end{array}}$$

• *Relative-state representation.*

By introducing a reference qubit R and the orthogonal maximally entangled states

31 $$\begin{aligned} |\phi^+\rangle_{RA} &= \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \\ |\phi^-\rangle_{RA} &= \tfrac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \\ |\psi^+\rangle_{RA} &= \tfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \\ |\psi^-\rangle_{RA} &= \tfrac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \end{aligned}$$

when the channel acts only on A, we get

32 $$|\phi^+\rangle\langle\phi^+| \mapsto (1-p)|\phi^+\rangle\langle\phi^+| + \tfrac{p}{3}|\psi^+\rangle\langle\psi^+| + \tfrac{p}{3}|\psi^-\rangle\langle\psi^-| + \tfrac{p}{3}|\phi^-\rangle\langle\phi^-|.$$

• *Bloch-sphere representation.*

33 $$\varrho = \tfrac{1}{2}(\mathbb{I} + \vec{P}\cdot\vec{\sigma}) \mapsto \mathcal{E}(\varrho) = \tfrac{1}{2}\Big(\mathbb{I} + (1-\tfrac{4}{3}p)\vec{P}\cdot\vec{\sigma}\Big)$$

■ **Example.** *Dephasing channel*

• *Unitary representation* $U_{A\to AE} : \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$.

By using a 3-dimensional environment, we define the scalar product preserving map

34 $$\begin{aligned} |0\rangle_A &\mapsto \sqrt{1-p}\,|0\rangle_A \otimes |0\rangle_E + \sqrt{p}\,|0\rangle_A \otimes |1\rangle_E \\ |1\rangle_A &\mapsto \sqrt{1-p}\,|1\rangle_A \otimes |0\rangle_E + \sqrt{p}\,|1\rangle_A \otimes |2\rangle_E \end{aligned}$$

• *Operator-sum representation.*

The Kraus operators are: $M_a = {}_E\langle a|\mathbf{U}$.

35 $$\varrho \mapsto \mathcal{E}(\varrho) = \sum_{a=0}^{2} M_a \varrho M_a^\dagger \quad \boxed{\begin{array}{l} M_0 = \sqrt{1-p}\,\mathbf{I} \\ M_1 = \sqrt{p}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ M_2 = \sqrt{p}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \end{array}}$$

Possible representation with two operators:

36 $$\varrho \mapsto \mathcal{E}(\varrho) = \sum_{a=0}^{1} M_a \varrho M_a^\dagger \quad \boxed{\begin{array}{l} M_0 = \sqrt{1-\tfrac{p}{2}}\,\mathbf{I} \\ M_1 = \sqrt{\tfrac{p}{2}}\,\sigma_3. \end{array}}$$

• *Density operator evolution*

37 $$\varrho = \begin{pmatrix} \varrho_{00} & \varrho_{01} \\ \varrho_{10} & \varrho_{11} \end{pmatrix} \mapsto \mathcal{E}(\varrho) = \begin{pmatrix} \varrho_{00} & (1-p)\varrho_{01} \\ (1-p)\varrho_{10} & \varrho_{11} \end{pmatrix}.$$

• *Bloch-sphere representation.*

38 $$\varrho = \tfrac{1}{2}(\mathbb{I} + \vec{P}\cdot\vec{\sigma}) \mapsto \mathcal{E}(\varrho) = \tfrac{1}{2}(\mathbb{I} + (1-p)P_1\sigma_1 + (1-p)P_2\sigma_2 + P_3\sigma_3).$$

■ **Example.** *Amplitude-damping channel*

• *Unitary representation* $U_{A\to AE} : \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$.

By using a 2-dimensional environment, we define the scalar product preserving map

39 $$\begin{aligned} |0\rangle_A \otimes |0\rangle_E &\mapsto |0\rangle_A \otimes |0\rangle_E \\ |1\rangle_A \otimes |0\rangle_E &\mapsto \sqrt{1-p}\,|1\rangle_A \otimes |0\rangle_E + \sqrt{p}\,|0\rangle_A \otimes |1\rangle_E \end{aligned}$$

• *Operator-sum representation.*

The Kraus operators are: $M_a = {}_E\langle a|\mathbf{U}$.

40 $$\varrho \mapsto \mathcal{E}(\varrho) = \sum_{a=0}^{1} M_a \varrho M_a^\dagger \quad \boxed{\begin{array}{l} M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} \\ M_1 = \sqrt{p}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{array}}$$

• *Density operator evolution* $\varrho \mapsto \mathcal{E}(\varrho)$:

41 $$\begin{pmatrix} \varrho_{00} & \varrho_{01} \\ \varrho_{10} & \varrho_{11} \end{pmatrix} \mapsto \begin{pmatrix} \varrho_{00} + p\,\varrho_{11} & \sqrt{1-p}\;\varrho_{01} \\ \sqrt{1-p}\;\varrho_{10} & (1-p)\;\varrho_{11} \end{pmatrix}.$$

J <u>AXIOMS OF QUANTUM MECHANICS</u> (for open systems)

1 *Open system*= an imperfectly isolated quantum system which changes energy and information with its unobserved environment.

■ *Axioms of quantum mechanics* (case of open systems).

• <u>*Axiom* **1'**.</u>

2 **State** = $\boxed{\text{a density operator } \varrho \text{ with } \begin{array}{l} \varrho \geq 0, \\ \mathrm{tr}\,\varrho = 1. \end{array}}$

• <u>*Axiom* **2'**.</u>

3 **Measurement**= $\boxed{\text{a POVM } \{E_a\} \text{ with } \begin{array}{l} E_a \geq 0, \\ \sum_a E_a = \mathbb{I}. \end{array}}$

When the measurement $\{E_a\}$ is performed on the state ϱ , the outcome a occurs with probability

4 $$\boxed{prob(a) = \mathrm{tr}(\varrho E_a)}.$$

• <u>*Axiom* **3'**.</u> **Dynamics.** Time evolution is described by a trace-preserving completely positive map (TPCP)

5 $$\varrho \mapsto \mathcal{E}(\varrho) \text{ with } \begin{array}{l} \mathrm{tr}\,\mathcal{E}(\varrho) = \mathrm{tr}\,\varrho, \\ \varrho\otimes\mathbb{I}_n \geq 0 \Rightarrow \mathcal{E}(\varrho)\otimes\mathbb{I}_n \geq 0. \end{array}$$

■ **Remarks:**

- closed system = limit case of open system.
- an open system $\mathcal{A}$ can be regarded as a part $\mathcal{A}$ of a composite system $\mathcal{A}\&\mathcal{B}$ when we observe subsystem $\mathcal{A}$ alone.
- every density operator has a purification in an extended system.
- every POVM can be realized by an orthogonal measurement in an extended system.
- every trace preserving completely positive map has an isometric Stinespring dilatation.

ADDITIONAL DEFINITIONS/RESULTS

#1 Exercise.

For a $\mathbb{K}$-vector space $\mathcal{V}$, $A\in\mathcal{L}(\mathcal{V})$ and $f:\mathcal{V}\to\mathbb{K}$:

$$\boxed{\begin{array}{r} Av=f(v)\,v,\\ \text{for any } v\end{array}} \Rightarrow \boxed{\begin{array}{l} f \text{ is a constant function,}\\ \quad f(v)=\lambda,\ \text{that is } A=\lambda\mathbb{I}\end{array}}$$

#2 Theorem (*Singular value decomposition*).

$$\mathbf{A}\in\mathcal{M}_{n\times m}(\mathbb{C}) \Rightarrow \begin{array}{c}\text{there}\\ \text{exist}\end{array} \begin{array}{c}\mathbf{U}\in U(n),\\ \mathbf{V}\in U(m),\\ \mathbf{D}\in\mathcal{M}_{n\times m}(\mathbb{C})\\ \text{diagonal}\end{array} \begin{array}{c}\text{such}\\ \text{that}\end{array} \boxed{\mathbf{A}=\mathbf{UDV}}$$

#3 Theorem (*Polar decomposition*).

$$\mathbf{A}\in\mathcal{M}_{n\times n}(\mathbb{C}) \Rightarrow \begin{array}{c}\text{there}\\ \text{exist}\end{array} \begin{array}{c}\mathbf{U}\in U(n),\\ \mathbf{P}\in\mathcal{M}_{n\times m}(\mathbb{C})\\ \text{with } \mathbf{P}\geq 0\end{array} \begin{array}{c}\text{such}\\ \text{that}\end{array} \boxed{\mathbf{A}=\mathbf{UP}}$$

$$\mathbf{A}\in\mathcal{M}_{n\times n}(\mathbb{C}) \Rightarrow \begin{array}{c}\text{there}\\ \text{exist}\end{array} \begin{array}{c}\mathbf{U}\in U(n),\\ \mathbf{P}\in\mathcal{M}_{n\times m}(\mathbb{C})\\ \text{with } \mathbf{P}\geq 0\end{array} \begin{array}{c}\text{such}\\ \text{that}\end{array} \boxed{\mathbf{A}=\mathbf{PU}}$$

#4 Definition.

A linear operator $A:\mathcal{H}\to\mathcal{H}$ is a *bounded operator* if there exists a constant $C\in(0,\infty)$ such that $||Ax||\leq C\,||x||,\ \forall\, x\in\mathcal{H}$.

#5 Theorem.

$$\mathcal{H} \text{ is finite-dimensional} \Rightarrow \boxed{\begin{array}{c}\text{Any linear operator}\\ A:\mathcal{H}\to\mathcal{H}\\ \text{is a } \textit{bounded operator.}\end{array}}$$

#6 Definition.

A *tight frame* in a Hilbert space $\mathcal{H}$ is a set of vectors $\{u_1,u_2,\dots,u_\ell\}$ such that $\boxed{\sum_{j=1}^{\ell}|u_j\rangle\langle u_j|=\mathbb{I}_{\mathcal{H}}}$

#7 Remark.

Any orthonormal basis is a tight frame.

#8 Exercise

$$\boxed{(\mathrm{e}^{A})^\dagger=\mathrm{e}^{A^\dagger}}.$$

#9 Exercise.

$$\boxed{A\in\mathcal{A}(\mathcal{H}) \quad\Rightarrow \mathrm{e}^{\mathrm{i}A}\in U(\mathcal{H})}.$$

#10 Theorem Discrete (finite) *Fourier transform.*

$$\begin{array}{c}F:\mathbb{C}^n\longrightarrow\mathbb{C}^n\\ \varphi\mapsto F[\varphi]\end{array} \quad \boxed{F[\varphi](k)=\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{n}kj}\varphi(j)}$$

defined by using the identification $\{\varphi:\{0,1,\dots,n-1\}\to\mathbb{C}\}\equiv\mathbb{C}^n$

$$\varphi\equiv(\varphi(0),\varphi(1),\dots,\varphi(n-1)),$$

is a unitary transformation. Its inverse is

$$F^{-1}=F^\dagger,\ \text{where } \boxed{F^\dagger[\varphi](k)=\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}\mathrm{e}^{\frac{2\pi\mathrm{i}}{n}kj}\varphi(j)}.$$

#11 Theorem Alternative definition of $\mathcal{H}_1\otimes\mathcal{H}_2$.

$$\mathcal{H}_1\otimes\mathcal{H}_2 \text{ can be defined as } \boxed{\begin{array}{c}\text{the space of all the}\\ \text{anti-linear operators}\\ T:\mathcal{H}_2\to\mathcal{H}_1.\end{array}}$$

#12 Theorem.

A tensor product of tight frames is a tight frame.

#13 Remark.

The superposition (liniar combination) of states

$$|\psi\rangle=\sum_{k=1}^{n}c_k|\psi_k\rangle$$

is a pure state with the density operator

$$\varrho=|\psi\rangle\langle\psi|=\sum_{k=1}^{n}|c_k|^2|\psi_k\rangle\langle\psi_k|+\sum_{k\neq j}c_k\,\bar{c}_j\,|\psi_k\rangle\langle\psi_j|.$$

The second sum contains the *interference terms.*

#14 Exercise.

Representations as a probabilistic ensemble of pure states of the maximally mixed state of a qubit:

$$\varrho=\begin{pmatrix}\frac12 & 0\\ 0 & \frac12\end{pmatrix}\Rightarrow$$

- $\varrho=\ \frac12|0\rangle\langle0|+\frac12|1\rangle\langle1|$,
- $\varrho=\ \frac12|+\rangle\langle+|+\frac12|-\rangle\langle-|$,
- $\varrho=\frac12|\uparrow_y\rangle\langle\uparrow_y|+\frac12|\downarrow_y\rangle\langle\downarrow_y|$,
- $\varrho=\ \frac14|0\rangle\langle0|+\frac14|1\rangle\langle1|$ $+\frac14|+\rangle\langle+|+\frac14|-\rangle\langle-|$,
- etc.

#15 Exercise.

$$\boxed{\begin{array}{l}\text{A mixed state } \varrho \text{ of any quantum}\\ \text{system can be realised as an}\\ \textit{ensemble} \text{ of orthogonal pure states.}\end{array}}$$

#16 Exercise.

$$\boxed{\begin{array}{l}\text{In the Schrödinger picture, the time}\\ \text{evolution of the density operator}\\ \qquad\qquad \varrho=\sum_{k=0}^{d-1}p_k|\psi_k\rangle\langle\psi_k|\\ \text{for a closed system is described by}\\ \qquad\qquad \mathrm{i}\hbar\frac{\partial}{\partial t}\varrho=[H,\varrho],\\ \text{where } H \text{ is the Hamiltonian of the system.}\end{array}}$$

MAIN PROOFS

A Axioms of Quantum Mechanics

A3 If $\{|\psi_j\rangle\}_{j=1}^n$ is an orthonormal basis of $\mathcal{H}$, then

$$\begin{aligned}\mathrm{tr}(E_k\,|\psi\rangle\langle\psi|) &= \sum_{j=1}^n \langle\psi_j|E_k|\psi\rangle\langle\psi|\psi_j\rangle\\ &= \sum_{j=1}^n \langle\psi|\psi_j\rangle\langle\psi_j|E_k|\psi\rangle\\ &= \langle\psi|\sum_{j=1}^n |\psi_j\rangle\langle\psi_j|E_k|\psi\rangle\\ &= \langle\psi|E_k|\psi\rangle\\ &= \langle\psi|E_k^2|\psi\rangle\\ &= \langle\psi|E_k^\dagger E_k|\psi\rangle\\ &= ||E_k|\psi\rangle||^2.\end{aligned}$$

A4

If $\{|\psi_j\rangle\}_{j=1}^n$ is an orthonormal basis of $\mathcal{H}$, then

$$\begin{aligned}\mathrm{tr}(A\,|\psi\rangle\langle\psi|) &= \sum_{j=1}^n \langle\psi_j|A|\psi\rangle\langle\psi|\psi_j\rangle\\ &= \sum_{j=1}^n \langle\psi|\psi_j\rangle\langle\psi_j|A|\psi\rangle\\ &= \langle\psi|\sum_{j=1}^n |\psi_j\rangle\langle\psi_j|A|\psi\rangle\\ &= \langle\psi|A|\psi\rangle.\end{aligned}$$

and

$$\begin{aligned}\langle A\rangle &= \sum_{j=1}^n a_k\,prob(a_k)\\ &= \sum_{j=1}^n a_k\,\langle\psi|E_k|\psi\rangle\\ &= \langle\psi|\sum_{j=1}^n a_k\,E_k|\psi\rangle\\ &= \langle\psi|A|\psi\rangle.\end{aligned}$$

A7

$$E_k = |\psi_k\rangle\langle\psi_k|$$
$$\Downarrow$$
$$prob(a_k) = \langle\psi|E_k|\psi\rangle = \langle\psi|\psi_k\rangle\langle\psi_k|\psi\rangle = |\langle\psi_k|\psi\rangle|^2.$$

B Density Operators

B1

- $E_\psi^\dagger = (|\psi\rangle\langle\psi|)^\dagger = |\psi\rangle\langle\psi| = E_\psi$.
- $\langle\varphi|E_\psi|\varphi\rangle = \langle\varphi|\psi\rangle\langle\psi|\varphi\rangle = |\langle\varphi|\psi\rangle|^2 \geq 0$.
- If we extend the set $\{|\psi\rangle\}$ up to an orthonormal basis $\{|\psi\rangle, |\varphi_1\rangle, |\varphi_2\rangle, ..., |\varphi_{n-1}\rangle\}$ of $\mathcal{H}$, then

$$\begin{aligned}\mathrm{tr}\,E_\psi &= \langle\psi|E_\psi|\psi\rangle + \sum_{j=1}^{n-1}\langle\varphi_j|E_\psi|\varphi_j\rangle\\ &= \langle\psi|\psi\rangle\langle\psi|\psi\rangle + \sum_{j=1}^{n-1}\langle\varphi_j|\psi\rangle\langle\psi|\varphi_j\rangle = 1.\end{aligned}$$

B5 $\varrho^\dagger = \varrho \;\Rightarrow\; \varrho$ is diagonalizable $\varrho = \sum_{k=1}^n p_k|\psi_k\rangle\langle\psi_k|$.

where $\{|\psi_1\rangle, |\psi_2\rangle, ..., |\psi_n\rangle\}$ is an orthonormal basis, that is $\langle\psi_j|\psi_k\rangle = \delta_{jk}$, $\sum_{k=1}^n |\psi_k\rangle\langle\psi_k| = \mathbb{I}$.

$$\left.\begin{aligned}&\varrho\geq 0 \;\Rightarrow\; p_k = \langle\psi_k|\varrho|\psi_k\rangle \geq 0,\\ &\mathrm{tr}\,\varrho = 1 \;\Rightarrow\; \sum_{k=1}^n p_k = \mathrm{tr}\,\varrho = 1\end{aligned}\right\} \;\Rightarrow\; 0\leq p_k\leq 1.$$
$$\Downarrow$$
$$p_k^2 \leq p_k.$$

$$\varrho^2 = \sum_{k=1}^n p_k^2|\psi_k\rangle\langle\psi_k| \;\Rightarrow\; \mathrm{tr}\,\varrho^2 = \sum_{k=1}^n p_k^2 \leq \sum_{k=1}^n p_k = 1.$$

The minimum value of $\mathrm{tr}\,\varrho^2$ is obtained in the case $p_1 = p_2 = ... = p_n = \frac{1}{n}$, and it is $n\,\frac{1}{n^2} = \frac{1}{n}$.

B7 (Case of a qubit).

If r lies on the line segment connecting two points $r', r'' \in \partial\Omega_3$, then there exists $\lambda\in(0,1)$ such that $r = (1-\lambda)r' + \lambda r''$, and consequently

$$\begin{aligned}\varrho(r) &= \tfrac{1}{2}(r_0\,\mathbb{I} + ((1-\lambda)r' + \lambda r'')\cdot\sigma)\\ &= (1-\lambda)\varrho(r') + \lambda\varrho(r'').\end{aligned}$$

B8

$$0\leq p_k\leq 1 \;\Rightarrow\; 0\leq \sum_{k=1}^n p_k|\langle\psi_k|\psi\rangle|^2 \leq \sum_{k=1}^n |\langle\psi_k|\psi\rangle|^2$$
$$\Downarrow$$
$$0\leq \sum_{k=1}^n p_k\langle\psi|\psi_k\rangle\langle\psi_k|\psi\rangle \leq \sum_{k=1}^n \langle\psi|\psi_k\rangle\langle\psi_k|\psi\rangle$$
$$\Downarrow$$
$$0\leq \langle\psi|\sum_{k=1}^n p_k|\psi_k\rangle\langle\psi_k|\psi\rangle \leq \langle\psi|\sum_{k=1}^n |\psi_k\rangle\langle\psi_k|\psi\rangle$$
$$\Downarrow$$
$$0\leq\langle\psi|\varrho|\psi\rangle\leq\langle\psi|\mathbb{I}|\psi\rangle$$
$$\Downarrow$$
$$0\leq\varrho\leq\mathbb{I}$$

B9

$$\begin{aligned}\varrho = \sum_{k=1}^n p_k|\psi_k\rangle\langle\psi_k| \;\Rightarrow\; \langle A\rangle_\varrho &= \sum_{k=1}^n p_k\langle\psi_k|A|\psi_k\rangle\\ &= \sum_{k=1}^n p_k\,\mathrm{tr}(A\,|\psi_k\rangle\langle\psi_k|)\\ &= \mathrm{tr}(A\sum_{k=1}^n p_k\,|\psi_k\rangle\langle\psi_k|)\\ &= \mathrm{tr}(A\,\varrho).\end{aligned}$$

B10, 11

We have $||\varrho||^2 = \langle\varrho,\varrho\rangle = \mathrm{tr}\,\varrho^2$.

- $\varrho = |\psi\rangle\langle\psi| \;\Rightarrow\; \varrho^2 = |\psi\rangle\langle\psi|\psi\rangle\langle\psi| = \varrho \;\Rightarrow\; \mathrm{tr}\,\varrho^2 = 1$.
- $\varrho^2 = \varrho \;\Rightarrow\; \sum_{k=1}^n p_k^2|\psi_k\rangle\langle\psi_k| = \sum_{k=1}^n p_k|\psi_k\rangle\langle\psi_k|$
  $\Rightarrow\; p_k\in\{0;1\}$, for any k
  $\Rightarrow$ there exists j such that $p_k = \delta_{kj}$
  $\Rightarrow\; \varrho = |\psi_j\rangle\langle\psi_j|$.
- there exists j such that $0<p_j<1$ $\;\Rightarrow\; p_j^2 < p_j \;\Rightarrow\; \mathrm{tr}\,\varrho^2 < \mathrm{tr}\,\varrho = 1$.

Consequently:

$\mathrm{tr}\,\varrho^2 = 1 \;\Rightarrow\; p_k\in\{0;1\}$, for any k
$\Rightarrow$ there exists j such that $p_k = \delta_{kj}$
$\Rightarrow\; \varrho = |\psi_j\rangle\langle\psi_j|$.

B12

$$\varrho\in\mathcal{D}(\mathcal{H}) \Rightarrow \begin{array}{c}\varrho^\dagger = \varrho,\\ \langle\psi|\varrho|\psi\rangle\geq 0,\ \forall\psi\\ \mathrm{tr}\,\varrho = 1\end{array} \Rightarrow \begin{array}{c}(K^\dagger\varrho K)^\dagger = K^\dagger\varrho K,\\ \langle\psi|K^\dagger\varrho K|\psi\rangle\geq 0,\ \forall\psi\\ \mathrm{tr}(K^\dagger\varrho K) = \mathrm{tr}(\varrho K K^\dagger) = 1.\end{array}$$

B13

$((1-\lambda)\varrho_1 + \lambda\varrho_2)^\dagger = (1-\lambda)\varrho_1 + \lambda\varrho_2$,
$\langle\psi|(1-\lambda)\varrho_1 + \lambda\varrho_2|\psi\rangle = (1-\lambda)\langle\psi|\varrho_1|\psi\rangle + \lambda\langle\psi|\varrho_2|\psi\rangle \geq 0$,
$\mathrm{tr}((1-\lambda)\varrho_1 + \lambda\varrho_2) = (1-\lambda)\mathrm{tr}\,\varrho_1 + \lambda\,\mathrm{tr}\,\varrho_2 = 1$.

B14

"$\Rightarrow$"

Let $\varrho = |\psi\rangle\langle\psi|$, and let $\varrho_1, \varrho_2\in\mathcal{D}(\mathcal{H})$, $\lambda\in[0,1]$ be such that $\varrho = (1-\lambda)\,\varrho_1 + \lambda\,\varrho_2$. We have

$$\varrho^2 = \varrho \Rightarrow\; \varrho = (1-\lambda)\,\varrho\varrho_1\varrho + \lambda\,\varrho\varrho_2\varrho,$$
$$\Downarrow$$
$$1 = (1-\lambda)\,\mathrm{tr}(\varrho\varrho_1\varrho) + \lambda\,\mathrm{tr}(\varrho\varrho_2\varrho).$$

Density operators ϱ_1 and ϱ_2 satisfy the relations

$$\mathrm{tr}(\varrho\varrho_i\varrho) = \mathrm{tr}(\varrho^2\varrho_i) = \mathrm{tr}(\varrho\varrho_i) = \langle\varrho,\varrho_i\rangle \le |\langle\varrho,\varrho_i\rangle| \le ||\varrho||\,||\varrho|| \le 1$$

$$\mathrm{tr}(\varrho\varrho_i\varrho) = \textstyle\sum_{j=1}^n\langle\psi_j,\varrho\varrho_i\varrho\psi_j\rangle = \sum_{j=1}^n\langle\varrho\psi_j,\varrho_i\varrho\psi_j\rangle \ge 0,$$

where $\{|\psi_1\rangle,|\psi_2\rangle,...,|\psi_n\rangle\}$ is an orthonormal basis of $\mathcal{H}$.
We have $\mathrm{tr}(\varrho\varrho_1\varrho) = \mathrm{tr}(\varrho\varrho_2\varrho) = 1$ because

$$\begin{array}{c}\mathrm{tr}(\varrho\varrho_1\varrho)<1\\ \text{or}\\ \mathrm{tr}(\varrho\varrho_2\varrho)<1\end{array} \Rightarrow (1-\lambda)\,\mathrm{tr}(\varrho\varrho_1\varrho)+\lambda\,\mathrm{tr}(\varrho\varrho_2\varrho)<1.$$

Consequently,

$$\langle\varrho,\varrho_i\rangle = \mathrm{tr}(\varrho\varrho_i) = \mathrm{tr}(\varrho^2\varrho_i) = \mathrm{tr}(\varrho\varrho_i\varrho) = 1.$$

Since $\langle\varrho,\varrho\rangle = \mathrm{tr}\,\varrho^2 \le 1$ and $\langle\varrho_i,\varrho_i\rangle = \mathrm{tr}\,\varrho_i^2 \le 1$ in Cauchy-Schwarz inequality

$$|\langle\varrho,\varrho_i\rangle|^2 \le \langle\varrho,\varrho\rangle\,\langle\varrho_i,\varrho_i\rangle$$

we have equality

$$|\langle\varrho,\varrho_i\rangle|^2 = \langle\varrho,\varrho\rangle\,\langle\varrho_i,\varrho_i\rangle.$$

This is possible only for $\varrho = c\,\varrho_i$, where c is a constant. But

$$\mathrm{tr}\,\varrho = \mathrm{tr}\,\varrho_i = 1 \quad\Rightarrow\quad \varrho = \varrho_i.$$

"$\Leftarrow$"
Let ϱ be an extremal point of $\mathcal{D}(\mathcal{H})$ with the spectral decomposition

$$\varrho = \sum_{j=1}^n p_j|\psi_j\rangle\langle\psi_j| = p_1|\psi_1\rangle\langle\psi_1| + \dots + p_n|\psi_n\rangle\langle\psi_n|.$$

Each p_k must be either 0 or 1 because

$$\begin{array}{c}0<p_k<1\\ \Downarrow\\ \varrho = p_k\,|\psi_k\rangle\langle\psi_k| + (1-p_k)\sum_{j\neq k}\frac{p_j}{1-p_k}|\psi_j\rangle\langle\psi_j|\end{array}$$

and this is possible only if

$$\varrho = |\psi_k\rangle\langle\psi_k| = \sum_{j\neq k}\frac{p_j}{1-p_k}|\psi_j\rangle\langle\psi_j|,$$

and this is impossible since

$$\begin{array}{c}|\psi_k\rangle\langle\psi_k| = \sum_{j\neq k}\frac{p_j}{1-p_k}|\psi_j\rangle\langle\psi_j|\\ \Downarrow\\ |\psi_k\rangle = |\psi_k\rangle\langle\psi_k|\psi_k\rangle = \sum_{j\neq k}\frac{p_j}{1-p_k}|\psi_j\rangle\langle\psi_j|\psi_k\rangle = 0.\end{array}$$

$\boxed{C}$ Qubits

$\boxed{C7}$

Eigenvalues of $A\in\mathcal{L}(\mathbb{C}^2)$ are the roots of the equation

$$\begin{vmatrix}\frac{r_0+r_3}{2}-\lambda & \frac{r_1-\mathrm{i}r_2}{2}\\ \frac{r_1+\mathrm{i}r_2}{2} & \frac{r_0-r_3}{2}-\lambda\end{vmatrix} = 0.$$

$\boxed{C12}$

$$A\ge 0 \;\Leftrightarrow\; \begin{array}{c}\lambda_1 = \frac{r_0+||r||}{2} \ge 0\\ \text{and}\\ \lambda_2 = \frac{r_0-||r||}{2} \ge 0\end{array} \;\Leftrightarrow\; ||r|| \le r_0.$$

$\boxed{C19}$

The diametrically opposite point corresponding to (ϱ,θ) has the angular coordinates $(\varrho+\pi,\theta-\pi)$.
The corresponding pure states

$$\cos\tfrac{\theta}{2}\,|0\rangle + \sin\tfrac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\phi}\,|1\rangle$$
$$\cos\tfrac{\pi-\theta}{2}\,|0\rangle + \sin\tfrac{\pi-\theta}{2}\,\mathrm{e}^{\mathrm{i}(\phi+\pi)}\,|1\rangle$$

are orthogonal.

$\boxed{C21}$

$$\begin{aligned}A^2=\mathbb{I} \;\Rightarrow\; \mathrm{e}^{\mathrm{i}tA} &= I + \tfrac{\mathrm{i}t}{1!}A + \tfrac{(\mathrm{i}t)^2}{2!}A^2 + \tfrac{(\mathrm{i}t)^3}{3!}A^3 + ...\\ &= \left(1 - \tfrac{t^2}{2!} + \tfrac{t^4}{4!} - \tfrac{t^6}{6!} + \dots\right)\mathbb{I}\\ &\quad +\mathrm{i}\left(\tfrac{t}{1!} - \tfrac{t^3}{3!} + \tfrac{t^5}{5!} - \tfrac{t^7}{7!} + \dots\right)A.\\ &= \cos t\,\mathbb{I} + \mathrm{i}\sin t\,A.\end{aligned}$$

$\boxed{C25}$

We have

$$\mathcal{R}_x(t)\begin{pmatrix}1+r_3 & r_1-\mathrm{i}r_2\\ r_1+\mathrm{i}r_2 & 1-r_3\end{pmatrix}\mathcal{R}_x^\dagger(t) = \begin{pmatrix}1+r_3' & r_1'-\mathrm{i}r_2'\\ r_1'+\mathrm{i}r_2' & 1-r_3'\end{pmatrix},$$

where

$$\begin{pmatrix}r_1'\\ r_2'\\ r_3'\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & \cos t & -\sin t\\ 0 & \sin t & \cos t\end{pmatrix}\begin{pmatrix}r_1\\ r_2\\ r_3\end{pmatrix}.$$

$\boxed{C26}$

We have

$$\mathcal{R}_y(t)\begin{pmatrix}1+r_3 & r_1-\mathrm{i}r_2\\ r_1+\mathrm{i}r_2 & 1-r_3\end{pmatrix}\mathcal{R}_y^\dagger(t) = \begin{pmatrix}1+r_3' & r_1'-\mathrm{i}r_2'\\ r_1'+\mathrm{i}r_2' & 1-r_3'\end{pmatrix},$$

where

$$\begin{pmatrix}r_1'\\ r_2'\\ r_3'\end{pmatrix} = \begin{pmatrix}\cos t & 0 & -\sin t\\ 0 & 1 & 0\\ \sin t & 0 & \cos t\end{pmatrix}\begin{pmatrix}r_1\\ r_2\\ r_3\end{pmatrix}.$$

$\boxed{C27}$

We have

$$\mathcal{R}_z(t)\begin{pmatrix}1+r_3 & r_1-\mathrm{i}r_2\\ r_1+\mathrm{i}r_2 & 1-r_3\end{pmatrix}\mathcal{R}_z^\dagger(t) = \begin{pmatrix}1+r_3' & r_1'-\mathrm{i}r_2'\\ r_1'+\mathrm{i}r_2' & 1-r_3'\end{pmatrix},$$

where

$$\begin{pmatrix}r_1'\\ r_2'\\ r_3'\end{pmatrix} = \begin{pmatrix}\cos t & -\sin t & 0\\ \sin t & \cos t & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}r_1\\ r_2\\ r_3\end{pmatrix}.$$

$\boxed{C28}$

A transform $U:\mathbb{C}^2\longrightarrow\mathbb{C}^2$ with the matrix

$$U = \begin{pmatrix}u_{00} & u_{01}\\ u_{10} & u_{11}\end{pmatrix}$$

is unitary if $U^\dagger U = UU^\dagger = I$, that is

$$\begin{array}{ll}|u_{00}|^2+|u_{01}|^2=1, & |u_{00}|^2+|u_{10}|^2=1,\\ |u_{10}|^2+|u_{11}|^2=1, & |u_{01}|^2+|u_{11}|^2=1,\end{array} \quad (*)$$

and

$$\bar{u}_{00}u_{10}+\bar{u}_{01}u_{11}=0, \qquad \bar{u}_{00}u_{01}+\bar{u}_{10}u_{11}=0.$$

Matrix U satisfies (*) if and only if it has the form

$$U = \begin{pmatrix}\cos\frac{\beta}{2}\,\mathrm{e}^{\mathrm{i}\frac{a}{2}} & -\sin\frac{\beta}{2}\,\mathrm{e}^{\mathrm{i}\frac{b}{2}}\\ \sin\frac{\beta}{2}\,\mathrm{e}^{\mathrm{i}\frac{c}{2}} & \cos\frac{\beta}{2}\,\mathrm{e}^{\mathrm{i}\frac{d}{2}}\end{pmatrix}.$$

This matrix satisfies the last two conditions if and only if $a-c=b-d$. Consequently, U must have the form

$$U = \begin{pmatrix}\cos\frac{\beta}{2}\,\mathrm{e}^{\mathrm{i}\frac{a}{2}} & -\sin\frac{\beta}{2}\,\mathrm{e}^{\mathrm{i}\frac{b}{2}}\\ \sin\frac{\beta}{2}\,\mathrm{e}^{\mathrm{i}\frac{c}{2}} & \cos\frac{\beta}{2}\,\mathrm{e}^{\mathrm{i}\frac{b+c-a}{2}}\end{pmatrix}.$$

But, this matrix can be written as

$$\begin{aligned}U &= \mathrm{e}^{\mathrm{i}\frac{b+c}{4}}\begin{pmatrix}\cos\frac{\beta}{2}\,\mathrm{e}^{\mathrm{i}\frac{(a-b)+(a-c)}{2}} & -\sin\frac{\beta}{2}\,\mathrm{e}^{\mathrm{i}\frac{(a-c)-(a-b)}{2}}\\ \sin\frac{\beta}{2}\,\mathrm{e}^{-\mathrm{i}\frac{(a-c)-(a-b)}{2}} & \cos\frac{\beta}{2}\,\mathrm{e}^{-\mathrm{i}\frac{(a-b)+(a-c)}{2}}\end{pmatrix}\\ &= \mathrm{e}^{\mathrm{i}\frac{b+c}{4}}\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}} & 0\\ 0 & \mathrm{e}^{\mathrm{i}\frac{\alpha}{2}}\end{pmatrix}\begin{pmatrix}\cos\frac{\beta}{2} & -\sin\frac{\beta}{2}\\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\frac{\gamma}{2}} & 0\\ 0 & \mathrm{e}^{\mathrm{i}\frac{\gamma}{2}}\end{pmatrix}\\ &= \mathrm{e}^{\mathrm{i}\theta}\,\mathcal{R}_z(\alpha)\,\mathcal{R}_y(\beta)\,\mathcal{R}_z(\gamma),\end{aligned}$$

where $\theta = \frac{b+c}{4}$, $\alpha = \frac{a-b}{2}$ and $\gamma = \frac{a-c}{2}$.

$\boxed{D}$ Composite systems

$\boxed{D8}$

$$\mathrm{tr}(|\varphi\rangle\langle\psi|) = \sum_j \langle j|\varphi\rangle\langle\psi|j\rangle = \sum_j \langle\psi|j\rangle\langle j|\varphi\rangle = \langle\psi|\varphi\rangle.$$

$\boxed{D9}$

Direct consequences of $\boxed{D3}$ and $\boxed{D4}$ (page 3).

$\boxed{D10}$

$$\langle k\ell| \sum_j {}_2\langle j|\varphi\psi\rangle \otimes |j\rangle_2 = \sum_k \langle kj|\varphi\psi\rangle\langle\ell|j\rangle = \sum_k \langle kj|\varphi\psi\rangle\,\delta_{\ell j} = \langle k\ell|\varphi\psi\rangle.$$

$\boxed{D11}$

$$\begin{aligned} A\otimes B &= \sum_{ijk\ell} |ik\rangle\langle ik|A\otimes B|j\ell\rangle\langle j\ell| \\ &= \sum_{ijk\ell} \langle ik|A\otimes B|j\ell\rangle\,|ik\rangle\langle j\ell| \\ &= \sum_{ijk\ell} (A\otimes B)^{ik}_{j\ell}\,|ik\rangle\langle j\ell| \\ &= \sum_{ijk\ell} A^i_j\,B^k_\ell\,|ik\rangle\langle j\ell| \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} {}_1\langle\varphi_1|A\otimes B|\varphi_2\rangle_1 &= \sum_{ijk\ell} A^i_j\,B^k_\ell\;{}_1\langle\varphi_1|ik\rangle\langle j\ell|\varphi_2\rangle_1 \\ &= \sum_{ijk\ell} A^i_j\,B^k_\ell\,\langle\varphi_1|i\rangle\langle j|\varphi_2\rangle|k\rangle\langle\ell| \\ &= \sum_{ij}\langle\varphi_1|i\rangle\langle i|A|j\rangle\langle j|\varphi_2\rangle \sum_{k\ell} B^k_\ell\,|k\rangle\langle\ell| \\ &= \langle\varphi_1|A|\varphi_2\rangle\,B, \end{aligned}$$

$$\begin{aligned} {}_2\langle\psi_1|A\otimes B|\psi_2\rangle_2 &= \sum_{ijk\ell} A^i_j\,B^k_\ell\;{}_2\langle\psi_1|ik\rangle\langle j\ell|\psi_2\rangle_2 \\ &= \sum_{ijk\ell} A^i_j\,B^k_\ell\,\langle\psi_1|k\rangle\langle\ell|\psi_2\rangle|i\rangle\langle j| \\ &= \sum_{k\ell}\langle\psi_1|k\rangle\langle k|B|\ell\rangle\langle\ell|\psi_2\rangle \sum_{ij} A^i_j\,|i\rangle\langle j| \\ &= \langle\psi_1|B|\psi_2\rangle\,A \end{aligned}$$

$\boxed{D12}$

$$\begin{aligned} \mathrm{tr}_1(A\otimes B) &= \sum_j {}_1\langle j|A\otimes B|j\rangle_1, \\ &\Downarrow \boxed{D9} \\ &= \sum_j \langle j|A|j\rangle\,B \\ &= (\mathrm{tr}\,A)\,B. \end{aligned}$$

$$\begin{aligned} \mathrm{tr}_2(A\otimes B) &= \sum_k {}_2\langle k|A\otimes B|k\rangle_2, \\ &\Downarrow \boxed{D9} \\ &= \sum_k \langle k|B|k\rangle\,A \\ &= (\mathrm{tr}\,B)\,A. \end{aligned}$$

$\boxed{D13}$

$$\begin{aligned} \langle ik|(|\varphi_1\rangle\langle\varphi_2| \otimes |\psi_1\rangle\langle\psi_2|)|j\ell\rangle &= \langle ik|(|\varphi_1\rangle\langle\varphi_2|j\rangle \otimes |\psi_1\rangle\langle\psi_2|\ell\rangle) \\ &= \langle i|\varphi_1\rangle\langle\varphi_2|j\rangle\langle k|\psi_1\rangle\langle\psi_2|\ell\rangle \\ &= \langle i|\varphi_1\rangle\langle k|\psi_1\rangle\langle\varphi_2|j\rangle\langle\psi_2|\ell\rangle \\ &= \langle ik|\varphi_1\psi_1\rangle\langle\varphi_2\psi_2|j\ell\rangle. \end{aligned}$$

$\boxed{D14}$

$$\begin{aligned} \mathrm{tr}_1(|\varphi_1\psi_1\rangle\langle\varphi_2\psi_2|) &= \mathrm{tr}_1(|\varphi_1\rangle\langle\varphi_2| \otimes |\psi_1\rangle\langle\psi_2|) \\ &\Downarrow \boxed{D13} \\ &= \mathrm{tr}(|\varphi_1\rangle\langle\varphi_2|)\,|\psi_1\rangle\langle\psi_2|) \\ &\Downarrow \boxed{D8} \\ &= \langle\psi_2|\psi_1\rangle\;\;|\varphi_1\rangle\langle\varphi_2|. \end{aligned}$$

$$\begin{aligned} \mathrm{tr}_2(|\varphi_1\psi_1\rangle\langle\varphi_2\psi_2|) &= \mathrm{tr}_2(|\varphi_1\rangle\langle\varphi_2| \otimes |\psi_1\rangle\langle\psi_2|) \\ &\Downarrow \boxed{D13} \\ &= \mathrm{tr}(|\psi_1\rangle\langle\psi_2|)\,|\varphi_1\rangle\langle\varphi_2| \\ &\Downarrow \boxed{D8} \\ &= \langle\varphi_2|\varphi_1\rangle\;\;|\psi_1\rangle\langle\psi_2|, \end{aligned}$$

$\boxed{D15}$

$$T = \sum_{ijk\ell} |ik\rangle\langle ik|T|j\ell\rangle\langle j\ell| = \sum_{ijk\ell} T^{ik}_{j\ell}\,|ik\rangle\langle j\ell|$$

$$\Downarrow \boxed{D7}$$

$$\begin{aligned} \mathrm{tr}_1 T &= \sum_r {}_1\langle r|T|r\rangle_1, \\ &= \sum_r\sum_{ijk\ell} T^{ik}_{j\ell}\;{}_1\langle r|ik\rangle\langle j\ell|r\rangle_1 \\ &= \sum_r\sum_{ijk\ell} T^{ik}_{j\ell}\,\delta_{ri}\,\delta_{rj}|k\rangle\langle\ell| \\ &= \sum_{jk\ell} T^{jk}_{j\ell}\,|k\rangle\langle\ell| \end{aligned}$$

$$\Downarrow$$

$$\langle k|\mathrm{tr}_1 T|\ell\rangle = \sum_j T^{jk}_{j\ell}$$

$$(\mathrm{tr}_1 T)^k_\ell = \sum_j T^{jk}_{j\ell}$$

$$T = \sum_{ijk\ell} |ik\rangle\langle ik|T|j\ell\rangle\langle j\ell| = \sum_{ijk\ell} T^{ik}_{j\ell}\,|ik\rangle\langle j\ell|$$

$$\Downarrow \boxed{D7}$$

$$\begin{aligned} \mathrm{tr}_2 T &= \sum_r {}_2\langle r|T|r\rangle_2, \\ &= \sum_r\sum_{ijk\ell} T^{ik}_{j\ell}\;{}_2\langle r|ik\rangle\langle j\ell|r\rangle_2 \\ &= \sum_r\sum_{ijk\ell} T^{ik}_{j\ell}\,\delta_{rk}\,\delta_{r\ell}|i\rangle\langle j| \\ &= \sum_{ijk} T^{ik}_{jk}\,|i\rangle\langle j| \end{aligned}$$

$$\Downarrow$$

$$\langle i|\mathrm{tr}_2 T|j\rangle = \sum_k T^{ik}_{jk}$$

$$(\mathrm{tr}_2 T)^i_j = \sum_k T^{ik}_{jk}$$

$\boxed{D16}$

Direct consequences of $\boxed{D15}$ (page 3).

$\boxed{D17}$

$$\begin{aligned} &\langle ab| \sum_{k,\ell} {}_2\langle k|T|\ell\rangle_2 \otimes |k\rangle_2\,{}_2\langle\ell|cd\rangle \\ &= \sum_{k,\ell} \langle ak|T|c\ell\rangle\,\langle b|k\rangle\langle\ell|d\rangle \\ &= \sum_{k,\ell} \langle ak|T|c\ell\rangle\,\delta_{bk}\delta_{\ell d} \\ &= \langle ab|T|cd\rangle. \end{aligned}$$

$\boxed{D18}$

$$\begin{aligned} \mathrm{tr}(|\varphi\rangle\langle\psi|A) &= \sum_j \langle j|(|\varphi\rangle\langle\psi|A)|j\rangle \\ &= \sum_j \langle j|\varphi\rangle\langle\psi|\,A|j\rangle \\ &= \sum_j \langle\psi|\,A|j\rangle\langle j|\varphi\rangle \\ &= \langle\psi|A|\varphi\rangle. \end{aligned}$$

$\boxed{D19}$

$$(\mathbb{I}\otimes(|b\rangle\langle b|))T = \sum_{ijk\ell}\sum_{rs}(\mathbb{I}\otimes(|b\rangle\langle b|)^{ik}_{rs}T^{rs}_{j\ell}\,|ik\rangle\langle j\ell|$$

$$\Downarrow$$

$$\begin{aligned} \mathrm{tr}_2((\mathbb{I}\otimes(|b\rangle\langle b|))T) &= \sum_{ijk}\sum_{rs}(\mathbb{I}\otimes(|b\rangle\langle b|)^{ik}_{rs}T^{rs}_{jk}\,|i\rangle\langle j| \\ &= \sum_{ijk}\sum_{rs}\delta^i_r\langle k|b\rangle\langle b|s\rangle T^{rs}_{jk}\,|i\rangle\langle j| \end{aligned}$$

$$=\sum_{ijk}\sum_s \langle b|s\rangle T^{is}_{jk}\langle k|b\rangle\,|i\rangle\langle j|$$
$$=\sum_{ijk}\sum_s \langle b|s\rangle\langle is|T|jk\rangle\langle k|b\rangle\,|i\rangle\langle j|$$
$$=\sum_{ij}\langle ib|T|jb\rangle\,|i\rangle\langle j|$$
$$=\sum_{ij}|i\rangle\langle ib|T|jb\rangle\langle j|$$
$$={}_2\langle b|T|b\rangle_2$$

$\boxed{D20}$

$$\begin{array}{l} A^i_k=\langle i|A|k\rangle\\ B^m_\ell=\langle m|B|\ell\rangle\\ T^{k\ell}_{im}=\langle k\ell|T|im\rangle\end{array} \Rightarrow \sum_{ik\ell m}A^i_k\,B^m_\ell\,T^{k\ell}_{im}=\sum_{ik}A^i_k\sum_{j\ell m}\delta^k_j\,B^m_\ell\,T^{j\ell}_{im}$$
$$\Downarrow$$
$$\sum_{ik\ell m}(A\otimes B)^{im}_{k\ell}\,T^{k\ell}_{im}=\sum_{ik}A^i_k\sum_{j\ell m}(\mathbb{I}\otimes B)^{km}_{j\ell}\,T^{j\ell}_{im}$$
$$\Downarrow$$
$$\sum_{im}((A\otimes B)T)^{im}_{im}=\sum_{ik}A^i_k\sum_m((\mathbb{I}\otimes B)T)^{km}_{im}$$
$$\Downarrow$$
$$\mathrm{tr}\,((A\otimes B)T)=\sum_{ik}A^i_k(\mathrm{tr}_2((\mathbb{I}\otimes B)T))^k_i$$
$$\Downarrow$$
$$\mathrm{tr}\,((A\otimes B)T)=\mathrm{tr}(A\,\mathrm{tr}_2((\mathbb{I}\otimes B)T)).$$

$\boxed{D21}$

$$\begin{array}{l} A^i_k=\langle i|A|k\rangle\\ B^m_\ell=\langle m|B|\ell\rangle\\ T^{k\ell}_{im}=\langle k\ell|T|im\rangle\end{array} \Rightarrow \sum_{ik\ell m}A^i_k\,B^m_\ell\,T^{k\ell}_{im}=\sum_{m\ell}B^m_\ell\sum_{ijk}A^i_k\,\delta^\ell_j\,T^{kj}_{im}$$
$$\Downarrow$$
$$\sum_{ik\ell m}(A\otimes B)^{im}_{k\ell}\,T^{k\ell}_{im}=\sum_{m\ell}B^m_\ell\sum_{ijk}(A\otimes\mathbb{I})^{i\ell}_{kj}\,T^{kj}_{im}$$
$$\Downarrow$$
$$\sum_{im}((A\otimes B)T)^{im}_{im}=\sum_{m\ell}B^m_\ell\sum_i((A\otimes\mathbb{I})T)^{i\ell}_{im}$$
$$\Downarrow$$
$$\mathrm{tr}\,((A\otimes B)T)=\sum_{m\ell}B^m_\ell(\mathrm{tr}_1((A\otimes\mathbb{I})T))^\ell_m$$
$$\Downarrow$$
$$\mathrm{tr}\,((A\otimes B)T)=\mathrm{tr}(B\,\mathrm{tr}_1((A\otimes\mathbb{I})T)).$$

$\boxed{D22}$

$$A\otimes B=(A\otimes\mathbb{I})(\mathbb{I}\otimes B)$$
$$\Downarrow$$
$$\mathrm{tr}\,((A\otimes B)T)=\mathrm{tr}\,((A\otimes\mathbb{I})(\mathbb{I}\otimes B)T)$$

$\boxed{D23}$

Direct consequence of $\boxed{D20}$ (page 3).

$\boxed{D24}$

Direct consequence of $\boxed{D21}$ (page 3).

$\boxed{D2}$

$${}_2\langle\varphi|\mathrm{tr}_1\varrho|\varphi\rangle_2={}_2\langle\varphi|\sum_j{}_1\langle j|\varrho|j\rangle_1|\varphi\rangle_2$$
$$=\sum_j{}_2\langle\varphi|_1\langle j|\varrho|j\rangle_1|\varphi\rangle_2$$
$$=\sum_j\langle j\varphi|\varrho|j\varphi\rangle\ge 0.$$
$$\mathrm{tr}(\mathrm{tr}_1\varrho)=\sum_k{}_2\langle k|\mathrm{tr}_1\varrho|k\rangle_2$$
$$=\sum_k{}_2\langle k|\sum_j{}_1\langle j|\varrho|j\rangle_1|k\rangle_2$$
$$=\sum_{j,k}\langle jk|\varrho|jk\rangle$$
$$=\mathrm{tr}\,\varrho=1.$$

$\boxed{D26}$

$$(\mathrm{tr}_1\varrho)^k_\ell=\langle k|\mathrm{tr}_1\varrho|\ell\rangle=\sum_{j=0}^1\varrho^{jk}_{j\ell}=\varrho^{0k}_{0\ell}+\varrho^{1k}_{1\ell},$$
$$(\mathrm{tr}_2\varrho)^i_j=\langle i|\mathrm{tr}_2\varrho|j\rangle=\sum_{k=0}^1\varrho^{ik}_{jk}=\varrho^{i0}_{j0}+\varrho^{i1}_{j1}.$$

$\boxed{D27}$ *Schmidt decomposition.*

$\{|0\rangle_1,|1\rangle_1,\dots,|n-1\rangle_1\}$ basis of $\mathcal{H}_1$
$\{|0\rangle_2,|1\rangle_2,\dots,|m-1\rangle_2\}$ basis of $\mathcal{H}_2$
$\Phi\in\mathcal{H}_1\otimes\mathcal{H}_2,$

$$\Downarrow$$
$${}_1\langle j|(\mathrm{tr}_2|\Phi\rangle\langle\Phi|)|k\rangle_1=\sum_{\ell=0}^{m-1}\langle j\ell|\Phi\rangle\langle\Phi|k\ell\rangle$$
$$=\sum_{\ell=0}^{m-1}\overline{\langle k\ell|\Phi\rangle\langle\Phi|j\ell\rangle}=\overline{{}_1\langle k|(\mathrm{tr}_2|\Phi\rangle\langle\Phi|)|j\rangle_1},$$
$${}_1\langle\varphi|(\mathrm{tr}_2|\Phi\rangle\langle\Phi|)|\varphi\rangle_1=\sum_{j,k=0}^{n-1}\sum_{i=0}^{m-1}{}_1\langle\varphi|j\rangle_1\langle ji|\Phi\rangle\langle\Phi|ki\rangle_1\langle k|\varphi\rangle_1$$
$$=\sum_{i=0}^{m-1}\langle\varphi\,i|\Phi\rangle\langle\Phi|\varphi\,i\rangle=\sum_{i=0}^{m-1}|\langle\varphi\,i|\Phi\rangle|^2\ge 0$$
$$\Downarrow$$
$$\boxed{\mathrm{tr}_2|\Phi\rangle\langle\Phi|:\mathcal{H}_1\to\mathcal{H}_1\text{ is self-adjoint and positive}}$$
$$\Downarrow$$

there exists $\{\varphi_0,\varphi_1,\dots,\varphi_{n-1}\}$ orthonormal basis of $\mathcal{H}_1$ such that
$$\mathrm{tr}_2|\Phi\rangle\langle\Phi|=\sum_{k=0}^{r-1}p_k|\varphi_k\rangle\langle\varphi_k|$$
and $p_0\ge p_1\ge\dots\ge p_{r-1}>0.$

$$\Downarrow$$
$$|\Phi\rangle=\sum_{k=0}^{n-1}\sum_{j=0}^{m-1}\Phi^{kj}|\varphi_k\,j\rangle$$
$$\Downarrow$$
$$|\Phi\rangle\langle\Phi|=\sum_{i,k=0}^{n-1}\sum_{j,\ell=0}^{m-1}\Phi^{kj}\,\overline{\Phi^{i\ell}}\,|\varphi_k\,j\rangle\langle\varphi_i\,\ell|$$
$$\Downarrow$$
$$\mathrm{tr}_2|\Phi\rangle\langle\Phi|=\sum_{i,k=0}^{n-1}\sum_{j=0}^{m-1}\Phi^{kj}\,\overline{\Phi^{ij}}\,|\varphi_k\rangle\langle\varphi_i|$$
$$\Downarrow$$
$$\sum_{k=0}^{r-1}p_k|\varphi_k\rangle\langle\varphi_k|=\sum_{i,k=0}^{n-1}\sum_{j=0}^{m-1}\Phi^{kj}\,\overline{\Phi^{ij}}\,|\varphi_k\rangle\langle\varphi_i|$$
$$\Downarrow$$
$$\boxed{\begin{array}{ll}\displaystyle\sum_{j=0}^{m-1}\Phi^{kj}\,\overline{\Phi^{ij}}=p_k\,\delta_{ik} & \text{for}\quad 0\le i,k\le r-1\\ \displaystyle\sum_{j=0}^{m-1}|\Phi^{kj}|^2=\sum_{j=0}^{m-1}\Phi^{kj}\,\overline{\Phi^{kj}}=0 & \text{for}\quad k>r-1\end{array}}$$
$$\Downarrow$$
$$\left\{|\psi_k\rangle=\frac{1}{\sqrt{p_k}}\sum_{j=0}^{m-1}\Phi^{kj}\,|j\rangle\;\middle|\;k\in\{0,1,\dots,r-1\}\right\}$$
is an orthonormal set and $\Phi^{kj}=0$ for $k>r-1$
$$\Downarrow$$
$$|\Phi\rangle=\sum_{k=0}^{n-1}\sum_{j=0}^{m-1}\Phi^{kj}|\varphi_k\,j\rangle$$
$$=\sum_{k=0}^{r-1}\sum_{j=0}^{m-1}\Phi^{kj}|\varphi_k\rangle\otimes|j\rangle$$
$$=\sum_{k=0}^{r-1}\sqrt{p_k}\,|\varphi_k\rangle\otimes|\psi_k\rangle=\sum_{k=0}^{r-1}\sqrt{p_k}\,|\varphi_k\psi_k\rangle.$$

The set $\{\,|\psi_k\rangle\;|\;k\in\{0,1,\dots,r-1\}\,\}$ can be extended up to an orthonormal basis $\{|\psi_0\rangle,|\psi_1\rangle,\dots,|\psi_{m-1}\rangle\}$ of $\mathcal{H}_2$.

$\boxed{D33}$

$$|\Phi\rangle = \sum_k \sqrt{p_k}\,|\varphi_k\rangle\otimes|\psi_k\rangle = \sum_k \sqrt{p_k}\,|\varphi_k\psi_k\rangle$$
$$\Downarrow$$
$$|\Phi\rangle\langle\Phi| = \sum_{kj} \sqrt{p_k p_j}\,|\varphi_k\psi_k\rangle\langle\varphi_j\psi_j|$$
$$\Downarrow$$
$$\mathrm{tr}_2|\Phi\rangle\langle\Phi| = \sum_{kj} \sqrt{p_k p_j}\ \mathrm{tr}_2(|\varphi_k\psi_k\rangle\langle\varphi_j\psi_j|)$$
$$\Downarrow \boxed{D12}$$
$$\mathrm{tr}_2|\Phi\rangle\langle\Phi| = \sum_{kj} \sqrt{p_k p_j}\ \langle\psi_j|\psi_k\rangle\,|\varphi_k\rangle\langle\varphi_j|$$
$$\Downarrow$$
$$\mathrm{tr}_2|\Phi\rangle\langle\Phi| = \sum_{kj} \sqrt{p_k p_j}\ \delta_{jk}\,|\varphi_k\rangle\langle\varphi_j|$$
$$\Downarrow$$
$$\mathrm{tr}_2|\Phi\rangle\langle\Phi| = \sum_k p_k\,|\varphi_k\rangle\langle\varphi_k|.$$

$\boxed{E}$ **Entanglement**

....

$\boxed{F}$ **Orthogonal measurements**

$\boxed{F2}$

$$\left.\begin{array}{r} P_j P_k = \delta_{jk} P_j, \\ P_j = P_j^\dagger, \\ \sum_{j=0}^{n-1} P_j = \mathbb{I} \end{array}\right\} \Rightarrow \begin{array}{c} U:\mathbb{C}^n\otimes\mathbb{C}^n \to \mathbb{C}^n\otimes\mathbb{C}^n \\ U = \sum_{j,k} P_j \otimes |k+j\rangle\langle k| \\ \text{is a unitary operator.} \end{array}$$

Indeed,

$$\begin{aligned} UU^\dagger &= \left(\sum_{j,k} P_j \otimes |k+j\rangle\langle k|\right)\left(\sum_{a,b} P_a^\dagger \otimes |b\rangle\langle b+a|\right) \\ &= \sum_{j,k,a} P_j P_a^\dagger \otimes |k+j\rangle\langle k+a| \\ &= \sum_{j,k,a} P_j P_a \otimes |k+j\rangle\langle k+a| \\ &= \sum_{j,k,a} \delta_{ja} P_j \otimes |k+j\rangle\langle k+a| \\ &= \sum_{j,k} P_j \otimes |k+j\rangle\langle k+j| \\ &= \sum_j P_j \otimes \sum_k |k+j\rangle\langle k+j| \\ &= \sum_j P_j \otimes \mathbb{I} \\ &= \mathbb{I}\otimes\mathbb{I}. \end{aligned}$$

$\boxed{G}$ **Generalized measurements**

$\boxed{G3}$

$$\varrho = |\psi\rangle\langle\psi| \ \Rightarrow\ \begin{aligned} p_k &= \mathrm{tr}(M_k|\psi\rangle\langle\psi|M_k^\dagger) \\ &= \mathrm{tr}(M_k^\dagger M_k|\psi\rangle\langle\psi|) \\ &= \langle\psi|M_k^\dagger M_k|\psi\rangle \\ &= ||M_k|\psi\rangle||^2. \end{aligned}$$

and

$\frac{M_k|\psi\rangle\langle\psi|M_k^\dagger}{p_k}$ is the density operator of $\frac{M_k|\psi\rangle}{\sqrt{p_k}}$.

$\boxed{G4}$

See $G15$.

$\boxed{G5}$

$$\frac{||M_k M_j|\psi\rangle||^2}{||M_k|\psi\rangle||^2} = \delta_{kj} \quad \text{for any } |\psi\rangle$$
$$\Downarrow$$
$$M_k M_j = \delta_{kj}\, M_k, \quad \text{up to a phase factor.}$$

$\boxed{G6}$

We choose:

- an arbitrary space $\mathcal{H}_B$ of dimension m,
- an arbitrary $|\omega\rangle\in\mathcal{H}_B$ with $\langle\omega|\omega\rangle = 1$,
- an arbitrary orthonormal basis $\{|k\rangle\}_{k=0}^{m-1}$ of $\mathcal{H}_B$

By using the given operators $\{M_k:\mathcal{H}_A\to\mathcal{H}_A\}_{k=0}^{m-1}$, we define the surjective map

$$U:\mathcal{V}\to\mathcal{W}, \quad U(|\psi\rangle\otimes|\omega\rangle) = \sum_{k=0}^{m-1} M_k|\psi\rangle\otimes|k\rangle,$$

where the sets

$$\mathcal{V} \equiv \mathcal{H}_A\otimes|\omega\rangle = \{\ |\psi\rangle\otimes|\omega\rangle \ |\ \ |\psi\rangle\in\mathcal{H}_A\ \}$$
$$\mathcal{W} = \left\{\ \sum_{k=0}^{m-1} M_k|\psi\rangle\otimes|k\rangle \ \middle|\ \ |\psi\rangle\in\mathcal{H}_A \right\}$$

are vector subspaces of $\mathcal{H}_A\otimes\mathcal{H}_B$.

The map U is a unitary transform because

$$\begin{aligned} &\langle U(|\psi\rangle\otimes|\omega\rangle), U(|\varphi\rangle\otimes|\omega\rangle)\rangle = \\ &= \left\langle \sum_{k=0}^{m-1} M_k|\psi\rangle\otimes|k\rangle, \sum_{\ell=0}^{m-1} M_\ell|\varphi\rangle\otimes|\ell\rangle \right\rangle \\ &= \sum_{k,\ell=0}^{m-1} \langle M_k\psi, M_\ell\varphi\rangle\,\langle k|\ell\rangle \\ &= \sum_{k=0}^{m-1} \langle M_k\psi, M_k\varphi\rangle \\ &= \sum_{k=0}^{m-1} \langle M_k^\dagger M_k\psi, \varphi\rangle \\ &= \left\langle \sum_{k=0}^{m-1} M_k^\dagger M_k\psi, \varphi \right\rangle \\ &= \langle\psi,\varphi\rangle = \langle(|\psi\rangle\otimes|\omega\rangle), (|\varphi\rangle\otimes|\omega\rangle)\rangle\,. \end{aligned}$$

Consequently, $U:\mathcal{V}\to\mathcal{W}$ is also injective.

It is an isomorphism and $\dim\mathcal{W} = \dim\mathcal{V} = \dim\mathcal{H}_{\mathcal{A}} \equiv n$.

The unitary transform $U:\mathcal{V}\to\mathcal{W}$ can be extended up to a unitary transform $\tilde{U}:\mathcal{H}_A\otimes\mathcal{H}_B\to\mathcal{H}_A\otimes\mathcal{H}_B$ by following the steps:

- we choose an orthonormal basis $\{|\Psi_0\rangle, |\Psi_1\rangle, ..., |\Psi_{n-1}\rangle\}$ of $\mathcal{V}$ and then extend it up to an orthonormal basis $\{|\Psi_0\rangle, |\Psi_1\rangle, ..., |\Psi_{n-1}\rangle, ..., |\Psi_{nm-1}\rangle\}$ of $\mathcal{H}_A\otimes\mathcal{H}_B$,
- we extend the orthonormal basis $\{|\Phi_0\rangle = U|\Psi_0\rangle, |\Phi_1\rangle = U|\Psi_1\rangle, ..., |\Phi_{n-1}\rangle = U|\Psi_{n-1}\rangle\}$ of $\mathcal{W}$ up to an orthonormal basis $\{|\Phi_0\rangle, |\Phi_1\rangle, ..., |\Phi_{n-1}\rangle, ..., |\Phi_{nm-1}\rangle\}$ of $\mathcal{H}_A\otimes\mathcal{H}_B$,
- we define $\tilde{U}:\mathcal{H}_A\otimes\mathcal{H}_B\to\mathcal{H}_A\otimes\mathcal{H}_B$ as $\tilde{U}\Psi_k = \Phi_k$.

  We have
  $${}_B\langle k|U|\psi\omega\rangle = {}_B\langle k|\sum_{k=0}^{m-1} M_\ell|\psi\rangle\otimes|\ell\rangle = M_k|\psi\rangle.$$

  In the case of a density operator
  $$\varrho_A = \sum_{j=0}^{n-1} \lambda_j|\psi_j\rangle\langle\psi_j|$$
  we get

$$\begin{aligned} &\mathrm{tr}(U\varrho_A\otimes\varrho_B U^\dagger \mathbb{I}\otimes|k\rangle\langle k|) \\ &\Downarrow \boxed{D19} \\ &= \mathrm{tr}_A({}_B\langle k|U\varrho_A\otimes\varrho_B U^\dagger|k\rangle_B) \\ &= \mathrm{tr}_A({}_B\langle k|U\varrho_A\otimes\varrho_B U^\dagger|k\rangle_B) \\ &= \sum_{j=0}^{n-1} \lambda_j \mathrm{tr}_A({}_B\langle k|U|\psi_j\rangle\langle\psi_j|\otimes|\omega\rangle\langle\omega|U^\dagger|k\rangle_B) \\ &\Downarrow \boxed{D13} \\ &= \sum_{j=0}^{n-1} \lambda_j \mathrm{tr}_A({}_B\langle k|U|\psi_j\omega\rangle\langle\psi_j\omega|U^\dagger|k\rangle_B) \\ &= \sum_{j=0}^{n-1} \lambda_j \mathrm{tr}_A(M_k|\psi_j\rangle\langle\psi_j|M_k^\dagger) \\ &= \mathrm{tr}_A(M_k \sum_{j=0}^{n-1} \lambda_j|\psi_j\rangle\langle\psi_j|M_k^\dagger) \\ &= \mathrm{tr}_A(M_k\varrho_A M_k^\dagger) = p_k, \end{aligned}$$

$$\frac{1}{p_k}\mathrm{tr}_B(U\varrho_A\otimes\varrho_B U^\dagger \mathbb{I}\otimes|k\rangle\langle k|)$$
$$=\frac{1}{p_k}\,{}_B\langle k|U\varrho_A\otimes\varrho_B U^\dagger|k\rangle_B)$$
$$=\frac{M_k\varrho_A M_k^\dagger}{p_k}=\varrho_{Ak}.$$

$\boxed{G7}$

By using the given unitary transform U, we define the operators $\{M_k:\mathcal{H}_A\to\mathcal{H}_A\}_{k=0}^{m-1}$, where

$$M_k|\psi\rangle={}_B\langle k|U|\psi\omega\rangle.$$

They satisfy

$$\langle\varphi|\sum_{k=0}^{m-1}M_k^\dagger M_k|\psi\rangle=\sum_{k=0}^{m-1}\langle\varphi|M_k^\dagger M_k|\psi\rangle$$
$$=\sum_{k=0}^{m-1}\langle\varphi\omega|U^\dagger|k\rangle\langle k|U|\psi\omega\rangle$$
$$=\langle\varphi\omega|U^\dagger\sum_{k=0}^{m-1}|k\rangle\langle k|U|\psi\omega\rangle$$
$$=\langle\varphi\omega|U^\dagger U|\psi\omega\rangle$$
$$=\langle\varphi\omega|\psi\omega\rangle$$
$$=\langle\varphi|\psi\rangle\,\langle\omega|\omega\rangle$$
$$=\langle\varphi|\psi\rangle\quad\text{for any }\varphi,\,\psi\in\mathcal{H}_A$$
$$\Downarrow$$
$$\sum_{k=0}^{m-1}M_k^\dagger M_k|\psi\rangle=|\psi\rangle\quad\text{for any }\psi\in\mathcal{H}_A$$
$$\Downarrow$$
$$\sum_{k=0}^{m-1}M_k^\dagger M_k=\mathbb{I}.$$

In the case of a density operator

$$\varrho_A=\sum_{j=0}^{n-1}\lambda_j|\psi_j\rangle\langle\psi_j|$$

we get

$$p_k=\mathrm{tr}(U\varrho_A\otimes\varrho_B U^\dagger\mathbb{I}\otimes P_k)$$
$$\Downarrow\boxed{D19}$$
$$=\mathrm{tr}_A({}_B\langle k|U\varrho_A\otimes\varrho_B U^\dagger|k\rangle_B)$$
$$=\sum_{j=0}^{n-1}\lambda_j\mathrm{tr}_A({}_B\langle k|U|\psi_j\rangle\langle\psi_j|\otimes|\omega\rangle\langle\omega|U^\dagger|k\rangle_B)$$
$$\Downarrow\boxed{D13}$$
$$=\sum_{j=0}^{n-1}\lambda_j\mathrm{tr}_A({}_B\langle k|U|\psi_j\omega\rangle\langle\psi_j\omega|U^\dagger|k\rangle_B)$$
$$=\sum_{j=0}^{n-1}\lambda_j\mathrm{tr}_A(M_k|\psi_j\rangle\langle\psi_j|M_k^\dagger)$$
$$=\mathrm{tr}_A(M_k\sum_{j=0}^{n-1}\lambda_j|\psi_j\rangle\langle\psi_j|M_k^\dagger)$$
$$=\mathrm{tr}_A(M_k\varrho_A M_k^\dagger),$$

$$\varrho_{Ak}=\frac{1}{p_k}\,{}_B\langle k|U\varrho_A\otimes\varrho_B U^\dagger|k\rangle_B)=\frac{M_k\varrho_A M_k^\dagger}{p_k}.$$

$\boxed{G13,14}$

$$U(|\psi\rangle\otimes|0\rangle)=E_0|\psi\rangle\otimes|0\rangle+E_1|\psi\rangle\otimes|1\rangle$$
$$=\alpha|0\rangle\otimes|0\rangle+\beta|1\rangle\otimes|1\rangle$$
$$=\frac{1}{\sqrt{2}}(\alpha|0\rangle+\beta|1\rangle)\otimes|0'\rangle$$
$$+\frac{1}{\sqrt{2}}(\alpha|0\rangle-\beta|1\rangle)\otimes|1'\rangle$$
$$=M_0|\psi\rangle\otimes|0'\rangle+M_1|\psi\rangle\otimes|1'\rangle,$$

where

$$M_0=\frac{1}{\sqrt{2}}\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad M_1=\frac{1}{\sqrt{2}}\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

$\boxed{H}$ Positive operator-valued measures

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$\boxed{I}$ Quantum channels

$\boxed{I2}$

- $$\mathcal{E}(\alpha\varrho_1+\beta\varrho_2)=\sum_{j=0}^{m-1}K_j(\alpha\varrho_1+\beta\varrho_2)K_j^\dagger$$
$$=\alpha\sum_{j=0}^{m-1}K_j\varrho_1K_j^\dagger+\beta\sum_{j=0}^{m-1}K_j\varrho_2K_j^\dagger$$
$$=\alpha\,\mathcal{E}(\varrho_1)+\beta\,\mathcal{E}(\varrho_2).$$

- $$\mathrm{tr}\,\varrho=1$$
$$\Downarrow$$
$$\mathrm{tr}\,\mathcal{E}(\varrho)=\sum_j\mathrm{tr}(K_j\varrho K_j^\dagger)$$
$$\Downarrow\boxed{B20}$$
$$=\sum_j\mathrm{tr}(K_j^\dagger K_j\varrho)$$
$$=\mathrm{tr}\sum_j K_j^\dagger K_j\varrho$$
$$=\mathrm{tr}\,\varrho.$$

- $$\varrho=\sum_k\lambda_k\,|\psi_k\rangle\langle\psi_k|\quad\text{with }\lambda_k\geq 0$$
$$\Downarrow$$
$$\langle\varphi|\mathcal{E}(\varrho)|\varphi\rangle=\sum_{k,j}\lambda_k\,\langle\varphi|K_j|\psi_k\rangle\langle\psi_k|K_j^\dagger|\varphi\rangle$$
$$=\sum_{k,j}\lambda_k\,\langle\varphi,K_j\psi_k\rangle\langle\psi_k,K_j^\dagger\varphi\rangle$$
$$=\sum_{k,j}\lambda_k\,\langle\varphi,K_j\psi_k\rangle\langle K_j\psi_k,\varphi\rangle$$
$$=\sum_{k,j}\lambda_k\,\langle\varphi,K_j\psi_k\rangle\,\overline{\langle\varphi,K_j\psi_k\rangle}$$
$$=\sum_{k,j}\lambda_k\,|\langle\varphi,K_j\psi_k\rangle|^2\geq 0,\quad\forall\,\varphi\in\mathcal{H}_{A'}.$$

- $$C_1\in\mathcal{L}(\mathcal{H}_A),\quad C_2\in\mathcal{L}(\mathcal{H})$$
$$\Downarrow$$
$$(\mathcal{E}\otimes\mathbb{I}_{\mathcal{L}(\mathcal{H})})(C_1\otimes C_2)=\mathcal{E}(C_1)\otimes C_2$$
$$=\sum_j K_jC_1K_j^\dagger\otimes C_2$$
$$=\sum_j(K_j\otimes\mathbb{I}_{\mathcal{H}})(C_1\otimes C_2)(K_j^\dagger\otimes\mathbb{I}_{\mathcal{H}})$$
$$\Downarrow$$
$$(\mathcal{E}\otimes\mathbb{I}_{\mathcal{L}(\mathcal{H})})(C)=\sum_j(K_j\otimes\mathbb{I}_{\mathcal{H}})C(K_j^\dagger\otimes\mathbb{I}_{\mathcal{H}})$$
$$\Downarrow$$
$$\langle\Psi|(\mathcal{E}\otimes\mathbb{I}_{\mathcal{L}(\mathcal{H})})(C)|\Psi\rangle=\sum_j\langle\Psi|(K_j\otimes\mathbb{I}_{\mathcal{H}})C(K_j^\dagger\otimes\mathbb{I}_{\mathcal{H}})|\Psi\rangle$$
$$\Downarrow C\geq 0$$
$$\langle\Psi|(\mathcal{E}\otimes\mathbb{I}_{\mathcal{L}(\mathcal{H})})(C)|\Psi\rangle\geq 0,\quad\forall\,\Psi\in\mathcal{L}(\mathcal{H}_{A'})\otimes\mathcal{H}.$$

$\boxed{I3}$

- $\boxed{\mathcal{E} \text{ channel } \Rightarrow (\mathcal{E}\otimes\mathbb{I}_B)(|\Phi\rangle\langle\Phi|) \text{ density operator}}$

$$|\Phi\rangle\langle\Phi| \geq 0$$
$$\Downarrow \mathcal{E}\otimes\mathbb{I}_B \geq 0$$
$$(\mathcal{E}\otimes\mathbb{I}_B)(|\Phi\rangle\langle\Phi|) \geq 0, \quad \text{and}$$

$$\begin{aligned}
\operatorname{tr}(\mathcal{E}\otimes\mathbb{I}_B)(|\Phi\rangle\langle\Phi|) &= \tfrac{1}{n}\sum_{k,j=0}^{n-1} \operatorname{tr}(\mathcal{E}\otimes\mathbb{I}_B)(|kk\rangle\langle jj|)\\
&= \tfrac{1}{n}\sum_{k,j=0}^{n-1} \operatorname{tr}(\mathcal{E}\otimes\mathbb{I}_B)(|k\rangle\langle j|\otimes|k\rangle\langle j|)\\
&= \tfrac{1}{n}\sum_{k,j=0}^{n-1} \operatorname{tr}(\mathcal{E}(|k\rangle\langle j|)\otimes(|k\rangle\langle j|)\\
&= \tfrac{1}{n}\sum_{k,j=0}^{n-1} \operatorname{tr}\mathcal{E}(|k\rangle\langle j|)\operatorname{tr}(|k\rangle\langle j|)\\
&= \tfrac{1}{n}\sum_{k,j=0}^{n-1} \operatorname{tr}(|k\rangle\langle j|)\,\langle j|k\rangle\\
&= \tfrac{1}{n}\sum_{k,j=0}^{n-1} \operatorname{tr}(|k\rangle\langle j|)\,\delta_{jk}\\
&= \tfrac{1}{n}\sum_{k=0}^{n-1} \operatorname{tr}(|k\rangle\langle k|) = 1.
\end{aligned}$$

- $\boxed{\mathcal{E} \mapsto (\mathcal{E}\otimes\mathbb{I}_B)(|\Phi\rangle\langle\Phi|) \text{ is an injective map}}$

We have

$$|\varphi\rangle_A = \sum_k \varphi_k |k\rangle_A$$
$$\Downarrow$$
$$|\varphi\rangle_A = \sum_k \varphi_k\, {}_B\langle k|\Phi\rangle_{AB}$$
$$|\varphi\rangle_A = {}_B\langle\bar\varphi|\Phi\rangle_{AB} \text{ where } {}_B\langle\bar\varphi| = \sum_k \varphi_k\, {}_B\langle k|.$$

Since

$$\begin{aligned}
\mathcal{E}(|\varphi\rangle\langle\varphi|) &= \mathcal{E}({}_B\langle\bar\varphi|\Phi\rangle\langle\Phi|\bar\varphi\rangle_B)\\
&= \mathcal{E}({}_B\langle\bar\varphi|\sum_{k,j}|kk\rangle\langle jj|\bar\varphi\rangle_B)\\
&= \mathcal{E}({}_B\langle\bar\varphi|\sum_{k,j}|k\rangle\langle j|\otimes|k\rangle\langle j|\bar\varphi\rangle_B)\\
&= \mathcal{E}(\sum_{k,j}\varphi_k\,\bar\varphi_j|k\rangle\langle j|)\\
&= \sum_{k,j}\varphi_k\,\bar\varphi_j\mathcal{E}(|k\rangle\langle j|)\\
&= {}_B\langle\bar\varphi|\sum_{k,j}\mathcal{E}(|k\rangle\langle j|)\otimes|k\rangle\langle j|\bar\varphi\rangle_B\\
&= {}_B\langle\bar\varphi|(\mathcal{E}\otimes\mathbb{I}_B)(\sum_{k,j}|k\rangle\langle j|\otimes|k\rangle\langle j|)|\bar\varphi\rangle_B\\
&= {}_B\langle\bar\varphi|(\mathcal{E}\otimes\mathbb{I}_B)(\sum_{k,j}|kk\rangle\langle jj|)|\bar\varphi\rangle_B\\
&= {}_B\langle\bar\varphi|(\mathcal{E}\otimes\mathbb{I}_B)(|\Phi\rangle\langle\Phi|)|\bar\varphi\rangle_B,
\end{aligned}$$

we have

$$(\mathcal{E}\otimes\mathbb{I}_B)(|\Phi\rangle\langle\Phi|) = (\tilde{\mathcal{E}}\otimes\mathbb{I}_B)(|\Phi\rangle\langle\Phi|)$$
$$\Downarrow$$
$${}_B\langle\bar\varphi|(\mathcal{E}\otimes\mathbb{I}_B)(|\Phi\rangle\langle\Phi|)|\bar\varphi\rangle_B = {}_B\langle\bar\varphi|(\tilde{\mathcal{E}}\otimes\mathbb{I}_B)(|\Phi\rangle\langle\Phi|)|\bar\varphi\rangle_B$$
$$\Downarrow$$
$$\mathcal{E}(|\varphi\rangle\langle\varphi|) = \tilde{\mathcal{E}}(|\varphi\rangle\langle\varphi|), \text{ for all } |\varphi\rangle$$
$$\Downarrow$$
$$\mathcal{E}(\varrho) = \tilde{\mathcal{E}}(\varrho), \text{ for all } \varrho$$
$$\Downarrow$$
$$\mathcal{E} = \tilde{\mathcal{E}}.$$

- $\boxed{\mathcal{E} \mapsto (\mathcal{E}\otimes\mathbb{I}_B)(|\Phi\rangle\langle\Phi|) \text{ is a surjective map}}$

For a density operator on $\mathcal{H}_{A'}\otimes\mathcal{H}_B$

$$\Omega = \sum_{j=0}^{m-1} p_j\,|\omega_j\rangle\langle\omega_j|$$

there exist

$$K_j : \mathcal{H}_A \to \mathcal{H}_{A'}$$
$$K_j|k\rangle_A = \sqrt{n\,p_j}\,{}_B\langle k|\omega_j\rangle_{A'B}$$

such that

$$\mathcal{E}(\varrho) = \sum_j K_j\varrho K_j^\dagger$$

satisfies

$$\begin{aligned}
(\mathcal{E}\otimes\mathbb{I}_B)(|\Phi\rangle\langle\Phi|) &= \tfrac{1}{n}\sum_{k,\ell}(\mathcal{E}\otimes\mathbb{I}_B)(|kk\rangle\langle\ell\ell|)\\
&= \tfrac{1}{n}\sum_{k,\ell}(\mathcal{E}\otimes\mathbb{I}_B)(|k\rangle\langle\ell|\otimes|k\rangle\langle\ell|)\\
&= \tfrac{1}{n}\sum_{k,\ell}\mathcal{E}(|k\rangle\langle\ell|)\otimes|k\rangle\langle\ell|\\
&= \tfrac{1}{n}\sum_{k,\ell}\sum_j K_j|k\rangle\langle\ell|K_j^\dagger\otimes|k\rangle\langle\ell|\\
&= \sum_{k,\ell}\sum_j p_j\,{}_B\langle k|\omega_j\rangle\langle\omega_j|\ell\rangle_B\otimes|k\rangle\langle\ell|\\
&= \sum_{k,\ell}{}_B\langle k|\Omega|\ell\rangle_B\otimes|k\rangle_B\,{}_B\langle\ell|\\
&\Downarrow \boxed{D17}\text{ (page 3).}\\
&= \Omega.
\end{aligned}$$

but,

$$\begin{aligned}
{}_B\langle k|\sum_j K_j^\dagger K_j|\ell\rangle_B &= n\sum_j p_j\,\langle\omega_j|k\rangle_B\,{}_B\langle\ell|\omega_j\rangle\\
&= n\sum_j\sum_{a,b}p_j\,\langle\omega_j|ak\rangle_{A'}\langle a|b\rangle_{A'}\,\langle b\ell|\omega_j\rangle\\
&= n\sum_j\sum_{a,b}p_j\,\langle\omega_j|ak\rangle\,\delta_{ab}\,\langle b\ell|\omega_j\rangle\\
&= n\sum_j\sum_a p_j\,\langle a\ell|\omega_j\rangle\langle\omega_j|ak\rangle\\
&= n\sum_a\langle a\ell|\Omega|ak\rangle\\
&= n\,{}_B\langle\ell|\operatorname{tr}_{A'}\Omega|k\rangle_B.
\end{aligned}$$

$\boxed{I4}$

"$\Uparrow$" See $I2$.

"$\Downarrow$"

$$|\Phi\rangle = \tfrac{1}{\sqrt{n}}\sum_{k=0}^{n-1}|k\rangle_A\otimes|k\rangle_B$$
$$\Downarrow$$
$$|k\rangle_A = \sqrt{n}\,{}_B\langle k|\Phi\rangle_{AB}$$
$$\Downarrow$$
$$\begin{aligned}
|\varphi\rangle_A &= \sum_k\varphi_k|k\rangle_A\\
&= \sqrt{n}\sum_k\varphi_k\,{}_B\langle k|\Phi\rangle_{AB}\\
&= \sqrt{n}\,{}_B\langle\bar\varphi|\Phi\rangle_{AB} \text{ where } {}_B\langle\bar\varphi| = \sum_k\varphi_k\,{}_B\langle k|
\end{aligned}$$
$$\Downarrow$$

$$\Downarrow$$

$$\begin{aligned}
\mathcal{E}(|\varphi\rangle\langle\varphi|) &= n\,\mathcal{E}({}_B\langle\bar{\varphi}|\Phi\rangle\langle\Phi|\bar{\varphi}\rangle_B)\\
&= n\,\mathcal{E}({}_B\langle\bar{\varphi}|\sum_{k,j}|kk\rangle\langle jj|\bar{\varphi}\rangle_B)\\
&= n\,\mathcal{E}({}_B\langle\bar{\varphi}|\sum_{k,j}|k\rangle\langle j|\otimes|k\rangle\langle j|\bar{\varphi}\rangle_B)\\
&= n\,\mathcal{E}(\sum_{k,j}\varphi_k\,\bar{\varphi}_j|k\rangle\langle j|)\\
&= n\,\sum_{k,j}\varphi_k\,\bar{\varphi}_j\mathcal{E}(|k\rangle\langle j|)\\
&= n\,{}_B\langle\bar{\varphi}|\sum_{k,j}\mathcal{E}(|k\rangle\langle j|)\otimes|k\rangle\langle j|\bar{\varphi}\rangle_B\\
&= n\,{}_B\langle\bar{\varphi}|(\mathcal{E}\otimes\mathbb{I}_B)(\sum_{k,j}|k\rangle\langle j|\otimes|k\rangle\langle j|)|\bar{\varphi}\rangle_B\\
&= n\,{}_B\langle\bar{\varphi}|(\mathcal{E}\otimes\mathbb{I}_B)(\sum_{k,j}|kk\rangle\langle jj|)|\bar{\varphi}\rangle_B\\
&= n\,{}_B\langle\bar{\varphi}|(\mathcal{E}\otimes\mathbb{I}_B)(|\Phi\rangle\langle\Phi|)|\bar{\varphi}\rangle_B\\
&\Downarrow \boxed{I3}\\
&= n\,\sum_j p_j\,{}_B\langle\bar{\varphi}|\Psi_j\rangle\langle\Psi_j|\bar{\varphi}\rangle_B\\
&= \sum_j K_j|\varphi\rangle\langle\varphi|K_j^\dagger.
\end{aligned}$$

$$\Downarrow$$

$$\boxed{\mathcal{E}(\varrho)=\sum_j K_j\,\varrho K_j^\dagger}$$

where $\quad K_j:\mathcal{H}_A\to\mathcal{H}_{A'}$

$\qquad K_j|\varphi\rangle_A=\sqrt{np_j}\;{}_B\langle\bar{\varphi}|\Psi_j\rangle_{A'B}$.

In addition,

$$\begin{aligned}
K_j|\varphi\rangle_A &= \sqrt{np_j}\,{}_B\langle\bar{\varphi}|\Psi_j\rangle_{A'B}\\
&\Downarrow\\
K_j|i\rangle_A &= \sqrt{np_j}\;{}_B\langle i|\Psi_j\rangle_{A'B}\\
&= \sqrt{np_j}\sum_\ell|\ell\rangle_{A'}\;\langle\ell i|\Psi_j\rangle\\
&\Downarrow\\
{}_{A'}\langle\ell|K_j|i\rangle_A &= \sqrt{np_j}\;\langle\ell i|\Psi_j\rangle\\
&\Downarrow\\
{}_A\langle i|K_j^\dagger|\ell\rangle_{A'} &= \sqrt{np_j}\;\langle\Psi_j|\ell i\rangle\\
&\Downarrow\\
K_j^\dagger|\ell\rangle_{A'} &= \sqrt{np_j}\sum_i\langle\Psi_j|\ell k\rangle_{A'B}\;|k\rangle_A
\end{aligned}$$

and

$$\begin{aligned}
K_j^\dagger K_j|k\rangle_A &= \sqrt{np_j}\sum_\ell K_j^\dagger|\ell\rangle_{A'}\;\langle\ell k|\Psi_j\rangle\\
&= np_j\sum_{\ell,i}\langle\ell k|\Psi_j\rangle\langle\Psi_j|\ell i\rangle_{A'B}\;|i\rangle_A\\
&\Downarrow\\
\sum_j K_j^\dagger K_j|k\rangle_A &= n\sum_{\ell,i}\langle\ell k|(\mathcal{E}\otimes\mathbb{I}_B)(|\Phi\rangle\langle\Phi|)|\ell i\rangle\;|i\rangle_A\\
&= n\sum_{\ell,i}\sum_{a,b}\langle\ell k|(\mathcal{E}\otimes\mathbb{I}_B)|aa\rangle\langle bb|)|\ell i\rangle\;|i\rangle_A\\
&= \sum_{\ell,i}\sum_{a,b}\langle\ell k|\mathcal{E}(|a\rangle\langle b|)\otimes|a\rangle\langle b|\ell i\rangle\;|i\rangle_A\\
&= \sum_{\ell,i}\sum_{a,b}\langle\ell|\mathcal{E}(|a\rangle\langle b|)|\ell\rangle\langle k|a\rangle\langle b|i\rangle\;|i\rangle_A\\
&= \sum_{\ell,i}\sum_{a,b}\langle\ell|\mathcal{E}(|a\rangle\langle b|)|\ell\rangle\delta_{ka}\delta_{bi}\langle b|i\rangle\;|i\rangle_A\\
&= \sum_i\operatorname{tr}\mathcal{E}(|k\rangle\langle i|)|\;|i\rangle_A\\
&= \sum_i\operatorname{tr}(|k\rangle\langle i|)|\;|i\rangle_A\\
&= \sum_i\langle i|k\rangle\;|i\rangle_A\\
&= |k\rangle_A,\quad\text{for any }k\\
&\Downarrow
\end{aligned}$$

$$\boxed{\sum_j K_j^\dagger K_j=\mathbb{I}}.$$

$\boxed{I5}$

" $\Downarrow$ "

We choose:

- an arbitrary space $\mathcal{H}_B$ of dimension m,
- an arbitrary orthonormal basis $\{|k\rangle\}_{k=0}^{m-1}$ of $\mathcal{H}_B$

By using the operators $\{K_j:\mathcal{H}_A\to\mathcal{H}_{A'}\}_{j=0}^{m-1}$,
we define the surjective map

$$U_0:\mathcal{V}\to\mathcal{W},\quad U(|\psi\rangle\otimes|0\rangle)=\sum_{j=0}^{m-1}K_j|\psi\rangle\otimes|j\rangle,$$

where the sets

$$\mathcal{V}\equiv\mathcal{H}_A\otimes|0\rangle=\{\;|\psi\rangle\otimes|0\rangle\;|\;\;|\psi\rangle\in\mathcal{H}_A\;\}$$

$$\mathcal{W}=\left\{\sum_{j=0}^{m-1}K_j|\psi\rangle\otimes|j\rangle\;\middle|\;|\psi\rangle\in\mathcal{H}_A\right\}$$

are vector subspaces of $\mathcal{H}_A\otimes\mathcal{H}_B$ and $\mathcal{H}_{A'}\otimes\mathcal{H}_B$.
The map U is a unitary transform because

$$\begin{aligned}
&\langle U_0(|\psi\rangle\otimes|0\rangle),U_0(|\varphi\rangle\otimes|0\rangle)\rangle=\\
&\quad=\left\langle\sum_{j=0}^{m-1}K_j|\psi\rangle\otimes|j\rangle,\sum_{\ell=0}^{m-1}K_\ell|\varphi\rangle\otimes|\ell\rangle\right\rangle\\
&\quad=\sum_{j,\ell=0}^{m-1}\langle K_j\psi,K_\ell\varphi\rangle\;\langle j|\ell\rangle\\
&\quad=\sum_{j=0}^{m-1}\langle K_j\psi,K_j\varphi\rangle\\
&\quad=\sum_{j=0}^{m-1}\langle K_j^\dagger K_j\psi,\varphi\rangle\\
&\quad=\left\langle\sum_{j=0}^{m-1}K_j^\dagger K_j\psi,\varphi\right\rangle\\
&\quad=\langle\psi,\varphi\rangle\\
&\quad=\langle(|\psi\rangle\otimes|0\rangle),(|\varphi\rangle\otimes|0\rangle)\rangle\;.
\end{aligned}$$

Consequently, $U_0:\mathcal{V}\to\mathcal{W}$ is also injective.
It is an isomorphism and $\dim\mathcal{W}=\dim\mathcal{V}=\dim\mathcal{H}_A\equiv n$.
The unitary transform $U_0:\mathcal{V}\to\mathcal{W}$ can be extended
up to a unitary transform $U:\mathcal{H}_A\otimes\mathcal{H}_B\to\mathcal{H}_{A'}\otimes\mathcal{H}_B$
by following the steps:

- we choose an orthonormal basis $\{|\tilde{\psi}_0\rangle,\,|\tilde{\psi}_1\rangle,\,...,\,|\tilde{\psi}_{n-1}\rangle\}$ of $\mathcal{V}$ and then extend it up to an orthonormal basis $\{|\tilde{\psi}_0\rangle,\,|\tilde{\psi}_1\rangle,...,\,|\tilde{\psi}_{n-1}\rangle,...,\,|\tilde{\psi}_{nm-1}\rangle\}$ of $\mathcal{H}_A\otimes\mathcal{H}_B$,
- we extend the orthonormal basis $\{|\tilde{\phi}_0\rangle=U_0|\tilde{\psi}_0\rangle,\,|\tilde{\phi}_1\rangle=U_0|\tilde{\psi}_1\rangle,\,...,\,|\tilde{\phi}_{n-1}\rangle=U_0|\tilde{\psi}_{n-1}\rangle\}$ of $\mathcal{W}$ up to an orthonormal basis $\{|\tilde{\phi}_0\rangle,\,|\tilde{\phi}_1\rangle,...,\,|\tilde{\phi}_{n-1}\rangle,...,\,|\tilde{\phi}_{nm-1}\rangle\}$ of $\mathcal{H}_{A'}\otimes\mathcal{H}_B$,
- we define $\;U:\mathcal{H}_A\otimes\mathcal{H}_B\to\mathcal{H}_{A'}\otimes\mathcal{H}_B\;$ as $\;U\psi_k=\phi_k$.

.... We have

$$ {}_B\langle k|U|\psi\omega\rangle = {}_B\langle k|\sum_{k=0}^{m-1} M_\ell|\psi\rangle\otimes|\ell\rangle = M_k|\psi\rangle. $$

In the case of a density operator

$$ \varrho_A = \sum_{j=0}^{n-1} \lambda_j |\psi_j\rangle\langle\psi_j| $$

we get

$$
\begin{aligned}
&\mathrm{tr}(U\varrho_A\otimes\varrho_B U^\dagger \mathbb{I}\otimes|k\rangle\langle k|)\\
&\quad\Downarrow \boxed{D19}\\
&= \mathrm{tr}_A\big({}_B\langle k|U\varrho_A\otimes\varrho_B U^\dagger|k\rangle_B\big)\\
&= \mathrm{tr}_A\big({}_B\langle k|U\varrho_A\otimes\varrho_B U^\dagger|k\rangle_B\big)\\
&= \sum_{j=0}^{n-1}\lambda_j \mathrm{tr}_A\big({}_B\langle k|U|\psi_j\rangle\langle\psi_j|\otimes|\omega\rangle\langle\omega|U^\dagger|k\rangle_B\big)\\
&\quad\Downarrow \boxed{D13}\\
&= \sum_{j=0}^{n-1}\lambda_j \mathrm{tr}_A\big({}_B\langle k|U|\psi_j\omega\rangle\langle\psi_j\omega|U^\dagger|k\rangle_B\big)\\
&= \sum_{j=0}^{n-1}\lambda_j \mathrm{tr}_A(M_k|\psi_j\rangle\langle\psi_j|M_k^\dagger)\\
&= \mathrm{tr}_A(M_k\sum_{j=0}^{n-1}\lambda_j|\psi_j\rangle\langle\psi_j|M_k^\dagger)\\
&= \mathrm{tr}_A(M_k\varrho_A M_k^\dagger) = p_k,
\end{aligned}
$$

$\boxed{I8}$

By changing the basis $|j\rangle = \sum_k S_{jk}|k\rangle_1$, we get

$$ U(|\psi\rangle\otimes|0\rangle) = \sum_j M_j|\psi\rangle\otimes|n\rangle = \sum_k N_k|\psi\rangle\otimes|k\rangle_1, $$

with the new Kraus operators $N_k = \sum_j S_{jk}M_j$.

$\boxed{I10}$

" $\Leftarrow$ "

$$
\begin{aligned}
&\mathcal{E}(\varrho) = U\varrho U^\dagger\\
&\quad\Downarrow\\
&\tilde{\mathcal{E}}(\varrho) = U^\dagger\varrho U \text{ satisfies}\\
&\qquad\qquad \tilde{\mathcal{E}}(\mathcal{E}(\varrho)) = U^\dagger U\varrho U^\dagger U = \varrho.
\end{aligned}
$$

" $\Rightarrow$ "

- If for the channel $\mathcal{E}(\varrho) = \sum_j K_j\varrho K^\dagger$ there exists a channel $\tilde{\mathcal{E}}(\varrho) = \sum_k M_k\varrho M_k^\dagger$ such that $\tilde{\mathcal{E}}(\mathcal{E}(\varrho)) = \varrho$, for any state ϱ, then $\sum_{j,k} M_kK_j|\psi\rangle\langle\psi|K_j^\dagger M_k^\dagger = |\psi\rangle\langle\psi|$, for any pure state $|\psi\rangle$.
- Since $M_jN_k|\psi\rangle\langle\psi|N_k^\dagger M_j^\dagger \geq 0$, the relation
$$ \sum_{j,k} M_kK_j|\psi\rangle\langle\psi|K_j^\dagger M_k^\dagger = |\psi\rangle\langle\psi| $$
is a representation of the pure state $|\psi\rangle$ as a convex-linear combination of pure states.
- From $\boxed{B14}$ (page 1), it follows that
$$ M_kK_j|\psi\rangle = f_{kj}(\psi)\,|\psi\rangle, \text{ for any pure state } |\psi\rangle. $$
- From $\boxed{\#B1}$ (page 8), it follows that each of the functions $f_{kj}:\mathcal{H}\to\mathbb{C}$ is a constant function, $f_{kj}(\psi) = \lambda_{kj}$, and
$$ M_kK_j = \lambda_{kj}\mathbb{I}. $$
- From the relation
$$
\begin{aligned}
K_\ell^\dagger K_j &= K_\ell^\dagger\sum_k M_k^\dagger M_kK_j\\
&= \sum_k \lambda_{k\ell}^*\lambda_{kj}\mathbb{I} \equiv \beta_{\ell j}\mathbb{I},
\end{aligned}
$$
by using the polar decomposition (#F4, pag. 8), we get
$$ K_j = U_j\sqrt{K_j^\dagger K_j} = \sqrt{\beta_{jj}}U_j $$
and consequently $K_j^\dagger K_j = K_jK_j^\dagger$
$$
\begin{aligned}
&K_\ell^\dagger K_j = \beta_{\ell j}\mathbb{I} &\qquad \sum_j|\beta_{\ell j}|^2\mathbb{I} &= \sum_j K_\ell^\dagger K_jK_j^\dagger K_\ell\\
&\quad\Downarrow & &= \sum_j K_\ell^\dagger K_j^\dagger K_jK_\ell\\
&U_j = \tfrac{\beta_{\ell j}}{\sqrt{\beta_{\ell\ell}\beta_{jj}}}U_\ell & &= K_\ell^\dagger K_\ell\\
&\quad\Downarrow & &= \beta_{\ell\ell}\mathbb{I}.\\
&K_j = \tfrac{\beta_{\ell j}}{\sqrt{\beta_{\ell\ell}}}U_\ell & &
\end{aligned}
$$
- The channel $\mathcal{E}$ can be expressed as
$$ \mathcal{E}(\varrho) = \sum_j K_j\varrho K_j^\dagger = \tfrac{1}{\beta_{\ell\ell}}\sum_j|\beta_{\ell j}|^2U_\ell\varrho U_\ell^\dagger = U_\ell\varrho U_\ell^\dagger. $$

$\boxed{J}$ **Axioms of Quantum Mechanics** (open systems)

....

Additional definitions/results

#1

We have

$$\begin{aligned} &A(\alpha v)=f(\alpha v)\,\alpha v,\\ &\alpha\;Av=f(\alpha v)\,\alpha v,\\ &\alpha\;f(v)\,v=f(\alpha v)\,\alpha v,\\ &\qquad\Downarrow \text{ for } \alpha\neq 0 \text{ and } v\neq 0\\ &f(\alpha v)=f(v).\end{aligned}$$

If the vectors v and w are linearly independent, then

$$\begin{gathered} A(v+w)=Av+Aw\\ f(v+w)\,(v+w)=f(v)\,v+f(w)\,w\\ \Downarrow\\ \begin{cases} f(v+w)=f(v)\\ f(v+w)=f(w)\end{cases}\\ \Downarrow\\ f(v)=f(w).\end{gathered}$$

If $\{v_0, v_1, ..., v_{n-1}\}$ is a basis of $\mathcal{V}$, then

$$f(v_0)=f(v_1)=\cdots=f(v_{n-1})$$

and

$$\begin{aligned} A\sum_{j=0}^{n-1}\alpha_j v_j &= \sum_{j=0}^{n-1}\alpha_j\,Av_j\\ &=\sum_{j=0}^{n-1}\alpha_j\,f(v_j)\,v_j\\ &=f(v_0)\sum_{j=0}^{n-1}\alpha_j v_j.\end{aligned}$$

#5

If $\{|e_j\rangle\}$ is an orthonormal basis, then

$$\begin{aligned} ||Ax|| &= ||A(\textstyle\sum|e_j\rangle\langle e_j|x\rangle)||\\ &=\textstyle\sum|\langle e_j|x\rangle)|\;||A|e_j\rangle||\\ &\leq\sqrt{\textstyle\sum|\langle e_j|x\rangle)|^2}\;\sqrt{\textstyle\sum||A|e_j\rangle||^2}\\ &\leq||x||\;\sqrt{\textstyle\sum||A|e_j\rangle||^2}\\ &=C\,||x||.\end{aligned}$$

#8

$$\begin{aligned}(\mathrm{e}^A)^\dagger &= \left(\sum_{k=0}^{\infty}\frac{A^k}{k!}\right)^\dagger=\lim_{m\to\infty}\left(\sum_{k=0}^{m}\frac{A^k}{k!}\right)^\dagger\\ &=\lim_{m\to\infty}\sum_{k=0}^{m}\frac{A^{\dagger k}}{k!}=\sum_{k=0}^{\infty}\frac{A^{\dagger k}}{k!}=\mathrm{e}^{A^\dagger}.\end{aligned}$$

#9

$$(\mathrm{e}^{\mathrm{i}A})^\dagger\mathrm{e}^{\mathrm{i}A}=\mathrm{e}^{-\mathrm{i}A^\dagger}\mathrm{e}^{\mathrm{i}A}=\mathrm{e}^0=\mathbb{I}.$$

#10

$$\begin{aligned}(F^\dagger F)[\varphi](k) &= \tfrac{1}{n}\sum_{j=0}^{n-1}\mathrm{e}^{\frac{2\pi\mathrm{i}}{n}kj}\sum_{m=0}^{n-1}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{n}jm}\varphi(m)\\ &=\tfrac{1}{n}\sum_{m=0}^{n-1}\sum_{j=0}^{n-1}\mathrm{e}^{\frac{2\pi\mathrm{i}}{n}(k-m)j}\varphi(m)\\ &=\tfrac{1}{n}\sum_{m=0}^{n-1}\sum_{j=0}^{n-1}\left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{n}(k-m)}\right)^j\;\varphi(m)\\ &=\tfrac{1}{n}\sum_{m=0}^{n-1}n\,\delta_{km}\,\varphi(m)=\varphi(k).\end{aligned}$$

#11

$$\begin{matrix}\varphi\otimes\psi:\mathcal{H}_2\to\mathcal{H}_1,\\ (\varphi\otimes\psi)(\eta)=\langle\eta|\psi\rangle\varphi\end{matrix}\quad\text{is anti-linear, for any}\quad\begin{matrix}\varphi\in\mathcal{H}_1,\\ \psi\in\mathcal{H}_2.\end{matrix}$$

These particular operators span the considered space,

$$\begin{gathered}(\varphi_1+\varphi_2)\otimes\psi=\varphi_1\otimes\psi+\varphi_2\otimes\psi\\ \varphi\otimes(\psi_1+\psi_2)=\varphi\otimes\psi_1+\varphi\otimes\psi_2\\ \lambda(\varphi\otimes\psi)=(\lambda\varphi)\otimes\psi=\varphi\otimes(\lambda\psi).\end{gathered}$$

#12

Since $|u_iv_j\rangle\langle u_iv_j|=|u_i\rangle\langle u_i|\otimes|v_j\rangle\langle v_j|$, we have

$$\begin{matrix}\displaystyle\sum_{i=1}^{k}|u_i\rangle\langle u_i|=\mathbb{I}_{\mathcal{H}_1}\\ \displaystyle\sum_{j=1}^{\ell}|v_j\rangle\langle v_j|=\mathbb{I}_{\mathcal{H}_2}\end{matrix}\quad\Rightarrow\quad\begin{aligned}&\sum_{i=1}^{k}\sum_{j=1}^{\ell}|u_iv_j\rangle\langle u_iv_j|=\\ &\qquad=\sum_{i=1}^{k}|u_i\rangle\langle u_i|\otimes\sum_{j=1}^{\ell}|v_j\rangle\langle v_j|\\ &\qquad=\mathbb{I}_{\mathcal{H}_1}\otimes\mathbb{I}_{\mathcal{H}_2}=\mathbb{I}_{\mathcal{H}_1\otimes\mathcal{H}_2}.\end{aligned}$$

1

The superposition (liniar combination) of states

$$|\psi\rangle=\sum_{k=1}^{n}c_k|\psi_k\rangle$$

is a pure state with the density operator

$$\varrho=|\psi\rangle\langle\psi|=\sum_{k=1}^{n}|c_k|^2|\psi_k\rangle\langle\psi_k|+\sum_{k\neq j}c_k\,\bar{c}_j\,|\psi_k\rangle\langle\psi_j|.$$

The second sum contains the *interference terms.*

2

Example. Representations as a probabilistic ensemble of pure states of the maximally mixed state of a qubit:

$$\varrho=\begin{pmatrix}\frac12 & 0\\ 0 & \frac12\end{pmatrix}\quad\Rightarrow$$

- $\varrho=\;\frac12|0\rangle\langle 0|+\frac12|1\rangle\langle 1|,$
- $\varrho=\;\frac12|+\rangle\langle +|+\frac12|-\rangle\langle -|,$
- $\varrho=\frac12|\uparrow_y\rangle\langle\uparrow_y|+\frac12|\downarrow_y\rangle\langle\downarrow_y|,$
- $\varrho=\;\frac14|0\rangle\langle 0|+\frac14|1\rangle\langle 1|$ $+\frac14|+\rangle\langle +|+\frac14|-\rangle\langle -|,$
- etc.

#15

$\varrho^\dagger=\varrho\;\;\Rightarrow\;\;$ ϱ is diagonalizable, that is, there exists an orthonormal basis $\{|\psi_k\rangle\}$ such that

$$\varrho=\sum_{k=1}^{n}p_k|\psi_k\rangle\langle\psi_k|.$$

$$\varrho\geq 0\quad\Rightarrow\quad p_k=\langle\psi_k|\varrho|\psi_k\rangle\geq 0.$$

$$\mathrm{tr}\,\varrho=1\quad\Rightarrow\quad\sum_{k=1}^{n}p_k=\mathrm{tr}\,\varrho=1.$$

#16

$$\begin{aligned}&\mathrm{i}\,\tfrac{\partial|\psi\rangle}{\partial t}=H\,|\psi\rangle\quad\Rightarrow\quad-\mathrm{i}\,\tfrac{\partial\langle\psi|}{\partial t}=\langle\psi|H\\ \Rightarrow\;&\mathrm{i}\,\tfrac{\partial}{\partial t}\varrho=\sum_{k=0}^{d-1}p_k\mathrm{i}\,\tfrac{\partial|\psi_k\rangle}{\partial t}\langle\psi_k|-\sum_{k=0}^{d-1}p_k|\psi_k\rangle\mathrm{i}\,\tfrac{\partial\langle\psi_k|}{\partial t},\\ &\mathrm{i}\,\tfrac{\partial}{\partial t}\varrho=\sum_{k=0}^{d-1}p_k\,H|\psi_k\rangle\langle\psi_k|-\sum_{k=0}^{d-1}p_k|\psi_k\rangle\langle\psi_k|H,\\ &\mathrm{i}\,\tfrac{\partial}{\partial t}\varrho=H\varrho-\varrho H.\end{aligned}$$

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