# An overview on $\boxed{\text{COMPLEX ANALYSIS}}$

Nicolae Cotfas, version 27 Apr 2020 (for future updates see https://unibuc.ro/user/nicolae.cotfas/ )

## COMPLEX NUMBERS $\mathbb{C}=\{z=x+y\mathrm{i}\mid x,y\in\mathbb{R}\}$

**Definition**.

For $\boxed{z=x+y\,\mathrm{i}}$

$\mathfrak{Re}\,z\stackrel{\text{def}}{=}x$ *real part of* $z$

$\mathfrak{Im}\,z\stackrel{\text{def}}{=}y$ *imaginary part of* $z$

$\bar{z}\stackrel{\text{def}}{=}x-y\,\mathrm{i}$ *complex conjugate of* $z$

$|z|\stackrel{\text{def}}{=}\sqrt{x^2+y^2}$ *modulus of* $z$

$\mathbb{R}\subset\mathbb{C}$, $x\equiv x+0\mathrm{i}$

*Identification* $\mathbb{C}\equiv\mathbb{R}^2$ (as normed spaces and metric spaces).

$\mathbb{C}\longrightarrow\mathbb{R}^2$, $x+y\,\mathrm{i}\mapsto(x,y)$ is linear bijective and $\boxed{|x+y\,\mathrm{i}|=\sqrt{x^2+y^2}=\|(x,y)\|}$

*Distance in* $\mathbb{C}$

$|z_1-z_2|=$ distance between $z_1$ and $z_2$

$|z|=$ distance between $z$ and $0$

$\boxed{B_r(z_0)=\{z\mid |z-z_0|<r\}}$ open ball of center $z_0$ and radius $r$.

$D\subset\mathbb{C}$ is open set $\stackrel{\text{def}}{\Longleftrightarrow}$ for any $z\in D$ there exists $r>0$ such that $B_r(z)\subset D$.

*Convergence to* $\infty$. $\lim\limits_{n\to\infty}z_n=\infty\stackrel{\text{def}}{\Longleftrightarrow}\lim\limits_{n\to\infty}|z_n|=\infty$.

*A fundamental inequality.*

For any $z=x+y\,\mathrm{i}$, we have $\left.\begin{matrix}|x|\\|y|\end{matrix}\right\}\leq|x+y\,\mathrm{i}|\leq|x|+|y|$

*Convergence of a sequence* $\lim\limits_{n\to\infty}(x_n+y_n\mathrm{i})=\alpha+\beta\mathrm{i}\ \Leftrightarrow\ \begin{cases}\lim\limits_{n\to\infty}x_n=\alpha\\ \lim\limits_{n\to\infty}y_n=\beta\end{cases}$

$|z|>1\ \Rightarrow\ \lim\limits_{n\to\infty}z^n=\infty$. $|z|<1\ \Rightarrow\ \lim\limits_{n\to\infty}z^n=0$.

$|z|<1\ \Rightarrow\ 1+z+z^2+\ldots=\lim\limits_{n\to\infty}(1+z+z^2+\ldots+z^n)=\frac{1}{1-z}$.

*Euler's formula.* $\boxed{\mathrm{e}^{\mathrm{i}t}\stackrel{\text{def}}{=}\cos t+\mathrm{i}\sin t}$

*Argument of a complex number.*

For $z\neq0$ there exists $\boxed{\arg z\in(-\pi,\pi]}$ such that $z=|z|\,(\cos(\arg z)+\mathrm{i}\,\sin(\arg z))=|z|\,\mathrm{e}^{\mathrm{i}\arg z}$

## COMPLEX FUNCTIONS (Notation: $\mathbb{C}^*=\mathbb{C}\backslash\{0\}$)

*Complex function* = a complex-valued function.

*Examples:* $f:\mathbb{C}\longrightarrow\mathbb{C}$, $f(z)=z^3=z\,z\,z$,

$f:\mathbb{C}^*\longrightarrow\mathbb{C}$, $f(z)=\frac{1}{z}$, $\frac{1}{x+y\mathrm{i}}=\frac{x}{x^2+y^2}-\frac{y}{x^2+y^2}\mathrm{i}$

$f:\mathbb{C}\longrightarrow\mathbb{C}$, $f(z)=\mathrm{e}^z$, $\mathrm{e}^{x+y\mathrm{i}}=\mathrm{e}^x\cos y+\mathrm{i}\mathrm{e}^x\sin y$,

$f:\mathbb{C}\longrightarrow\mathbb{C}$, $f(z)=\cos z\stackrel{\text{def}}{=}\frac{\mathrm{e}^{\mathrm{i}z}+\mathrm{e}^{-\mathrm{i}z}}{2}$,

$f:\mathbb{C}\longrightarrow\mathbb{C}$, $f(z)=\sin z\stackrel{\text{def}}{=}\frac{\mathrm{e}^{\mathrm{i}z}-\mathrm{e}^{-\mathrm{i}z}}{2\mathrm{i}}$.

**Definition**. Let $D\subseteq\mathbb{C}$ be an open set, and $z_0\in D$.

A function $f:D\longrightarrow\mathbb{C}$ is **complex-differentiable** ($\mathbb{C}$-differentiable) at $z_0$ $\stackrel{\text{def}}{\Longleftrightarrow}$ there exists and is finite $f'(z_0)\stackrel{\text{def}}{=}\lim\limits_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$.

**Theorem** (*Cauchy-Riemann*).

A function $f:D\longrightarrow\mathbb{C}$, $f(x+y\mathrm{i})=u(x,y)+v(x,y)\mathrm{i}$ defined on an open set $D\subseteq\mathbb{C}\equiv\mathbb{R}^2$ is $\mathbb{C}$-differentiable at $z_0\in D$ $\Leftrightarrow$ $u,v:D\longrightarrow\mathbb{R}$ are differentiable at $(x_0,y_0)$ and satisfy the relations $\frac{\partial u}{\partial x}(x_0,y_0)=\frac{\partial v}{\partial y}(x_0,y_0)$, $\frac{\partial u}{\partial y}(x_0,y_0)=-\frac{\partial v}{\partial x}(x_0,y_0)$

If these conditions are satisfied, then $\boxed{f'(x_0+y_0\mathrm{i})=\frac{\partial u}{\partial x}(x_0,y_0)+\frac{\partial v}{\partial x}(x_0,y_0)\,\mathrm{i}}$

**Definition**.

$f:D\to\mathbb{C}$ defined on an open set $D$ is called $\mathbb{C}$-*differentiable* if (or *holomorphic function*) $\boxed{f \text{ is } \mathbb{C}\text{-differentiable at any point } z_0\in D.}$

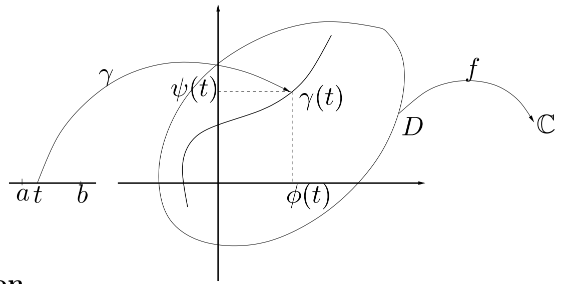
*Examples:* $\boxed{(z^n)'=nz^{n-1}}$ $\boxed{(\mathrm{e}^z)'=\mathrm{e}^z}$ $\boxed{(\sin z)'=\cos z}$

## COMPLEX LINE INTEGRAL

Let $D\subseteq\mathbb{C}$ be an open set,

$f:D\longrightarrow\mathbb{C}$ be a continuous function,

$\gamma:[a,b]\longrightarrow D$ be a path of class $C^1$ ($\gamma$ and $\gamma'$ are continuous).




**Definition**.

**The complex line integral of $f$ along $\gamma$** is $\boxed{\int\limits_\gamma f(z)\,dz=\int\limits_a^b f(\gamma(t))\,\gamma'(t)dt}$

Definition can be extended to the case when $\gamma$ is piecewise of class $C^1$.

Definition does not depend on the parametrization of $\gamma$ we use.

$g:D\to\mathbb{C}$ is a primitive of $f$ (that is $g'=f$) $\Rightarrow$ $\boxed{\int_\gamma f(z)\,dz=g(\gamma(b))-g(\gamma(a))}$

If $f:D\to\mathbb{C}$ admits a primitive and $\gamma$ is closed, then $\int_\gamma f(z)\,dz=0$.

**Definition**.

Two paths $\gamma_0,\gamma_1:[a,b]\to D$ with the same endpoints are **homotopic** in $D$ if one of them can be continuosly deformed into the other inside $D$.

Path **homotopic to zero** in $D$ = path homotopic to a constant path.

**Theorem** (Cauchy). For an open set $D$:

$f:D\longrightarrow\mathbb{C}$ is a holomorphic function, $\gamma:[a,b]\longrightarrow D$ is a path homotopic to zero in $D$ $\Rightarrow$ $\boxed{\int\limits_\gamma f(z)\,dz=0}$

**Theorem** For an open set $D$:

$f:D\longrightarrow\mathbb{C}$ is a holomorphic function, $\gamma_0,\gamma_1$ are paths homotopic in $D$ $\Rightarrow$ $\boxed{\int_{\gamma_0}f(z)\,dz=\int_{\gamma_1}f(z)\,dz}$

**Definition**. Let $D$ be an open set and $f:D\longrightarrow\mathbb{C}$ holomorphic.

$z_0\in\mathbb{C}\backslash D$ is *an isolated singular point* if there exists $r>0$ such that $\{z\mid 0<|z-z_0|<r\}\subset D$.

$z_0\in D$ is *a zero of multiplicity $k$* if $f(z_0)=f'(z_0)=\ldots=f^{(k-1)}(z_0)=0$ and $f^{(k)}(z_0)\neq0$.

$z_0\in\mathbb{C}\backslash D$ is *a pole of order $k$ of $f$* if $z_0$ is a zero of multiplicity $k$ of $\frac{1}{f}$.

**Theorem**. Let $D$ be an open set and $f:D\longrightarrow\mathbb{C}$ holomorphic.

$z_0$ is a an isolated singular point and $\{z\mid 0<|z-z_0|<r\}\subset D$ $\Rightarrow$ there exists a unique Laurent series such that $f(z)=\sum\limits_{n=-\infty}^{\infty}a_n\,(z-z_0)^n$, for $z$ satisfying $0<|z-z_0|<r$.

The number $\boxed{\mathbf{Res}_{z_0}f\stackrel{\text{def}}{=}a_{-1}}$ is the **residue** of $f$ in $z_0$.

$z_0$ pole of order $k\Rightarrow$ $f(z)=\frac{a_{-k}}{(z-z_0)^k}+\ldots+\frac{a_{-1}}{(z-z_0)}+a_0+a_1(z-z_0)+\ldots$ and $\mathbf{Res}_{z_0}f=\frac{1}{(k-1)!}\lim\limits_{z\to z_0}\left((z-z_0)^k\,f(z)\right)^{(k-1)}$

**Definition**. (*Index* of a point $z_0$ with respect to a path).

For a closed path $\gamma$ not passing through $z_0$, $\boxed{n(z_0,\gamma)\stackrel{\text{def}}{=}\frac{1}{2\pi\mathrm{i}}\int_\gamma\frac{1}{z-z_0}dz}$ shows how many turns around $z_0$, $\gamma$ makes.

**Theorem of Residues**. For an open set $D\subseteq\mathbb{C}$:

$f:D\longrightarrow\mathbb{C}$ is an holomorphic function, $S$ is the set of isolated singular points, $\gamma:[a,b]\to D$ homotopic to zero in $D\cup S$ $\Rightarrow$ $\boxed{\int\limits_\gamma f(z)dz=2\pi\mathrm{i}\sum\limits_{z\in S}n(z,\gamma)\mathbf{Res}_zf.}$