LINEAR ALGEBRA - Solved Exercises 4

(Real and complex finite-dimensional vector spaces)

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A PARTIAL EVALUATION A

Exercise 1

Find the dimension of the real vector space

A1 $\mathcal{V}_1=\{\ x=(x_1,x_2,x_3)\in\mathbb{R}^3\ |\ 2x_1+x_2-x_3=0\ \}$

and the dimension of the complex vector spaces

A2 $\mathcal{V}_2=\left\{\ x=(x_1,x_2,x_3)\in\mathbb{C}^3\ \middle|\ \begin{array}{c} \mathrm{i}\,x_1-x_3=0\\ x_2+x_3=0\end{array}\right\}$,

A3 $\mathcal{V}_3=\mathrm{span}\{\ (1,1,1),\ (\mathrm{i},1,-1),\ (0,1+\mathrm{i},1-\mathrm{i})\ \}$,

A4 $\mathcal{V}_4=\mathcal{V}_3\cap\{\ x=(x_1,x_2,x_3)\in\mathbb{C}^3\ |\ x_2+x_3=0\ \}$.

Exercise 2

A5 In $\mathbb{R}^3$, find the coordinates of $v=(1,2,3)$ in the basis $\{\ e_1=(1,0,0),\ e_2=(0,1,0),\ e_3=(0,0,1)\ \}$.

A6 In $\mathbb{R}^3$, find the coordinates of $v=(1,2,3)$ in the basis $\{\ v_1=(1,1,1),\ v_2=(0,1,1),\ v_3=(0,0,1)\ \}$.

A7 In $\mathbb{C}^2$, find the change of basis matrix from the basis
$\mathcal{B}=\{\ v_1=(1,1),\ v_2=(1,-1)\ \}$
to the basis
$\mathcal{B}'=\{\ v_1'=(1,\mathrm{i}),\ v_2'=(1,-\mathrm{i})\ \}$.

A8 Prove that the space

$$\mathcal{W}=\left\{\ A=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\in\mathcal{M}_{2\times2}(\mathbb{C})\ \middle|\ A^\dagger=A\ \right\}$$

where

$$A^\dagger=\bar{A}^T=\begin{pmatrix}\bar{a}_{11} & \bar{a}_{21}\\ \bar{a}_{12} & \bar{a}_{22}\end{pmatrix}\ \text{is the } \textit{adjoint} \text{ of } A=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix},$$

is a real vector space of dimension 4 with respect to the usual matrix addition and scalar multiplication.

Exercise 3

A9 Find the eigenvalues of the linear operator
$A:\mathbb{R}^2\longrightarrow\mathbb{R}^2,\quad A(x_1,x_2)=(x_1+x_2,x_1-x_2)$

A10 Describe the eigenspaces of A.

A11 Compute $\mathrm{Tr}\,A$ and $\det A$.

A12 Prove that the linear operator
$A:\mathbb{R}^3\longrightarrow\mathbb{R}^3,\quad A(x_1,x_2,x_3)=(x_2,x_1,2x_3)$
is diagonalizable, and indicate the diagonal form of A.

Some definitions, notations and results

1 **Definition**. In a vector space over the field $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$,

$$\boxed{\mathrm{span}\{v_1,v_2,...,v_n\}\overset{\text{def}}{=}\{\alpha_1v_1+\alpha_2v_2+...+\alpha_nv_n\ |\ \alpha_k\in\mathbb{K}\}}$$

2 **Definition**. In the case of vector space over $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$,

λ is an **eigenvalue** of a linear operator $A:\mathcal{V}\to\mathcal{V}$ if
- $\lambda\in\mathbb{K}$ and
- there exists $v\in\mathcal{V}$, $v\neq0$ (**eigenvector** corresp. to λ) such that $\boxed{Av=\lambda\,v}$.

The **eigenspace** corresponding to λ is $\boxed{\{\ x\ |\ Ax=\lambda x\ \}}$.

3 **Theorem**. In the case of vector space over $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$,

λ is an eigenvalue of a linear operator $A:\mathcal{V}\to\mathcal{V}$ $\Leftrightarrow$
- $\lambda\in\mathbb{K}$ and
- λ is a root of the equation

$$\begin{vmatrix} a_1^1-\lambda & a_2^1 & \cdots & a_n^1\\ a_1^2 & a_2^2-\lambda & \cdots & a_n^2\\ \vdots & \vdots & \ddots & \vdots\\ a_1^n & a_2^n & \cdots & a_n^n-\lambda\end{vmatrix}=0$$

4 **Definition**. In the case of of a linear operator $A:\mathcal{V}\to\mathcal{V}$

$$\boxed{\mathrm{Im\,A}=\{\ Ax\ |\ x\in\mathcal{V}\}}\qquad\boxed{\mathrm{Ker\,A}=\{\ x\ |\ Ax=0\}}$$

B PARTIAL EVALUATION B

Exercise 1

In the real Hilbert space $\mathbb{R}^3$:

B1 Compute $\langle(1,2,-1),(1,2,3)\rangle$.

B2 Find the projection of $x=(1,2,3)$ on $w=(1,2,-1)$.

B3 Find an orthonormal basis in the subspace
$\mathcal{H}=\{(x_1,x_2,x_3)\ \ |\ \ x_1-x_3=0\ \}$.

Exercise 2

In the complex Hilbert space $\mathbb{C}^2$, let

$$A:\mathbb{C}^2\to\mathbb{C}^2,\qquad A(x_1,x_2)=((1+\mathrm{i})x_2,(1-\mathrm{i})x_1)\,.$$

B4 Prove that A is self-adjoint (also called Hermitian).

B5 Find the eigenvalues of A.

B6 Diagonalize A. Indicate the used basis.

Exercise 3

In the complex Hilbert space $\mathbb{C}^2$, let

$$U:\mathbb{C}^2\to\mathbb{C}^2,\quad U(x_1,x_2)=\left(\tfrac{1}{\sqrt2}(x_1-x_2),\tfrac{1}{\sqrt2}(x_1+x_2)\right).$$

B7 Prove that U is a unitary transformation.

B8 Find the eigenvalues of U.

B9 Diagonalize U. Indicate the used basis.

Exercise 4

In the complex Hilbert space $\mathbb{C}^2$, let

$$|v_0\rangle=\begin{pmatrix}\sqrt{\frac{2}{3}}\\ 0\end{pmatrix},\quad |v_1\rangle=\begin{pmatrix}-\frac{1}{\sqrt6}\\ \frac{1}{\sqrt2}\end{pmatrix},\quad |v_2\rangle=\begin{pmatrix}-\frac{1}{\sqrt6}\\ -\frac{1}{\sqrt2}\end{pmatrix}.$$

B10 Compute $|v_0\rangle\langle v_0|+|v_1\rangle\langle v_1|+|v_2\rangle\langle v_2|$.

B11 Prove that

$$\left\{|w_0\rangle=\begin{pmatrix}\sqrt{\frac{2}{3}}\\ -\frac{1}{\sqrt6}\\ -\frac{1}{\sqrt6}\end{pmatrix},\quad |w_1\rangle=\begin{pmatrix}0\\ \frac{1}{\sqrt2}\\ -\frac{1}{\sqrt2}\end{pmatrix}\right\}$$

is an orthonormal basis in $\mathcal{H}=\mathrm{span}\{|w_0\rangle,\ |w_1\rangle\}\subset\mathbb{C}^3$.

B12 In $\mathbb{C}^3$, find the orthog. projector corresponding to $\mathcal{H}$.

Exercise 5

Let $\mathcal{H}$ be a complex Hilbert space $\mathcal{H}$ and $P:\mathcal{H}\to\mathcal{H}$ satisfying $\quad P^\dagger=P\quad$ and $\quad P^2=P$.

B13 Prove that eigenvalues of P belong to $\{0,\ 1\}$.

B14 Prove that there exists a subspace $\mathcal{K}\subset\mathcal{H}$ such that P is the orthogonal projector corresponding to $\mathcal{K}$.

B15 Compute $\langle A,U\rangle$ for A and U from the Ex.2 and Ex.3.

5 **Definition** In a Hilbert space $\mathcal{H}$:

An operator $A:\mathcal{H}\to\mathcal{H}$ is a **self-adjoint operator** if $\boxed{\langle Ax,y\rangle=\langle x,Ay\rangle}$ $\forall x,y\in\mathcal{H}$.

6 **Definition**. In a complex Hilbert space $\mathcal{H}$:

An operator $A:\mathcal{H}\to\mathcal{H}$ is a **unitary operator** if $\boxed{\langle Ax,Ay\rangle=\langle x,y\rangle}$ $\forall x,y\in\mathcal{H}$.

7 **Theorem** In a complex Hilbert space $\mathcal{H}$:

$A:\mathcal{H}\to\mathcal{H}$ is *self-adjoint* $\Leftrightarrow$ its matrix in an orthonormal basis is Hermitian ($A^\dagger=A$).

$A:\mathcal{H}\to\mathcal{H}$ is *unitary* $\Leftrightarrow$ its matrix in an orthonormal basis is unitary ($A^\dagger A=\mathbb{I}$).

SOLUTIONS for **PARTIAL EVALUATION A**

$\boxed{A1}$ $\mathcal{V}_1=\{(x_1,x_2,2x_1+x_2)|x_1,x_2\in\mathbb{R}$
$=\{x_1(1,0,2)+x_2(0,1,1)|x_1,x_2\in\mathbb{R}\}=\mathrm{span}\{(1,0,2),(0,1,1)\}$.
Since $(1,0,2)$ and $(0,1,1)$ are lin. independent,
it follows that $\dim\mathcal{V}=2$ (a plane in $\mathbb{R}^3$).

$\boxed{A2}$ $\mathcal{V}_2=\{(x_1,-\mathrm{i}x_1,\mathrm{i}x_1)|x_1\in\mathbb{R}\}=\mathrm{span}\{(1,-\mathrm{i},\mathrm{i})\}$

$$\Downarrow$$

$\dim\mathcal{V}_2=1$ (a line in $\mathbb{R}^3$).

$\boxed{A3}$ $(0,1+\mathrm{i},1-\mathrm{i})=(1,1,1)+\mathrm{i}(\mathrm{i},1,-1)$

$$\Downarrow$$

$\{(1,1,1),(\mathrm{i},1,-1)\}$ basis $\Rightarrow\dim\mathcal{V}_3=2$.

$\boxed{A4}$
$$\left.\begin{array}{l}(1,1,1)\notin\mathcal{V}_4\ \Rightarrow\ \dim\mathcal{V}_4<2,\\ (\mathrm{i},1,-1)\in\mathcal{V}_4\ \Rightarrow\ \dim\mathcal{V}_4\geq 1\end{array}\right\}\Rightarrow\dim\mathcal{V}_4=1.$$

$\boxed{A5}$ $v=(1,2,3)=1\,e_1+2\,e_2+3\,e_3\ \Rightarrow$ coordinates are $(1,2,3)$.

$\boxed{A6}$ $v=(1,2,3)=x_1\,(1,1,1)+x_2\,(0,1,1)+x_3\,(0,0,1)$

$$\Downarrow$$
$$(1,2,3)=(x_1,x_1+x_2,x_1+x_2+x_3)$$
$$\Downarrow$$
$$x_1=1,\quad x_2=1,\quad x_3=1.$$

$\boxed{A7}$
$$\left\{\begin{array}{l}v_1'=\alpha_1^1\,v_1+\alpha_1^2\,v_2,\\ v_2'=\alpha_2^1\,v_1+\alpha_2^2\,v_2,\end{array}\right.\ \Rightarrow\ \left\{\begin{array}{l}(1,\mathrm{i})=\alpha_1^1\,(1,1)+\alpha_1^2\,(1,-1),\\ (1,-\mathrm{i})=\alpha_2^1\,(1,1)+\alpha_2^2\,(1,-1),\end{array}\right.$$
$$\Rightarrow\begin{array}{l}\left\{\begin{array}{l}\alpha_1^1+\alpha_1^2=1,\\ \alpha_1^1-\alpha_1^2=\mathrm{i}\end{array}\right.\\ \left\{\begin{array}{l}\alpha_2^1+\alpha_2^2=1,\\ \alpha_2^1-\alpha_2^2=-\mathrm{i}\end{array}\right.\end{array}\Rightarrow S=\begin{pmatrix}\alpha_1^1&\alpha_1^2\\ \alpha_2^1&\alpha_2^2\end{pmatrix}=\begin{pmatrix}\frac{1+\mathrm{i}}{2}&\frac{1-\mathrm{i}}{2}\\ \frac{1-\mathrm{i}}{2}&\frac{1+\mathrm{i}}{2}\end{pmatrix}$$
is the change of basis matrix.

$\boxed{A8}$
$$\left.\begin{array}{l}A,\,B\in\mathcal{W}\\ \alpha,\,\beta\in\mathbb{R}\end{array}\right\}\Rightarrow(\alpha A+\beta B)^\dagger=\alpha A+\beta B\Rightarrow\alpha A+\beta B\in\mathcal{W}.$$
$$\mathcal{W}=\left\{A=\begin{pmatrix}a_{11}&a_{12}\\ a_{21}&a_{22}\end{pmatrix}\ \middle|\ \begin{pmatrix}\bar a_{11}&\bar a_{21}\\ \bar a_{12}&\bar a_{22}\end{pmatrix}=\begin{pmatrix}a_{11}&a_{12}\\ a_{21}&a_{22}\end{pmatrix}\right\}$$
$$=\left\{\begin{pmatrix}a_{11}&a_{12}\\ a_{21}&a_{22}\end{pmatrix}\ \middle|\ \begin{array}{l}\bar a_{11}=a_{11}\\ \bar a_{21}=a_{12}\\ \bar a_{22}=a_{22}\end{array}\right\}=\left\{\begin{pmatrix}a_{11}&a_{12}\\ a_{21}&a_{22}\end{pmatrix}\ \middle|\ \begin{array}{l}\bar a_{11}\in\mathbb{R}\\ \bar a_{21}=a_{12}\\ \bar a_{22}\in\mathbb{R}\end{array}\right\}.$$
By denoting $a_{11}=\alpha$, $a_{12}=\beta+\gamma\mathrm{i}$, $a_{22}=\delta$, we get
$$\mathcal{W}=\left\{A=\begin{pmatrix}\alpha&\beta+\gamma\mathrm{i}\\ \beta-\gamma\mathrm{i}&\delta\end{pmatrix}\ \middle|\ \alpha,\,\beta,\,\gamma\in\mathbb{R}\right\}$$
$$=\left\{\alpha\begin{pmatrix}1&0\\0&0\end{pmatrix}+\beta\begin{pmatrix}0&1\\1&0\end{pmatrix}+\gamma\begin{pmatrix}0&\mathrm{i}\\-\mathrm{i}&0\end{pmatrix}+\delta\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$$
$$\Rightarrow\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}0&\mathrm{i}\\-\mathrm{i}&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$$
is a basis in the real space $\mathcal{W}$ $\Rightarrow$ $\dim_{\mathbb{R}}\mathcal{W}=4$.

$\boxed{A9}$ In the canonical basis $\{(1,0),(0,1)\}$, the matrix of A is
$$A=\begin{pmatrix}1&1\\1&-1\end{pmatrix}.\qquad\begin{vmatrix}1-\lambda&1\\1&-1-\lambda\end{vmatrix}=0\Rightarrow\begin{array}{l}\lambda_1=\sqrt{2},\\ \lambda_2=-\sqrt{2}.\end{array}$$

$\boxed{A10}$
$\{\,(x_1,x_2)\mid A(x_1,x_2)=\sqrt{2}\,(x_1,x_2)\,\}=\mathrm{span}\{(1,-1+\sqrt{2})\}$
$\{\,(x_1,x_2)\mid A(x_1,x_2)=-\sqrt{2}\,(x_1,x_2)\,\}=\mathrm{span}\{(1,-1-\sqrt{2})\}$.

$\boxed{A11}$ $\mathrm{Tr}\,A=1-1=\lambda_1+\lambda_2=0$,
$\det A=1\,.\,(-1)-1\,.\,1=\lambda_1\,.\,\lambda_2=-2$.

$\boxed{A12}$
In the canonical basis $\{(1,0,0),(0,1,0),(0,0,1)\}$, matrix of A is
$$A=\begin{pmatrix}0&1&0\\1&0&0\\0&0&2\end{pmatrix}.\qquad\begin{vmatrix}-\lambda&1&0\\1&-\lambda&0\\0&0&2-\lambda\end{vmatrix}=0\Rightarrow\begin{array}{l}\lambda_1=-1,\\ \lambda_2=1,\\ \lambda_3=2.\end{array}$$
This shows that all the roots of the characteristic polynomial belong to $\mathbb{R}$, and for each of them, the algebraic and geometric multiplicities are equal (to 1). Therefore A is diagonalizable. More exactly,
$$S^{-1}AS=\begin{pmatrix}-1&0&0\\0&1&0\\0&0&2\end{pmatrix},\quad\text{where}\ S=\begin{pmatrix}1&1&0\\-1&1&0\\0&0&1\end{pmatrix}.$$

SOLUTIONS for **PARTIAL EVALUATION B**

$\boxed{B1}$ $\langle(1,2,-1),(1,2,3)\rangle=1\,.\,1+2\,.\,2+(-1)\,.\,3=2$

$\boxed{B2}$ $\mathcal{P}_w x=\frac{\langle w,x\rangle}{\langle w,w\rangle}w=\frac{\langle(1,2,-1),(1,2,3)\rangle}{\langle(1,2,-1),(1,2,-1)\rangle}(1,2,-1)$
$=\frac{2}{6}(1,2,-1)=\left(\frac{1}{3},\frac{2}{3},-\frac{1}{3}\right)$.

$\boxed{B3}$
$\mathcal{H}=\{(x_1,x_2,x_3)\mid x_1-x_3=0\}=\{(x_1,x_2,x_1)\mid x_1,x_2\in\mathbb{R}\}$
$=\{x_1(1,0,1)+x_2(0,1,0)\mid x_1,x_2\in\mathbb{R}\}=\mathrm{span}\{(1,0,1),(0,1,0)\}$.
Since $\langle(1,0,1),(0,1,0)\rangle=0$, it follows that
$$\left\{\frac{(1,0,1)}{||(1,0,1)||},\frac{(0,1,0)}{||(0,1,0)||}\right\}=\left\{\frac{(1,0,1)}{\sqrt{2}},(0,1,0),\right\}$$
$$=\left\{\left(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right),(0,1,0),\right\}\ \text{is an orthonormal basis.}$$

$\boxed{B4}$
The matrix of A in the orthonormal basis $\{(1,0),(0,1)\}$, namely
$A=\begin{pmatrix}0&1+\mathrm{i}\\1-\mathrm{i}&0\end{pmatrix}$ is a Hermitian matrix because $A^\dagger=A$.

$\boxed{B5}$ $\begin{vmatrix}-\lambda&1+\mathrm{i}\\1-\mathrm{i}&-\lambda\end{vmatrix}=0\ \Rightarrow\ \begin{array}{l}\lambda_1=\sqrt{2}\\ \lambda_2=-\sqrt{2}.\end{array}$

$\boxed{B6}$ $\{\,(x_1,x_2)\mid A(x_1,x_2)=\sqrt{2}\,(x_1,x_2)\,\}=\mathrm{span}\{(1,\frac{1-\mathrm{i}}{\sqrt{2}})\}$
$\{\,(x_1,x_2)\mid A(x_1,x_2)=-\sqrt{2}\,(x_1,x_2)\,\}=\mathrm{span}\{(1,\frac{-1+\mathrm{i}}{\sqrt{2}})\}$.
In the basis $\{(1,\frac{1-\mathrm{i}}{\sqrt{2}}),(1,\frac{-1+\mathrm{i}}{\sqrt{2}})\}$ the matrix of A is $\begin{pmatrix}\sqrt{2}&0\\0&-\sqrt{2}\end{pmatrix}$.

$\boxed{B7}$
The matrix of U in the orthonormal basis $\{(1,0),(0,1)\}$, namely
$U=\begin{pmatrix}\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{pmatrix}$ is a unitary matrix because $U^\dagger U=\begin{pmatrix}1&0\\0&1\end{pmatrix}$.

$\boxed{B8}$ $\begin{vmatrix}\frac{1}{\sqrt{2}}-\lambda&-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}-\lambda\end{vmatrix}=0\ \Rightarrow\ \begin{array}{l}\lambda_1=\frac{1+\mathrm{i}}{\sqrt{2}}\\ \lambda_2=\frac{1-\mathrm{i}}{\sqrt{2}}.\end{array}$

$\boxed{B9}$ $\{\,(x_1,x_2)\mid U(x_1,x_2)=\frac{1+\mathrm{i}}{\sqrt{2}}\,(x_1,x_2)\,\}=\mathrm{span}\{(1,-\mathrm{i})\}$
$\{\,(x_1,x_2)\mid U(x_1,x_2)=\frac{1-\mathrm{i}}{\sqrt{2}}\,(x_1,x_2)\,\}=\mathrm{span}\{(1,\mathrm{i})\}$.
In the basis $\{(1,-\mathrm{i}),(1,\mathrm{i})\}$ the matrix of U is $\begin{pmatrix}\frac{1+\mathrm{i}}{\sqrt{2}}&0\\0&\frac{1-\mathrm{i}}{\sqrt{2}}\end{pmatrix}$.

$\boxed{B10}$
$$|v_0\rangle\langle v_0|+|v_1\rangle\langle v_1|+|v_2\rangle\langle v_2|=\begin{pmatrix}\sqrt{\frac{2}{3}}\\0\end{pmatrix}\begin{pmatrix}\sqrt{\frac{2}{3}}&0\end{pmatrix}$$
$$+\begin{pmatrix}-\frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}-\frac{1}{\sqrt{6}}&\frac{1}{\sqrt{2}}\end{pmatrix}+\begin{pmatrix}-\frac{1}{\sqrt{6}}\\ -\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}-\frac{1}{\sqrt{6}}&-\frac{1}{\sqrt{2}}\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

$\boxed{B11}$ We have $\langle w_0,w_0\rangle=1$, $\langle w_1,w_1\rangle=1$, $\langle w_0,w_1\rangle=0$.

$\boxed{B12}$ $|w_0\rangle\langle w_0|+|w_1\rangle\langle w_1|=\begin{pmatrix}\frac{2}{3}&-\frac{1}{3}&-\frac{1}{3}\\ -\frac{1}{3}&\frac{2}{3}&-\frac{1}{3}\\ -\frac{1}{3}&-\frac{1}{3}&\frac{2}{3}\end{pmatrix}$.

$\boxed{B13}$ If λ is an eigenvalue of P, then there exists $x\neq0$ such that $Px=\lambda x$. Consequently, $P^2x=\lambda Px=\lambda^2x$.
But $P^2=P\ \Rightarrow\ \lambda^2x=\lambda x\ \Rightarrow\ (\lambda^2-\lambda)x=0$
$\Rightarrow\ \lambda^2-\lambda=0\ \Rightarrow\ \lambda=0$ or $\lambda=1$.

$\boxed{B14}$ $\left.\begin{array}{l}Px\in\mathrm{Im}\,P\\ y\in\mathrm{Ker}\,P\end{array}\right\}\Rightarrow\ \langle Px,y\rangle=\langle x,Py\rangle=\langle x,0\rangle=0.$
Therefore $\mathrm{Im}\,P\perp\mathrm{Ker}\,P$.
Since $P(x-Px)=Px-P^2x=Px-Px=0$
any x admits the decomposition $x=Px+(x-Px)$
with $\left\{\begin{array}{l}Px\in\mathrm{Im}\,P\\ x-Px\in\mathrm{Ker}\,P.\end{array}\right.$ Therefore, $\mathcal{H}\to\mathcal{H}:x\mapsto Px$
is the orthogonal projector corresponding to $\mathcal{K}=\mathrm{Im}\,P$.

$\boxed{B15}$ $\langle A,U\rangle=\left\langle\begin{pmatrix}0&1+\mathrm{i}\\1-\mathrm{i}&0\end{pmatrix},\begin{pmatrix}\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{pmatrix}\right\rangle$
$=\overline{1+\mathrm{i}}\,(-\frac{1}{\sqrt{2}})+\overline{1-\mathrm{i}}\,\frac{1}{\sqrt{2}}=\mathrm{i}\sqrt{2}$.