ON A DISCRETE VERSION OF THE WEYL-WIGNER DESCRIPTION

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Abstract. This is a tutorial[1] on a discrete version of the Weyl-Wigner description of quantum systems. The definitions and results, presented in a very concise way, are followed by proofs in detail. An extension to composite systems and a discrete version of some results concerning the reduced density matrix, recently presented by Gosson [arXiv:2106.14056], are also included.

A A WEYL-WIGNER DESCRIPTION IN THE ODD-DIMENSIONAL CASE ($d=2s+1$)

■ **Definition**. Alternative representations of the Hilbert space $\mathcal{H}$ of dimension d:

- $\mathcal{H}\equiv\mathbb{C}^d=\{\ x=(x_0,x_1,...,x_{d-1})\ |\ x_k\in\mathbb{C}\ \}$
- $\mathcal{H}\equiv\{\psi:\{0,1,...,d-1\}\to\mathbb{C}\ |\ \psi \text{ is a function}\}$
- $\mathcal{H}\equiv\{\ \psi:\mathbb{Z}_d\to\mathbb{C}\ |\ \psi \text{ is a function}\ \}$
- $\mathcal{H}\equiv\{\psi:\{-s,-s+1,...,s\}\to\mathbb{C}\ |\ \psi \text{ is a function}\}$

1 $$\langle\varphi|\psi\rangle=\sum_{n=-s}^{s}\overline{\varphi(n)}\,\psi(n)\equiv\sum_{n\in\mathbb{Z}_d}\overline{\varphi(n)}\,\psi(n).$$

2 The *canonical orthonormal basis* is formed by $\delta_{-s},\delta_{-s+1},...,\delta_s:\{-s,-s+1,...,s-1,s\}\to\mathbb{C}$,
$$\delta_m(n)=\delta_{nm}=\begin{cases}1 & \text{for} \quad n=m,\\ 0 & \text{for} \quad n\neq m.\end{cases}$$
By using the notation $|m\rangle=|\delta_m\rangle$, we have
$$\boxed{\langle j|m\rangle=\delta_{jm}\quad,\quad\sum_{m=-s}^{s}|m\rangle\langle m|=\mathbb{I}}.$$
Alternatively, we can regard it as the set $\{\ \delta_m:\mathbb{Z}_d\to\mathbb{C}\ |\ m\in\mathbb{Z}_d\ \}$, where
$$\delta_m(n)=\delta_{nm}^{[d]}=\begin{cases}1 & \text{for} \quad n=m \bmod(d),\\ 0 & \text{for} \quad n\neq m \bmod(d).\end{cases}$$

3 *Finite Fourier transform* $F:\mathcal{H}\to\mathcal{H}$ is
$$\begin{array}{c}\psi:\mathbb{Z}_d\longrightarrow\mathbb{C}\\ \downarrow\\ F[\psi]:\mathbb{Z}_d\longrightarrow\mathbb{C}\end{array}\qquad \boxed{F[\psi](k)=\frac{1}{\sqrt{d}}\sum_{n\in\mathbb{Z}_d}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}kn}\,\psi(n)}$$

4 A *density operator* is a Hermitian operator $\varrho:\mathcal{H}\to\mathcal{H}$ satisfying $\boxed{\varrho\geq 0,\ \mathrm{tr}\,\varrho=1.}$

5 *Mean value of an observable* A in a state $\varrho:\mathcal{H}\to\mathcal{H}$ $\quad\boxed{\langle A\rangle_\varrho=\mathrm{tr}(\varrho A)}$

6 *Position* operator $Q:\mathcal{H}\to\mathcal{H}$, $\boxed{Q\psi(n)=\tilde{n}\psi(n)}$

7 *Momentum* operator $P:\mathcal{H}\to\mathcal{H}$, $\boxed{P=F^\dagger QF}$

where $\tilde{n}$ is the representative modulo d of n satisfying $-s\leq\tilde{n}\leq s$.

8 *Translation* operators $(n,k\in\mathbb{Z}_d)$ $\quad\mathcal{H}\to\mathcal{H}$, $\quad\boxed{\begin{array}{l}\psi\mapsto\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}\psi\\ \psi\mapsto\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}\psi\end{array}}$

9 *Displacement* operators $(-s\leq n,k\leq s)$ $\quad\boxed{D(n,k)=\mathrm{e}^{-\frac{\pi\mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}}$

10 *Parity operators* $(n,k\in\mathbb{Z}_d)$ $\quad\boxed{\Pi(n,k)=D(n,k)\,\Pi\,D(n,k)^\dagger}$ where $\Pi\psi(n)=\psi(-n)$.

11 *Weyl transform* of a linear operator $A:\mathcal{H}\to\mathcal{H}$ is $\tilde{A}:\mathbb{Z}_d\times\mathbb{Z}_d\longrightarrow\mathbb{C}$, $\quad\boxed{\tilde{A}(n,k)\overset{\text{def}}{=}\mathrm{tr}(A\,\Pi(n,k))}$

12 *Wigner function* of a linear operator $A:\mathcal{H}\to\mathcal{H}$ is $\mathcal{W}_A:\mathbb{Z}_d\times\mathbb{Z}_d\longrightarrow\mathbb{C}$, $\ \mathcal{W}_A=\frac{1}{d}\,\tilde{A}$, $\quad\boxed{\mathcal{W}_A(n,k)\overset{\text{def}}{=}\frac{1}{d}\,\mathrm{tr}(A\,\Pi(n,k))}$

[1] Version 20 Feb. 2022 (for future updates see https://unibuc.ro/user/nicolae.cotfas/)

■ **Remark.**

13 The definition of $D(n,k)$ is not modulo d invariant:
$$D(n+d,k)=(-1)^k\,D(n,k),$$
$$D(n,k+d)=(-1)^n\,D(n,k),$$
but, the definition of $\Pi(n,k)$ is modulo d invariant.

■ **Lemma.** For odd $d=2s+1$, we have:

14 $$\boxed{\begin{array}{c}2n=0\ (\mathrm{mod}\,d)\\ \Downarrow\\ n=0\ (\mathrm{mod}\,d)\end{array}}\qquad\boxed{\begin{array}{c}\mathbb{Z}_d\times\mathbb{Z}_d\to\mathbb{Z}_d\times\mathbb{Z}_d\\ (n,m)\mapsto(n+m,n-m)\\ \text{is a one-to-one map}\end{array}}$$

15 $$\boxed{\frac{1}{d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{\pm\frac{2\pi\mathrm{i}}{d}km}=\delta_0(k)}\qquad\boxed{\frac{1}{d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{\pm\frac{4\pi\mathrm{i}}{d}km}=\delta_0(k)}$$

■ **Theorem.**

16 Finite Fourier transform is a unitary transform:
$$\boxed{F^{-1}=F^\dagger}\qquad\boxed{F^\dagger[\psi](k)=\frac{1}{\sqrt{d}}\sum_{n\in\mathbb{Z}_d}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kn}\,\psi(n)}$$

17 $$\boxed{P\psi(n)=\frac{1}{d}\sum_{k,m}\tilde{k}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k(n-m)}\psi(m)}$$

18 $$\boxed{\begin{array}{l}Q^\dagger=Q\\ P^\dagger=P\end{array}}\qquad\boxed{\begin{array}{l}\langle m|Q|j\rangle=\tilde{m}\,\delta_{mj}\\ \langle m|P|j\rangle=\frac{1}{d}\sum_k\tilde{k}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k(m-j)}\end{array}}$$

19 $$\boxed{\begin{array}{l}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}\psi(n)=\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kn}\psi(n),\\ \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}\psi(m)=\psi(m-n)\end{array}}\qquad\boxed{\begin{array}{l}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}|m\rangle=\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}km}|m\rangle,\\ \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}|m\rangle=|m+n\rangle\end{array}}$$

20 $$\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}=\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}nk}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ},$$

21 $$\begin{aligned}D(n,k)&\equiv\mathrm{e}^{-\frac{\pi\mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}\\ &=\mathrm{e}^{\frac{\pi\mathrm{i}}{d}nk}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}\end{aligned}$$

22 $$\boxed{\begin{array}{l}D(n,k)\psi(m)=\mathrm{e}^{-\frac{\pi\mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}km}\,\psi(m-n)\\ \Pi(n,k)\psi(m)=\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}2k(n-m)}\,\psi(2n-m)\end{array}}$$

23 $$\boxed{\begin{array}{l}\langle m|D(n,k)|j\rangle=\mathrm{e}^{\frac{\pi\mathrm{i}}{d}k(n+2j)}\,\delta_m(j+n)\\ \langle m|\Pi(n,k)|j\rangle=\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k(m-j)}\,\delta_{2n}(m+j)\end{array}}$$

24 $$D(n,k)\,D(m,j)=\mathrm{e}^{-\frac{\pi\mathrm{i}}{d}(nj-km)}D(\widetilde{n+m},\widetilde{k+j}).$$

25 $$D(n,k)^\dagger=D(-n,-k)=D^{-1}(n,k),$$

26 $\mathcal{L}(\mathcal{H})\equiv\left\{A:\mathcal{H}\to\mathcal{H}\,\middle|\,\begin{array}{c}\text{linear}\\ \text{operator}\end{array}\right\}$ with $\langle A,B\rangle=\mathrm{tr}(A^\dagger B)$ is a d^2-dimensional complex Hilbert space.

27 $\left\{\frac{1}{\sqrt{d}}D(n,k)\right\}$, $\left\{\frac{1}{\sqrt{d}}\Pi(n,k)\right\}$ are orthonormal bases in $\mathcal{L}(\mathcal{H})$.

28 $$A\in\mathcal{L}(\mathcal{H})\ \Rightarrow\ \boxed{\begin{array}{l}\tilde{A}(n,k)=\langle\Pi(n,k),A\rangle\\ \mathcal{W}_A(n,k)=\frac{1}{d}\langle\Pi(n,k),A\rangle\end{array}}$$

29 $$\boxed{\begin{array}{l}\langle A^\dagger,B\rangle=\overline{\langle A,B^\dagger\rangle}\\ \Pi(n,k)^\dagger=\Pi(n,k)\end{array}}\qquad\boxed{\begin{array}{l}\widetilde{A^\dagger}(n,k)=\overline{\tilde{A}(n,k)}\\ \mathcal{W}_{A^\dagger}(n,k)=\overline{\mathcal{W}_A(n,k)}\end{array}}$$

30 $A\in\mathcal{L}(\mathcal{H}) \;\Rightarrow\; \boxed{A=\sum_{n,k}\mathcal{W}_A(n,k)\,\Pi(n,k)}$

31 $A\in\mathcal{L}(\mathcal{H}) \Rightarrow \boxed{\mathcal{W}_A(n,k)=\frac{1}{d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\,\langle n+m|A|n-m\rangle}$

32 Pure state $\psi\in\mathcal{H}$ $\Rightarrow \boxed{\mathcal{W}_\psi(n,k)=\frac{1}{d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\,\psi(n+m)\,\overline{\psi(n-m)}}$

33 $A:\mathcal{H}\to\mathcal{H}$ linear operator $\Rightarrow \boxed{\sum_{n,k\in\mathbb{Z}_d}\mathcal{W}_A(n,k)=\mathrm{tr}\,A}$

34 $A,B\in\mathcal{L}(\mathcal{H}) \Rightarrow$
$$\boxed{\begin{aligned}\mathrm{tr}(AB)&=\frac{1}{d}\sum_{n,k}\tilde{A}(n,k)\,\tilde{B}(n,k)\\&=\sum_{n,k}\tilde{A}(n,k)\,\mathcal{W}_B(n,k)\\&=d\sum_{n,k}\mathcal{W}_A(n,k)\,\mathcal{W}_B(n,k)\end{aligned}}$$

35 $\varrho_\psi=|\psi\rangle\langle\psi|$, $\varrho_\varphi=|\varphi\rangle\langle\varphi|$ $\Rightarrow \boxed{\mathrm{tr}(\varrho_\psi\varrho_\varphi)=d\sum_{n,k}\mathcal{W}_\psi(n,k)\,\mathcal{W}_\varphi(n,k)=|\langle\psi|\varphi\rangle|^2}$

36 $A:\mathcal{H}\to\mathcal{H}$, $\varrho=|\psi\rangle\langle\psi|$ $\Rightarrow \boxed{\langle A\rangle_\varrho\equiv\mathrm{tr}(\,\varrho\,A)=\langle\psi|\,A|\psi\rangle.}$

37 $\boxed{\tilde{\mathbb{I}}(n,k)=1}$ $\boxed{\begin{array}{c}\tilde{Q}(n,k)=\tilde{n}\\ \tilde{P}(n,k)=\tilde{k}\end{array}}$ $\boxed{\begin{array}{c}\langle Q\rangle_\varrho=\sum\tilde{n}\,\mathcal{W}_\varrho(n,k)\\ \langle P\rangle_\varrho=\sum\tilde{k}\,\mathcal{W}_\varrho(n,k)\end{array}}$

38 $\mathcal{A}(\mathcal{H})\equiv\{A\in\mathcal{L}(\mathcal{H})\ |\ A^\dagger=A\}$ with $\langle A,B\rangle=\mathrm{tr}(AB)$ is a d^2-dimensional real Hilbert space.

39 $\left\{\frac{1}{\sqrt{d}}\Pi(n,k)\right\}$ is an orthonormal basis in $\mathcal{A}(\mathcal{H})$.

40 $A\in\mathcal{A}(\mathcal{H}) \;\Rightarrow\;$ $\tilde{A}:\mathbb{Z}_d\times\mathbb{Z}_d\longrightarrow\mathbb{R}$ is a real function, $\mathcal{W}_A:\mathbb{Z}_d\times\mathbb{Z}_d\longrightarrow\mathbb{R}$ is a real function.

41 Pure state $\psi\in\mathcal{H}$ $\Rightarrow \boxed{\sum_{k\in\mathbb{Z}_d}\mathcal{W}_\psi(n,k)=|\psi(n)|^2}$

42 Pure state $\psi\in\mathcal{H}$ $\Rightarrow \boxed{\sum_{n\in\mathbb{Z}_d}\mathcal{W}_\psi(n,k)=|F[\psi](k)|^2}$

43 $\varphi(n)=\psi(n-a) \;\Rightarrow\; \boxed{\mathcal{W}_\varphi(n,k)=\mathcal{W}_\psi(n-a,k)}$

44 $\varphi(n)=\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}bn}\,\psi(n) \;\Rightarrow\; \boxed{\mathcal{W}_\varphi(n,k)=\mathcal{W}_\psi(n,k-b)}$

45 Pure state $\psi\in\mathcal{H}$ $\Rightarrow \boxed{\sum_{n,k\in\mathbb{Z}_d}\mathcal{W}_\psi(n,k)^2=\frac{1}{d}}$

46 Pure state $\psi\in\mathcal{H}$ $\Rightarrow \boxed{|\mathcal{W}_\psi(n,k)|\le\frac{1}{d},\quad \text{for any } n,k}$

47 Even state $\psi(-n)=\psi(n)$ $\Rightarrow\; \mathcal{W}_\psi(0,0)|=\frac{1}{d}$.

48 Odd state $\psi(-n)=-\psi(n)$ $\Rightarrow\; \mathcal{W}_\psi(0,0)|=-\frac{1}{d}$.

■ **Remark.**
The corresponding *phase space* is
$\mathfrak{S}=\{-s,-s+1,...,s-1,s\}\times\{-s,-s+1,...,s-1,s\}$.
It is not $\mathbb{Z}_d\times\mathbb{Z}_d$ (see, for example, the relations 37).

B DISCRETE WIGNER FUNCTION OF THE REDUCED DENSITY MATRIX

■ **Definition** (in a particular representaton).

1 The *tensor product* of the Hilbert spaces
$\mathcal{H}_1=\{\ \psi:\mathbb{Z}_{d_1}\to\mathbb{C}\ |\ \psi$ is a function $\}$ of dim $d_1=2s_1+1$
$\mathcal{H}_2=\{\ \psi:\mathbb{Z}_{d_2}\to\mathbb{C}\ |\ \psi$ is a function $\}$ of dim $d_2=2s_2+1$
is the Hilbert space
$$\boxed{\mathcal{H}_1\otimes\mathcal{H}_2=\{\ \Psi:\mathbb{Z}_{d_1}\times\mathbb{Z}_{d_2}\to\mathbb{C}\ |\ \Psi \text{ is a function }\}}$$
where

2 $\langle\Psi,\Phi\rangle=\sum_{n\in\mathbb{Z}_{d_1}}\sum_{m\in\mathbb{Z}_{d_2}}\overline{\Psi(n,m)}\;\Phi(n,m)$ and

3 $(\psi\otimes\varphi)(n,m)=\psi(n)\,\varphi(m)$ for any $\psi\in\mathcal{H}_1$, $\varphi\in\mathcal{H}_2$

4 The *canonical basis* of $\mathcal{H}=\mathcal{H}_1\otimes\mathcal{H}_2$ is
$\{\ |nm\rangle\equiv|n\rangle_1\otimes|m\rangle_2\ \ |\ \ n\in\mathbb{Z}_{d_1},\ m\in\mathbb{Z}_{d_2}\}$
where $\{\ |n\rangle_1\ |\ \ n\in\mathbb{Z}_{d_1}\}$ = canonical basis of $\mathcal{H}_1$
$\{\ |m\rangle_2\ |\ \ m\in\mathbb{Z}_{d_2}\}$ = canonical basis of $\mathcal{H}_2$.

5 *Parity operators* $(n_1,k_1\in\mathbb{Z}_{d_1},\ n_2,k_2\in\mathbb{Z}_{d_2})$
$$\boxed{\Pi(n_1,n_2,k_1,k_2)=\Pi(n_1,k_1)\otimes\Pi(n_2,k_2)}$$

6 *Weyl transform* of a linear operator $A:\mathcal{H}\to\mathcal{H}$ is
$\tilde{A}:\mathbb{Z}_{d_1}\times\mathbb{Z}_{d_2}\times\mathbb{Z}_{d_1}\times\mathbb{Z}_{d_2}\longrightarrow\mathbb{C}$,
$$\boxed{\tilde{A}(n_1,n_2,k_1,k_2)=\mathrm{tr}(A\,\Pi(n_1,n_2,k_1,k_2))}$$

7 *Wigner function* of a linear operator $A:\mathcal{H}\to\mathcal{H}$ is
$\mathcal{W}_A:\mathbb{Z}_{d_1}\times\mathbb{Z}_{d_2}\times\mathbb{Z}_{d_1}\times\mathbb{Z}_{d_2}\longrightarrow\mathbb{C}$,
$$\boxed{\mathcal{W}_A(n_1,n_2,k_1,k_2)=\frac{1}{d_1d_2}\mathrm{tr}(A\,\Pi(n_1,n_2,k_1,k_2))}$$

■ **Lemma.**

8 $\mathrm{tr}(A\otimes B)=\mathrm{tr}A\ \mathrm{tr}B$,

9 $(A\otimes B)^\dagger=A^\dagger\otimes B^\dagger$.

■ **Theorem.**

10 $\left\{\frac{1}{\sqrt{d_1d_2}}\,\Pi(n_1,n_2,k_1,k_2)\right\}$ is an orthonormal basis in $\mathcal{A}(\mathcal{H}_1\otimes\mathcal{H}_2)$ and $\mathcal{L}(\mathcal{H}_1\otimes\mathcal{H}_2)$.

11 $A\in\mathcal{L}(\mathcal{H})\Rightarrow\boxed{\mathcal{W}_A(n_1,n_2,k_1,k_2)=\frac{1}{d_1d_2}\langle\Pi(n_1,n_2,k_1,k_2),A\rangle}$
$$\Downarrow$$
$$\boxed{A=\sum\mathcal{W}_A(n_1,n_2,k_1,k_2)\,\Pi(n_1,n_2,k_1,k_2)}$$

12 *Wigner function* of $A:\mathcal{H}_1\otimes\mathcal{H}_2\to\mathcal{H}_1\otimes\mathcal{H}_2$ is
$$\boxed{\begin{aligned}\mathcal{W}_A(n_1,n_2,k_1,k_2)=&\frac{1}{d_1d_2}\sum_{m_1\in\mathbb{Z}_{d_1}}\sum_{m_2\in\mathbb{Z}_{d_2}}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_1}k_1m_1}\,\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_2}k_2m_2}\\&\times\langle n_1+m_1,n_2+m_2|A|n_1-m_1,n_2-m_2\rangle\end{aligned}}$$

13 *Wigner function* of a pure state $\Psi\in\mathcal{H}_1\otimes\mathcal{H}_2$ is
$$\boxed{\begin{aligned}\mathcal{W}_\Psi(n_1,n_2,k_1,k_2)=&\frac{1}{d_1d_2}\sum_{m_1\in\mathbb{Z}_{d_1}}\sum_{m_2\in\mathbb{Z}_{d_2}}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_1}k_1m_1}\,\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_2}k_2m_2}\\&\times\Psi(n_1+m_1,n_2+m_2)\,\overline{\Psi(n_1-m_1,n_2-m_2)}\end{aligned}}$$

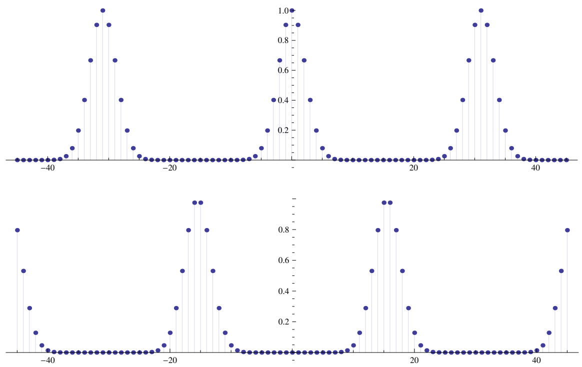

Figure 1: The functions g_1 and g_1^+ in the case $d\!=\!31$.

14 $\begin{array}{l}\psi\in\mathcal{H}_1\\ \varphi\in\mathcal{H}_2\end{array}\Rightarrow \boxed{\mathcal{W}_{\psi\otimes\varphi}(n_1,n_2,k_1,k_2)=\mathcal{W}_\psi(n_1,k_1)\,\mathcal{W}_\varphi(n_2,k_2)}$

15 Partial trace of a pure state (in terms of Schmidt form)

$$\Psi\in\mathcal{H}_1\otimes\mathcal{H}_2,\ \ \varrho=|\Psi\rangle\langle\Psi|$$
$$|\Psi\rangle=\sum_a\sqrt{\lambda_a}\,|\psi_a\rangle\otimes|\varphi_a\rangle$$
$$\Downarrow$$
$$\boxed{\varrho_1=\mathrm{tr}_2\,\varrho=\sum_a\lambda_a\,|\psi_a\rangle\langle\psi_a|}$$

16 Wigner function of a *partial trace*

$$\mathcal{H}_1\otimes\mathcal{H}_2\xrightarrow{\varrho}\mathcal{H}_1\otimes\mathcal{H}_2$$
linear operator
$$\Downarrow$$
$$\boxed{\boxed{\mathcal{W}_{\varrho_1}(n_1,k_1)=\sum_{n_2,k_2\in\mathbb{Z}_{d_2}}\mathcal{W}_\varrho(n_1,n_2,k_1,k_2)}}$$

17 $\boxed{\widetilde{A\otimes B}(n_1,n_2,k_1,k_2)=\tilde{A}(n_1,k_1)\,\tilde{B}(n_2,k_2)}$

18 $\boxed{\mathcal{W}_{A\otimes B}(n_1,n_2,k_1,k_2)=\mathcal{W}_A(n_1,k_1)\,\mathcal{W}_B(n_2,k_2)}$

19 $\boxed{\mathrm{tr}(\varrho(A\otimes\mathbb{I}))=\mathrm{tr}(\varrho_1\,A)}$

20 *Purification* Ψ of a state ϱ and Wigner function of ϱ

$\varrho:\mathcal{H}_1\to\mathcal{H}_1$ density operator

$\varrho=\sum\lambda_j|\varphi_j\rangle\langle\varphi_j|$ (spectral decomposition)

$$\Downarrow$$

$$\mathcal{W}_\varrho(n_1,k_1)=\sum_{n_2,k_2}\mathcal{W}_\Psi(n_1,n_2,k_1,k_2)$$
where
$$|\Psi\rangle=\sum\sqrt{\lambda_j}\,|\varphi_j\rangle\otimes|\varphi_j\rangle$$

C GAUSSIAN FUNCTIONS OF DISCRETE VARIABLE ($\kappa\in(0,\infty)$ is a parameter)

■ Definition.

1 Gaussian function $\quad g_\kappa:\mathbb{Z}_d\to\mathbb{R},\ \boxed{g_\kappa(n)=\sum_{\alpha=-\infty}^{\infty}\mathrm{e}^{-\frac{\kappa\pi}{d}(n+\alpha d)^2}}$

2 Complementary Gaussian function $\quad g_\kappa^+:\mathbb{Z}_d\to\mathbb{R},\ \boxed{g_\kappa^+(n)=\sum_{\alpha=-\infty}^{\infty}\mathrm{e}^{-\frac{\kappa\pi}{d}(n+(\alpha+\frac{1}{2})d)^2}}$

3 Discrete vacuum $\quad \boxed{|0\rangle=\frac{1}{\sqrt{\langle g_1|g_1\rangle}}|g_1\rangle}$

4 Discrete coherent states $\quad \boxed{|n,k\rangle=D(n,k)|0\rangle}$

5 Finite frame quantization $\quad \boxed{f:\mathfrak{S}\to\mathbb{C}}\mapsto \boxed{\begin{array}{l}A_f:\mathcal{H}\to\mathcal{H},\\ A_f=\frac{1}{d}\sum_{(n,k)\in\mathfrak{S}}f(n,k)\,|n,k\rangle\langle n,k|\end{array}}$

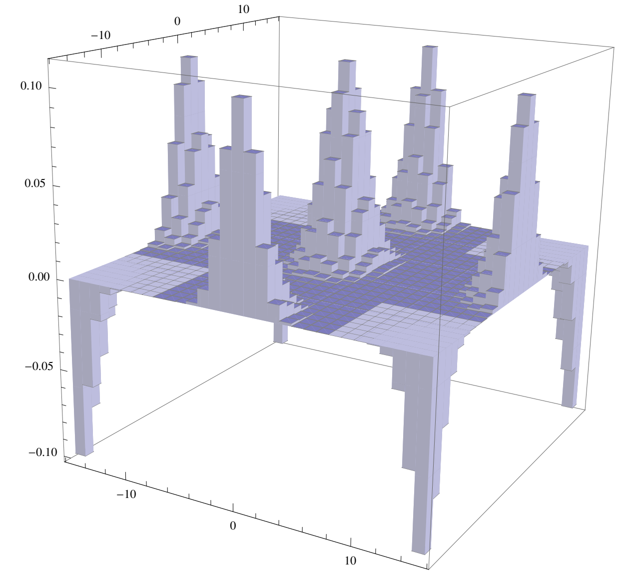

Figure 2: The Wigner function of g_1 in the case $d\!=\!31$.

6 Discrete version of Hermite-Gauss functions

The eigenfunctions of
$$H=\frac{1}{d}\sum_{(n,k)\in\mathfrak{S}}\frac{n^2+k^2}{2}\,|n,k\rangle\langle n,k|$$
considered in the increasing order of the number of sign alternations.

■ Theorem.

7 Resolution of the identity $\quad \boxed{\mathbb{I}=\frac{1}{d}\sum_{(n,k)\in\mathfrak{S}}|n,k\rangle\langle n,k|}$

Fourier transform

8 $\boxed{F[g_\kappa]=\frac{1}{\sqrt{\kappa}}\,g_{\frac{1}{\kappa}}}$

9 $\boxed{F[g_{2\kappa}^+](2k)=\frac{1}{\sqrt{2\kappa}}\left(g_{\frac{2}{\kappa}}(k)-g_{\frac{2}{\kappa}}^+(k)\right)}$

10 Wigner function of g_κ

Up to a normalizing constant,
$$\mathcal{W}_{g_\kappa}(n,k)=g_{2\kappa}(n)\left(g_{\frac{2}{\kappa}}(k)+g_{\frac{2}{\kappa}}^+(k)\right)$$
$$+g_{2\kappa}^+(n)\left(g_{\frac{2}{\kappa}}(k)-g_{\frac{2}{\kappa}}^+(k)\right)$$

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■ PROOFS

A A discrete Weyl-Wigner description

$\boxed{A14}$

$$
\begin{aligned}
&2n=0 \pmod d\\
&\Downarrow\\
&2n=kd\\
&\Downarrow\\
&k=2m \quad\Rightarrow\quad 2n=2md\\
&\qquad\qquad\qquad\ \Downarrow\\
&\qquad\qquad\qquad\ n=md\\
&\qquad\qquad\qquad\ \Downarrow\\
&\qquad\qquad\qquad\ n=0 \pmod d.
\end{aligned}
$$

Surjectivity

$$
\begin{cases} n+m=k\\ n-m=\ell \end{cases} \Rightarrow \begin{cases} n=(k+\ell)(-s)\\ m=(k-\ell)(-s). \end{cases}
$$

Injectivity

$$
\begin{aligned}
&\begin{cases} n+m=n'+m'\\ n-m=n'-m' \end{cases} \Rightarrow\ 2n=2n'\\
&\Updownarrow\\
&2(n-n')=0\\
&\Updownarrow\\
&2(n-n')\in\mathbb{Z}d\\
&\Updownarrow\\
&2(n-n')=2kd\\
&\Updownarrow\\
&n=n'+kd\\
&\Updownarrow\\
&n=n' \mod(d).
\end{aligned}
$$

$\boxed{A15}$

$$
k\neq 0 \ \Rightarrow\ \frac{1}{d}\sum_{m\in\mathbb{Z}_d} e^{\pm\frac{2\pi i}{d}km}=\frac{1}{d}\sum_{m=0}^{d-1} e^{\pm\frac{2\pi i}{d}km}=\frac{1}{d}\,\frac{1-\left(e^{\pm\frac{2\pi i}{d}k}\right)^d}{1-e^{\pm\frac{2\pi i}{d}k}}=0
$$

$$
\Rightarrow\ \frac{1}{d}\sum_{m\in\mathbb{Z}_d} e^{\pm\frac{2\pi i}{d}km}=\delta_0(k)\ \Rightarrow\ \frac{1}{d}\sum_{m\in\mathbb{Z}_d} e^{\pm\frac{4\pi i}{d}km}=\delta_0(2k)=\delta_0(k).
$$

$\boxed{A16}$

$$
\begin{aligned}
F[F^\dagger[\psi]](n)&=\frac{1}{\sqrt d}\sum_{k\in\mathbb{Z}_d} e^{-\frac{2\pi i}{d}kn}\,F^\dagger[\psi](k)\\
&=\frac{1}{d}\sum_{k,m\in\mathbb{Z}_d} e^{-\frac{2\pi i}{d}kn}\,e^{\frac{2\pi i}{d}km}\,\psi(m)\\
&=\sum_{m\in\mathbb{Z}_d}\frac{1}{d}\sum_{k\in\mathbb{Z}_d} e^{\frac{2\pi i}{d}k(m-n)}\,\psi(m)\\
&\Downarrow\ \boxed{A15}\\
&=\sum_{m\in\mathbb{Z}_d}\delta_{mn}\,\psi(m)\\
&=\psi(n).
\end{aligned}
$$

$$
\begin{aligned}
F^\dagger[F[\psi]](n)&=\frac{1}{\sqrt d}\sum_{k\in\mathbb{Z}_d} e^{\frac{2\pi i}{d}kn}\,F[\psi](k)\\
&=\frac{1}{d}\sum_{k,m\in\mathbb{Z}_d} e^{\frac{2\pi i}{d}kn}\,e^{-\frac{2\pi i}{d}km}\,\psi(m)\\
&=\sum_{m\in\mathbb{Z}_d}\frac{1}{d}\sum_{k\in\mathbb{Z}_d} e^{\frac{2\pi i}{d}k(n-m)}\,\psi(m)\\
&\Downarrow\ \boxed{A15}\\
&=\sum_{m\in\mathbb{Z}_d}\delta_{mn}\,\psi(m)\\
&=\psi(n).
\end{aligned}
$$

$\boxed{A17}$

$$
\begin{aligned}
P\psi(n)&=(F^\dagger QF)\psi(n)\\
&=F^\dagger(QF\psi)(n)\\
&=\frac{1}{\sqrt d}\sum_{k\in\mathbb{Z}_d} e^{\frac{2\pi i}{d}kn}(QF\psi)(k)\\
&=\frac{1}{\sqrt d}\sum_{k\in\mathbb{Z}_d}\tilde k e^{\frac{2\pi i}{d}kn}(F\psi)(k)\\
&=\frac{1}{d}\sum_{k,m\in\mathbb{Z}_d}\tilde k e^{\frac{2\pi i}{d}kn}e^{-\frac{2\pi i}{d}km}\psi(m)\\
&=\frac{1}{d}\sum_{k,m\in\mathbb{Z}_d}\tilde k e^{\frac{2\pi i}{d}k(n-m)}\psi(m).
\end{aligned}
$$

$\boxed{A18}$

$$
\begin{aligned}
\langle Q\psi|\varphi\rangle&=\sum_n \overline{Q\psi(n)}\,\varphi(n)\\
&=\sum_n \tilde n\,\overline{\psi(n)}\,\varphi(n) \qquad\Rightarrow\qquad Q^\dagger=Q.\\
&=\langle\psi|Q\varphi\rangle
\end{aligned}
$$

$$
P^\dagger=(F^\dagger QF)^\dagger=F^\dagger QF=P.
$$

$$
\begin{aligned}
\langle m|Q|j\rangle&=Q\delta_j(m)\\
&=\tilde m\,\delta_j(m)\\
&=\tilde m\,\delta_{mj}.
\end{aligned}
$$

$$
\begin{aligned}
\langle m|P|j\rangle&=P\delta_j(m)\\
&\Downarrow\ \boxed{A17}\\
&=\frac{1}{d}\sum_{k,n}\tilde k e^{\frac{2\pi i}{d}k(m-n)}\delta_j(n)\\
&=\frac{1}{d}\sum_{k}\tilde k e^{\frac{2\pi i}{d}k(m-j)}
\end{aligned}
$$

$\boxed{A19}$

$$
\begin{aligned}
e^{-\frac{2\pi i}{d}nP}\psi(m)&\equiv(e^{-\frac{2\pi i}{d}nP}\psi)(m)\\
&=(e^{-\frac{2\pi i}{d}nF^\dagger QF}\psi)(m)\\
&=(F^\dagger e^{-\frac{2\pi i}{d}nQ}F\psi)(m)\\
&=F^\dagger(e^{-\frac{2\pi i}{d}nQ}F\psi)(m)\\
&=\frac{1}{\sqrt d}\sum_k e^{\frac{2\pi i}{d}km}(e^{-\frac{2\pi i}{d}nQ}F\psi)(k)\\
&=\frac{1}{\sqrt d}\sum_k e^{\frac{2\pi i}{d}km}e^{-\frac{2\pi i}{d}nQ}(F\psi)(k)\\
&=\frac{1}{\sqrt d}\sum_k e^{\frac{2\pi i}{d}km}e^{-\frac{2\pi i}{d}nk}(F\psi)(k)\\
&=\frac{1}{d}\sum_{k,\ell} e^{\frac{2\pi i}{d}k(m-n)}e^{-\frac{2\pi i}{d}k\ell}\psi(\ell)\\
&=\frac{1}{d}\sum_{k,\ell} e^{\frac{2\pi i}{d}k(m-n-\ell)}\psi(\ell)\\
&\Downarrow\ \boxed{A15}\\
&=\sum_\ell \delta_{m-n}(\ell)\,\psi(\ell)\\
&=\psi(m-n),
\end{aligned}
$$

$$
\begin{aligned}
e^{-\frac{2\pi i}{d}nP}\delta_m(\ell)&=\delta_m(\ell-n)\\
&=\delta_{m+n}(\ell)\\
&\Downarrow\\
e^{-\frac{2\pi i}{d}nP}|m\rangle&=|m+n\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
e^{\frac{2\pi i}{d}kQ}\delta_m(\ell)&=e^{\frac{2\pi i}{d}k\ell}\delta_m(\ell)\\
&\Downarrow\\
e^{\frac{2\pi i}{d}kQ}|m\rangle&=e^{\frac{2\pi i}{d}km}|m\rangle
\end{aligned}
$$

$\boxed{A20}$

$$
\begin{aligned}
e^{\frac{2\pi i}{d}kQ}\,e^{-\frac{2\pi i}{d}nP}\psi(m)&\equiv((e^{\frac{2\pi i}{d}kQ}\,e^{-\frac{2\pi i}{d}nP})\psi)(m)\\
&=e^{\frac{2\pi i}{d}kQ}\,(e^{-\frac{2\pi i}{d}nP}\psi)(m)\\
&=e^{\frac{2\pi i}{d}km}\,(e^{-\frac{2\pi i}{d}nP}\psi)(m)\\
&=e^{\frac{2\pi i}{d}km}\,\psi(m-n)
\end{aligned}
$$

and

$$
\begin{aligned}
e^{-\frac{2\pi i}{d}nP}\,e^{\frac{2\pi i}{d}kQ}\psi(m)&\equiv((e^{-\frac{2\pi i}{d}nP}\,e^{\frac{2\pi i}{d}kQ})\psi)(m)\\
&=e^{-\frac{2\pi i}{d}nP}\,(e^{\frac{2\pi i}{d}kQ}\psi)(m)\\
&=(e^{\frac{2\pi i}{d}kQ}\psi)(m-n)\\
&=e^{\frac{2\pi i}{d}k(m-n)}\,\psi(m-n)\\
&=e^{-\frac{2\pi i}{d}kn}\,e^{\frac{2\pi i}{d}km}\,\psi(m-n)\\
&\Downarrow\\
e^{\frac{2\pi i}{d}kQ}\,e^{-\frac{2\pi i}{d}nP}&=e^{\frac{2\pi i}{d}nk}\,e^{-\frac{2\pi i}{d}nP}\,e^{\frac{2\pi i}{d}kQ},
\end{aligned}
$$

$\boxed{A21}$

$$
\begin{aligned}
D(n,k)&=e^{-\frac{\pi i}{d}nk}\,e^{\frac{2\pi i}{d}kQ}\,e^{-\frac{2\pi i}{d}nP}\\
&\Downarrow\ \boxed{A20}\\
D(n,k)&=e^{\frac{\pi i}{d}nk}\,e^{-\frac{2\pi i}{d}nP}\,e^{\frac{2\pi i}{d}kQ}.
\end{aligned}
$$

$\boxed{A22}$

$$
\begin{aligned}
D(n,k)\psi(m) &= \mathrm{e}^{-\frac{\pi \mathrm{i}}{d}nk}\,(\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}\psi)(m)\\
&= \mathrm{e}^{-\frac{\pi \mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}km}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}\psi(m)\\
&= \mathrm{e}^{-\frac{\pi \mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}km}\,\psi(m-n).\\
\Pi(n,k)\psi(m) &\equiv (\Pi(n,k)\psi)(m)\\
&= (D(n,k)\,\Pi\,D(n,k)^\dagger\psi)(m)\\
&= \mathrm{e}^{-\frac{\pi \mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}km}(\Pi\,D(n,k)^\dagger\psi)(m-n)\\
&= \mathrm{e}^{-\frac{\pi \mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}km}(D(n,k)^\dagger\psi)(n-m)\\
&= \mathrm{e}^{-\frac{\pi \mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}km}(D(-n,-k)\psi)(n-m)\\
&= \mathrm{e}^{-\frac{2\pi \mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}km}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}k(n-m)}\,\psi(2n-m)\\
&= \mathrm{e}^{-\frac{2\pi \mathrm{i}}{d}2k(n-m)}\,\psi(2n-m).
\end{aligned}
$$

$\boxed{A23}$

$$
\begin{aligned}
\langle m|D(n,k)|j\rangle &= \mathrm{e}^{-\frac{\pi \mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}km}\,\delta_j(m-n)\\
&= \mathrm{e}^{-\frac{\pi \mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k(j+n)}\,\delta_m(j+n)\\
&= \mathrm{e}^{\frac{\pi \mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kj}\,\delta_m(j+n)\\
&= \mathrm{e}^{\frac{\pi \mathrm{i}}{d}k(n+2j)}\,\delta_m(j+n)\\
\langle m|\Pi(n,k)|j\rangle &= \mathrm{e}^{-\frac{2\pi \mathrm{i}}{d}2k(n-m)}\,\delta_j(2n-m)\\
&= \mathrm{e}^{\frac{2\pi \mathrm{i}}{d}k(m-j)}\,\delta_{2n}(m+j).
\end{aligned}
$$

$\boxed{A24}$

$$
\begin{aligned}
D(n,k)\,D(m,j)\psi(\ell) &\equiv (D(n,k)\,D(m,j)\psi)(\ell)\\
&= D(n,k)(D(m,j)\psi)(\ell)\\
&= \mathrm{e}^{-\frac{\pi \mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}km}\,(D(m,j)\psi)(\ell-n)\\
&= \mathrm{e}^{-\frac{\pi \mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}km}\,\mathrm{e}^{-\frac{\pi \mathrm{i}}{d}mj}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}j(\ell-n)}\,\psi(\ell-n-m)\\
&= \mathrm{e}^{-\frac{\pi \mathrm{i}}{d}(nj-km)}D(\widetilde{n+m},\widetilde{k+j}).
\end{aligned}
$$

$\boxed{A25}$

$$
\begin{aligned}
D(n,k) &= \mathrm{e}^{-\frac{\pi \mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}\\
&\Downarrow\\
D(n,k)^\dagger &= \mathrm{e}^{\frac{\pi \mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}nP}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}kQ}\\
&\Downarrow\ \boxed{A21}\\
D(n,k)^\dagger &= \mathrm{e}^{-\frac{\pi \mathrm{i}}{d}nk}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}kQ}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}nP}\\
&\Downarrow\\
D(n,k)^\dagger &= D(-n,-k).\\
&\boxed{A24}\\
&\Downarrow\\
D(n,k)D(-n,-k) &= D(0,0)=\mathbb{I},\\
D(-n,-k)D(n,k) &= D(0,0)=\mathbb{I}
\end{aligned}
$$

$\boxed{A26}$

$$
\begin{aligned}
\langle A,B\rangle &= \sum_{n,m=-s}^{s}\overline{\langle n|A|m\rangle}\,\langle n|B|m\rangle\\
&= \sum_{n,m=-s}^{s}\langle m|A^\dagger|n\rangle\,\langle n|B|m\rangle\\
&= \sum_{m=-s}^{s}\langle m|A^\dagger B|m\rangle\\
&= \mathrm{tr}\,(A^\dagger B).
\end{aligned}
$$

$\boxed{A27}$

$$
\begin{aligned}
D(n,k),D(m,\ell)\rangle &= \mathrm{tr}(D(n,k)^\dagger D(m,\ell))\\
&= \mathrm{tr}(D(-n,-k)D(m,\ell))\\
&\Downarrow\ \boxed{A24}\\
&= \mathrm{tr}(\mathrm{e}^{\frac{\pi\mathrm{i}}{d}(n\ell-km)}D(\widetilde{m-n},\widetilde{\ell-k}))\\
&= \mathrm{e}^{\frac{\pi\mathrm{i}}{d}(n\ell-km)}\,\mathrm{tr}\,D(\widetilde{m-n},\widetilde{\ell-k})\\
&= \mathrm{e}^{\frac{\pi\mathrm{i}}{d}(n\ell-km)}\,\sum_a\langle a|D(\widetilde{m-n},\widetilde{\ell-k})|a\rangle\\
&\Downarrow\ \boxed{A23}\\
&= \mathrm{e}^{\frac{\pi\mathrm{i}}{d}(n\ell-km)}\,\sum_a \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}a(\ell-k)}\,\delta_a(a+m-n)\\
&= \mathrm{e}^{\frac{\pi\mathrm{i}}{d}(n\ell-km)}\,\sum_a \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}a(\ell-k)}\,\delta_m(n)\\
&\Downarrow\ \boxed{A15}\\
&= d\,\mathrm{e}^{\frac{\pi\mathrm{i}}{d}(n\ell-km)}\,\delta_{mn}\,\delta_{k\ell}.\\
&= d\,\delta_{mn}\,\delta_{k\ell}.
\end{aligned}
$$

$$
\begin{aligned}
\langle\Pi(n,k)\,\Pi(m,\ell)\rangle &= \mathrm{tr}(\Pi(n,k)\,\Pi(m,\ell))\\
&= \sum_{a,b}\langle a|\Pi(n,k)|b\rangle\langle b|\Pi(m,\ell)|a\rangle\\
&= \sum_{a,b}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k(a-b)}\,\delta_{2n}(a+b)\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}\ell(a-b)}\,\delta_{2m}(a+b)\\
&= \sum_{a,b}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(k-\ell)(a-b)}\,\delta_{2n}(a+b)\,\delta_{nm}\\
&= \sum_{a}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(k-\ell)(2a-2n)}\,\delta_{nm}\\
&= \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}n(k-\ell)}\,\sum_a\mathrm{e}^{\frac{4\pi\mathrm{i}}{d}a(k-\ell)}\,\delta_{nm}\\
&\Downarrow\ \boxed{A15}\\
&= \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}n(k-\ell)}\,d\,\delta_{k\ell}\,\delta_{nm}\\
&= d\,\delta_{nm}\,\delta_{k\ell}.
\end{aligned}
$$

$\boxed{A28}$

$$
\begin{aligned}
\tilde{A}(n,k) &= \mathrm{tr}(A\Pi(n,k))\\
&= \mathrm{tr}(\Pi(n,k)A)\\
&= \mathrm{tr}(\Pi(n,k)^\dagger A)\\
&= \langle\Pi(n,k),A\rangle.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{W}_A(n,k) &= \tfrac{1}{d}\mathrm{tr}(A\Pi(n,k))\\
&= \tfrac{1}{d}\tilde{A}(n,k)\\
&= \tfrac{1}{d}\langle\Pi(n,k),A\rangle.
\end{aligned}
$$

$\boxed{A29}$

$$
\begin{aligned}
\langle A^\dagger,B\rangle &= \mathrm{tr}(AB)\\
&= \mathrm{tr}(BA)\\
&= \overline{\langle B^\dagger,A\rangle}\\
&= \overline{\langle A,B^\dagger\rangle}.
\end{aligned}
$$

$$
\begin{aligned}
(\Pi\psi)(n) &= \psi(-n)\\
&\Downarrow\\
\langle\Pi\psi,\varphi\rangle &= \sum_n\overline{(\Pi\psi)(n)}\,\varphi(n)\\
&= \sum_n\overline{\psi(-n)}\,\varphi(n)\\
&\Downarrow\ n\mapsto -n\\
&= \sum_n\overline{\psi(n)}\,\varphi(-n)\\
&= \sum_n\overline{\psi(n)}\,(\Pi\varphi)(n)\\
&= \langle\psi,\Pi\varphi\rangle\\
&\Downarrow\\
\Pi^\dagger &= \Pi
\end{aligned}
$$

and

$$
\begin{aligned}
\Pi(n,k)^\dagger &= (D(n,k)\,\Pi\,D(n,k)^\dagger)^\dagger\\
&= D(n,k)\,\Pi^\dagger\,D(n,k)^\dagger\\
&= D(n,k)\,\Pi\,D(n,k)^\dagger\\
&= \Pi(n,k).
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{A^\dagger}(n,k) &= \langle\Pi(n,k),A^\dagger\rangle\\
&= \overline{\langle\Pi(n,k)^\dagger,A\rangle}\\
&= \overline{\langle\Pi(n,k),A\rangle}\\
&= \overline{\tilde{A}(n,k)}.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{W}_{A^\dagger}(n,k) &= \tfrac{1}{d}\widetilde{A^\dagger}(n,k)\\
&= \tfrac{1}{d}\overline{\tilde{A}(n,k)}\\
&= \overline{\mathcal{W}_A(n,k)}.
\end{aligned}
$$

$\boxed{A30}$

$$\left\{\tfrac{1}{\sqrt{d}}\Pi(n,k)\right\} = \text{orthonormal basis in } \mathcal{L}(\mathcal{H})$$
$$\Downarrow$$
$$\begin{aligned}A &= \sum_{n,k}\left\langle \tfrac{1}{\sqrt{d}}\Pi(n,k), A\right\rangle \tfrac{1}{\sqrt{d}}\Pi(n,k)\\ &= \sum_{n,k}\tfrac{1}{d}\ \langle \Pi(n,k), A\rangle\ \Pi(n,k)\\ &\Downarrow\ \boxed{A28}\\ &= \sum_{n,k}\mathcal{W}_A(n,k)\,\Pi(n,k).\end{aligned}$$

$\boxed{A31}$

$$\begin{aligned}\mathcal{W}_A(n,k) &= \tfrac{1}{d}\ \mathrm{tr}(A\,\Pi(n,k))\\ &= \tfrac{1}{d}\sum_{a\in\mathbb{Z}_d}\langle a|A\Pi(n,k)|a\rangle\\ &= \tfrac{1}{d}\sum_{a,b\in\mathbb{Z}_d}\langle a|A|b\rangle\langle b|\Pi(n,k)|a\rangle\\ &\Downarrow\ \boxed{A23}\\ &= \tfrac{1}{d}\sum_{a,b\in\mathbb{Z}_d}\langle a|A|b\rangle \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}k(a-b)}\ \delta_{2n}(a+b)\\ &\Downarrow\ \text{b=2n-a}\\ &= \tfrac{1}{d}\sum_{a\in\mathbb{Z}_d}\langle a|A|2n-a\rangle \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}k(a-n)}\\ &\Downarrow\ \text{m=a-n}\\ &= \tfrac{1}{d}\sum_{a\in\mathbb{Z}_d}\langle n+m|A|n-m\rangle \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\\ &= \tfrac{1}{d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\ \langle n+m|A|n-m\rangle.\end{aligned}$$

$\boxed{A32}$

$$\begin{aligned}&\boxed{A31}\\ &\Downarrow\\ \mathcal{W}_\varrho(n,k) &= \tfrac{1}{d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\ \langle n+m|\varrho|n-m\rangle\\ &\Downarrow\ \varrho=|\psi\rangle\langle\psi|\\ \mathcal{W}_\varrho(n,k) &= \tfrac{1}{d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\ \langle n+m|\psi\rangle\langle\psi|n-m\rangle\\ &= \tfrac{1}{d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\ \psi(n+m)\,\overline{\psi(n-m)}.\end{aligned}$$

$\boxed{A33}$

$$\begin{aligned}\sum_{n,k\in\mathbb{Z}_d}\mathcal{W}_A(n,k) &= \tfrac{1}{d}\sum_{n,k\in\mathbb{Z}_d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\ \langle n+m|A|n-m\rangle\\ &= \sum_{n,m\in\mathbb{Z}_d}\tfrac{1}{d}\sum_{k\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\ \langle n+m|A|n-m\rangle\\ &\Downarrow\ \boxed{A15}\\ &= \sum_{n,m\in\mathbb{Z}_d}\delta_{m0}\ \langle n+m|A|n-m\rangle\\ &= \sum_{n\in\mathbb{Z}_d}\langle n|A|n\rangle\\ &= \mathrm{tr}\,A.\end{aligned}$$

$\boxed{A34}$

$$\begin{aligned}\mathrm{tr}(AB) &= \langle A^\dagger, B\rangle\\ &\Downarrow\ \boxed{A27}\\ &= \tfrac{1}{d}\sum\overline{\langle\Pi(n,k),A^\dagger\rangle}\langle\Pi(n,k),B\rangle\\ &\Downarrow\ \boxed{A29}\\ &= \tfrac{1}{d}\sum\langle\Pi(n,k)^\dagger,A\rangle\langle\Pi(n,k),B\rangle\\ &= \tfrac{1}{d}\sum\langle\Pi(n,k),A\rangle\langle\Pi(n,k),B\rangle\\ &= \tfrac{1}{d}\sum\tilde{A}(n,k)\ \tilde{B}(n,k)\\ &= \sum\tilde{A}(n,k)\,\mathcal{W}_B(n,k)\\ &= d\sum\mathcal{W}_A(n,k)\,\mathcal{W}_B(n,k).\end{aligned}$$

$\boxed{A35}$

$$\begin{aligned}\mathrm{tr}(\varrho_\psi\varrho_\varphi) &= \mathrm{tr}(|\psi\rangle\langle\psi|\varphi\rangle\langle\varphi|)\\ &= \sum_n\langle n|\psi\rangle\langle\psi|\varphi\rangle\langle\varphi|n\rangle\\ &= \sum_n\langle\varphi|n\rangle\langle n|\psi\rangle\langle\psi|\varphi\rangle\\ &= \langle\varphi|\psi\rangle\langle\psi|\varphi\rangle\\ &= |\langle\psi|\varphi\rangle|^2.\end{aligned}$$

$\boxed{A36}$

$$\begin{aligned}\mathrm{tr}(\ \varrho\,A) &= \mathrm{tr}(\ |\psi\rangle\langle\psi|\ A\ )\\ &= \sum_{m=-s}^{s}\langle m|\psi\rangle\langle\psi|\ A|m\rangle\\ &= \sum_{m=-j}^{j}\langle\psi|\ A|m\rangle\langle m|\psi\rangle\\ &= \langle\psi|\ A|\psi\rangle.\end{aligned}$$

$\boxed{A37}$

$$\begin{aligned}\tilde{\mathbb{I}}(n,k) &= \mathrm{tr}(\mathbb{I}\Pi(n,k))\\ &= \mathrm{tr}(\Pi(n,k))\\ &= \sum_m\langle m|\Pi(n,k)|m\rangle\\ &\Downarrow\ \boxed{A23}\\ &= \sum_m\delta_{2n}(2m)\\ &= 1.\end{aligned}$$

$$\begin{aligned}\tilde{Q}(n,k) &= \mathrm{tr}(Q\,\Pi(n,k))\\ &= \mathrm{tr}(\Pi(n,k)Q)\\ &= \sum_m\langle m|\Pi(n,k)Q|m\rangle\\ &= \sum_m\tilde{m}\ \langle m|\Pi(n,k)|m\rangle\\ &\Downarrow\ \boxed{A23}\\ &= \sum_m\tilde{m}\,\delta_{nm}\\ &= \tilde{n}.\end{aligned}$$

$$\begin{aligned}\tilde{P}(n,k) &= \mathrm{tr}(P\,\Pi(n,k))\\ &= \sum_m\langle m|P\Pi(n,k)|m\rangle\\ &= \sum_{m,j}\langle m|P|j\rangle\langle \mathrm{j}|\Pi(n,k)|m\rangle\\ &= \tfrac{1}{d}\sum_{m,j,\ell}\tilde{\ell}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}\ell(m-j)}\ \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k(j-m)}\ \delta_{2n}(j+m)\\ &= \tfrac{1}{d}\sum_{m,\ell}\tilde{\ell}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}\ell(2m-2n)}\ \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k(2n-2m)}\\ &= \tfrac{1}{d}\sum_{m,\ell}\tilde{\ell}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}n(2k-2\ell)}\ \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}m(2\ell-2k)}\\ &= \sum_\ell\tilde{\ell}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}n(2k-2\ell)}\,\tfrac{1}{d}\sum_m\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}m(2\ell-2k)}\\ &\Downarrow\ \boxed{A15}\\ &= \sum_\ell\tilde{\ell}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}n(2k-2\ell)}\delta_{\ell k}\\ &= \tilde{k}.\end{aligned}$$

$\boxed{A38}$

$$\begin{aligned}\langle A,B\rangle &= \sum_{n,m=-s}^{s}\overline{\langle n|A|m\rangle}\ \langle n|B|m\rangle\\ &= \sum_{n,m=-s}^{s}\langle m|A^\dagger|n\rangle\ \langle n|B|m\rangle\\ &= \sum_{m=-s}^{s}\langle m|AB|m\rangle\\ &= \mathrm{tr}\,(AB).\end{aligned}$$

$\boxed{A39}$

See $\boxed{A27}$ at page 5.

$\boxed{A40}$

$$\boxed{A29} \Downarrow$$

$\mathcal{W}_A(n,k)$ are real numbers.
$\tilde{A}(n,k)$ are real numbers.

$\boxed{A41}$

$$\begin{aligned}\sum_{k\in\mathbb{Z}_d}\mathcal{W}_\psi(n,k) &= \frac{1}{d}\sum_{k\in\mathbb{Z}_d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\ \psi(n+m)\,\overline{\psi(n-m)}\\ &= \frac{1}{d}\sum_{m\in\mathbb{Z}_d}\sum_{k\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\ \psi(n+m)\,\overline{\psi(n-m)}\\ &\Downarrow \boxed{A15}\\ &= \sum_{m\in\mathbb{Z}_d}\delta_0(m)\,\psi(n+m)\,\overline{\psi(n-m)}\\ &= \psi(n)\,\overline{\psi(n)}\\ &= |\psi(n)|^2.\end{aligned}$$

$\boxed{A42}$

$$\begin{aligned}\sum_{n\in\mathbb{Z}_d}\mathcal{W}_\psi(n,k) &= \frac{1}{d}\sum_{n\in\mathbb{Z}_d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\ \psi(n+m)\,\overline{\psi(n-m)}\\ &\Downarrow \boxed{A14}\ (n,m)\mapsto(a,b)=(n+m,n-m)\ \text{bijective}\\ &= \frac{1}{d}\sum_{a\in\mathbb{Z}_d}\sum_{b\in\mathbb{Z}_d}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}k(a-b)}\ \psi(a)\,\overline{\psi(b)}\\ &= \frac{1}{\sqrt{d}}\sum_{a\in\mathbb{Z}_d}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}ka}\psi(a)\ \frac{1}{\sqrt{d}}\sum_{b\in\mathbb{Z}_d}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kb}\,\overline{\psi(b)}\\ &= F[\psi](k)\ \overline{F[\psi](k)}\\ &= |F[\psi](k)|^2.\end{aligned}$$

$\boxed{A43}$

$$\begin{aligned}\mathcal{W}_\varphi(n,k) &= \frac{1}{d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\ \varphi(n+m)\,\overline{\varphi(n-m)}\\ &= \frac{1}{d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\ \psi(n+m-a)\,\overline{\psi(n-m-a)}\\ &= \mathcal{W}_\psi(n-a,k).\end{aligned}$$

$\boxed{A44}$

$$\begin{aligned}\mathcal{W}_\varphi(n,k) &= \frac{1}{d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\ \varphi(n+m)\,\overline{\varphi(n-m)}\\ &= \frac{1}{d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\ \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}b(n+m)}\ \psi(n+m)\\ &\qquad\qquad \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}b(n-m)}\ \overline{\psi(n-m)}\\ &= \frac{1}{d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}(k-b)m}\ \psi(n+m)\,\overline{\psi(n-m)}\\ &= \mathcal{W}_\psi(n,k-b).\end{aligned}$$

$\boxed{A45}$

$$\begin{aligned}\mathcal{W}_\psi(n,k) &= \frac{1}{d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\ \psi(n+m)\,\overline{\psi(n-m)}\\ &= \frac{1}{d}\langle\varphi_1,\varphi_2\rangle,\end{aligned}$$

where $\varphi_1(m)=\psi(n-m)$
$\varphi_2(m)=\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\ \psi(n+m)$

and $||\varphi_1||^2=\sum_m|\varphi_1(m)|^2=\sum_m|\psi(n-m)|^2=||\psi||^2=1,$
$||\varphi_2||^2=\sum_m|\varphi_2(m)|^2=\sum_m|\psi(n+m)|^2=||\psi||^2=1.$

But, $\mathcal{W}_\psi(n,k)=\frac{1}{d}\langle\varphi_1,\varphi_2\rangle$

$$\Downarrow$$

$$|\mathcal{W}_\psi(n,k)|=\frac{1}{d}|\langle\varphi_1,\varphi_2\rangle|\le\frac{1}{d}||\varphi_1||\ ||\varphi_2||=\frac{1}{d}.$$

B Wigner function of a reduced density operator

$\boxed{B8}$

$$\begin{aligned}\mathrm{tr}(A\otimes B) &= \sum_{a,b}\langle ab|A\otimes B|ab\rangle\\ &= \sum_a\langle a|A|a\rangle\ \sum_b\langle b|B|b\rangle\\ &= \mathrm{tr}\,A\ \mathrm{tr}\,B\end{aligned}$$

$\boxed{B9}$

$$\begin{aligned}\langle\varphi_1\otimes\psi_1,(A\otimes B)^\dagger(\varphi_2\otimes\psi_2)\rangle &= \langle(A\otimes B)(\varphi_1\otimes\psi_1),\varphi_2\otimes\psi_2\rangle\\ &= \langle(A\varphi_1)\otimes(B\psi_1),\varphi_2\otimes\psi_2\rangle\\ &= \langle A\varphi_1,\varphi_2\rangle\langle B\psi_1,\psi_2\rangle\\ &= \langle\varphi_1,A^\dagger\varphi_2\rangle\langle\psi_1,B^\dagger\psi_2\rangle\\ &= \langle\varphi_1\otimes\psi_1,(A^\dagger\varphi_2)\otimes(B^\dagger\psi_2)\rangle\\ &= \langle\varphi_1\otimes\psi_1,(A^\dagger\otimes B^\dagger)(\varphi_2\otimes\psi_2)\rangle\end{aligned}$$

for any $\varphi_1,\varphi_2\in\mathcal{H}_1,\ \psi_1,\psi_2\in\mathcal{H}_2$

$$\Downarrow$$

$$\langle nk|(A\otimes B)^\dagger|m\ell\rangle=\langle nk|A^\dagger\otimes B^\dagger|m\ell\rangle$$

(Matrices of the two operators coincide).

$\boxed{B10}$

$$\begin{aligned}&\left\langle\frac{1}{\sqrt{d_1d_2}}\,\Pi(n_1,n_2,k_1,k_2),\ \frac{1}{\sqrt{d_1d_2}}\,\Pi(n_1',n_2',k_1',k_2')\right\rangle\\ &= \frac{1}{d_1d_2}\ \langle\Pi(n_1,k_1)\otimes\Pi(n_2,k_2),\Pi(n_1',k_1')\otimes\Pi(n_2',k_2')\rangle\\ &= \frac{1}{d_1d_2}\ \mathrm{tr}((\Pi(n_1,k_1)\otimes\Pi(n_2,k_2))(\Pi(n_1',k_1')\otimes\Pi(n_2',k_2')))\\ &= \frac{1}{d_1d_2}\ \mathrm{tr}((\Pi(n_1,k_1)\Pi(n_1',k_1')\otimes\Pi(n_2,k_2)\Pi(n_2',k_2'))\\ &= \frac{1}{d_1d_2}\ \mathrm{tr}((\Pi(n_1,k_1)\Pi(n_1',k_1'))\ \mathrm{tr}(\Pi(n_2,k_2)\Pi(n_2',k_2'))\\ &= \delta_{n_1n_1'}\ \delta_{k_1k_1'}\ \delta_{n_2n_2'}\ \delta_{k_2k_2'}.\end{aligned}$$

$\boxed{B11}$

$$\begin{aligned}\mathcal{W}_\varrho(n_1,n_2,k_1,k_2) &= \frac{1}{d_1d_2}\mathrm{tr}(\varrho\,\Pi(n_1,n_2,k_1,k_2)).\\ &= \frac{1}{d_1d_2}\langle\Pi(n_1,n_2,k_1,k_2),\varrho\rangle\end{aligned}$$

$\left\{\frac{1}{\sqrt{d_1d_2}}\,\Pi(n_1,n_2,k_1,k_2)\right\}$ is an orthonormal basis in $\mathcal{A}(\mathcal{H}_1\otimes\mathcal{H}_2)$.

$$\Downarrow$$

$$\begin{aligned}\varrho &= \sum_{n_1,n_2,k_1,k_2}\frac{1}{d_1d_2}\langle\Pi(n_1,n_2,k_1,k_2),\varrho\rangle\ \Pi(n_1,n_2,k_1,k_2)\\ &= \sum_{n_1,n_2,k_1,k_2}\mathcal{W}_\rho(n_1,n_2,k_1,k_2)\,\Pi(n_1,n_2,k_1,k_2)\end{aligned}$$

$\boxed{B12}$

$$\begin{aligned}\mathcal{W}_\varrho(n_1,n_2,k_1,k_2) &= \frac{1}{d_1d_2}\mathrm{tr}(\varrho\,\Pi(n_1,n_2,k_1,k_2))\\ &= \frac{1}{d_1d_2}\sum_{a_1,a_2,b_1,b_2}\langle a_1a_2|\varrho|b_1b_2\rangle\\ &\qquad \langle b_1b_2|\Pi(n_1,k_1)\otimes\Pi(n_2,k_2)|a_1a_2\rangle\\ &= \frac{1}{d_1d_2}\sum_{a_1,a_2,b_1,b_2}\langle a_1a_2|\varrho|b_1b_2\rangle\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_1}k_1(a_1-b_1)}\\ &\qquad \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_2}k_2(a_2-b_2)}\delta_{2n_1}(a_1+b_1)\delta_{2n_2}(a_2+b_2)\\ &\Downarrow\ b_1=2n_1-a_1,\ \ b_2=2n_2-a_2\\ &= \frac{1}{d_1d_2}\sum_{a_1,a_2,b_1,b_2}\langle a_1a_2|\varrho|2n_1-a_1,2n_2-a_2\rangle\\ &\qquad \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_1}k_1(a_1-b_1)}\ \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_2}k_2(a_2-b_2)}\\ &\Downarrow\ m_1=a_1-n_1,\ \ m_2=a_2-n_2\\ &= \frac{1}{d_1d_2}\sum_{m_1\in\mathbb{Z}_{d_1}}\sum_{m_2\in\mathbb{Z}_{d_2}}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_1}k_1m_1}\ \mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_2}k_2m_2}\\ &\quad\times\langle n_1+m_1,n_2+m_2|\varrho|n_1-m_1,n_2-m_2\rangle\end{aligned}$$

$\boxed{B13}$

$$\mathcal{W}_\varrho(n_1,n_2,k_1,k_2)=\tfrac{1}{d_1d_2}\sum_{m_1\in\mathbb{Z}_{d_1}}\sum_{m_2\in\mathbb{Z}_{d_2}}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_1}k_1m_1}\,\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_2}k_2m_2}\times\langle n_1{+}m_1,n_2{+}m_2|\varrho|n_1{-}m_1,n_2{-}m_2\rangle$$
$$\Downarrow\ \varrho=|\Psi\rangle\langle\Psi|$$
$$\mathcal{W}_\Psi(n_1,n_2,k_1,k_2)=\tfrac{1}{d_1d_2}\sum_{m_1\in\mathbb{Z}_{d_1}}\sum_{m_2\in\mathbb{Z}_{d_2}}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_1}k_1m_1}\,\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_2}k_2m_2}\times\Psi(n_1{+}m_1,n_2{+}m_2)\,\overline{\Psi(n_1{-}m_1,n_2{-}m_2)}$$

$\boxed{B14}$

$$\begin{aligned}\mathcal{W}_{\psi\otimes\varphi}(n_1,n_2,k_1,k_2)=&\\=&\tfrac{1}{d_1d_2}\sum_{m_1\in\mathbb{Z}_{d_1}}\sum_{m_2\in\mathbb{Z}_{d_2}}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_1}k_1m_1}\,\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_2}k_2m_2}\\&\times\psi(n_1{+}m_1)\,\varphi(n_2{+}m_2)\,\overline{\psi(n_1{-}m_1)}\,\overline{\varphi(n_2{-}m_2)}\\=&\mathcal{W}_\psi(n_1,k_1)\,\mathcal{W}_\varphi(n_2,k_2)\end{aligned}$$

$\boxed{B15}$

$$|\Psi\rangle=\sum_a\sqrt{\lambda_a}|\psi_a\rangle\otimes|\varphi_a\rangle\equiv\sum_a\sqrt{\lambda_a}|\psi_a\varphi_a\rangle\ \text{(Schmidt form)}$$
$$\Downarrow$$
$$\begin{aligned}|\Psi\rangle\langle\Psi|&=\sum_{a,b}\sqrt{\lambda_a\lambda_b}|\psi_a\varphi_a\rangle\langle\psi_b\varphi_b|\\&=\sum_{a,b}\sqrt{\lambda_a\lambda_b}|\psi_a\rangle\langle\psi_b|\otimes|\varphi_a\rangle\langle\varphi_b|\end{aligned}$$
$$\Downarrow$$
$$\begin{aligned}\mathrm{tr}_2|\Psi\rangle\langle\Psi|&=\sum_{a,b}\sqrt{\lambda_a\lambda_b}\ \langle\varphi_b|\varphi_a\rangle\ |\psi_a\rangle\langle\psi_b|\\&=\sum_{a,b}\sqrt{\lambda_a\lambda_b}\ \delta_{ab}\ |\psi_a\rangle\langle\psi_b|\\&=\sum_a\ \lambda_a\ |\psi_a\rangle\langle\psi_a|.\end{aligned}$$

$\boxed{B16}$

$$\mathcal{H}_1\otimes\mathcal{H}_2\xrightarrow{\varrho}\mathcal{H}_1\otimes\mathcal{H}_2$$
linear operator
$$\Downarrow\ \text{(spectral decomposition)}$$
$$\varrho=\sum_j p_j\,|\Psi_j\rangle\langle\Psi_j|$$
$$\Downarrow\ \text{(Schmidt form)}$$
$$\varrho=\sum_j p_j\sum_{a_j,b_j}\sqrt{\lambda_{a_j}\lambda_{b_j}}|\psi_{a_j}\varphi_{a_j}\rangle\langle\psi_{b_j}\varphi_{b_j}|$$
$$\Downarrow\ \boxed{B15}$$
$$\varrho_1=\mathrm{tr}_2\,\varrho=\sum_j p_j\sum_{a_j}\lambda_{a_j}\,|\psi_{a_j}\rangle\langle\psi_{a_j}|$$

and

$$\varrho=\sum_j p_j\sum_{a_j,b_j}\sqrt{\lambda_{a_j}\lambda_{b_j}}|\psi_{a_j}\varphi_{a_j}\rangle\langle\psi_{b_j}\varphi_{b_j}|$$
$$\Downarrow\ \boxed{B7}$$
$$\mathcal{W}_\varrho=\sum_j p_j\sum_{a_j,b_j}\sqrt{\lambda_{a_j}\lambda_{b_j}}\ \mathcal{W}_{|\psi_{a_j}\varphi_{a_j}\rangle\langle\psi_{b_j}\varphi_{b_j}|}$$
$$\Downarrow\ \boxed{B13}$$
$$\sum_{n_2,k_2=-s_2}^{s_2}\mathcal{W}_\varrho(n_1,n_2,k_1,k_2)=\tfrac{1}{d_1d_2}$$
$$\times\sum_j p_j\sum_{a_j,b_j}\sqrt{\lambda_{a_j}\lambda_{b_j}}\sum_{m_1\in\mathbb{Z}_{d_1}}\sum_{m_2\in\mathbb{Z}_{d_2}}\sum_{n_2,k_2=-s_2}^{s_2}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_1}k_1m_1}\,\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_2}k_2m_2}$$
$$\times\psi_{a_j}(n_1{+}m_1)\,\varphi_{a_j}(n_2{+}m_2)\,\overline{\psi_{b_j}(n_1{-}m_1)}\,\overline{\varphi_{b_j}(n_2{-}m_2)}$$
$$\Downarrow\ \boxed{A15}$$
$$=\tfrac{1}{d_1}\sum_j p_j\sum_{a_j,b_j}\sqrt{\lambda_{a_j}\lambda_{b_j}}\sum_{m_1\in\mathbb{Z}_{d_1}}\sum_{n_2=-s_2}^{s_2}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_1}k_1m_1}\times\psi_{a_j}(n_1{+}m_1)\,\varphi_{a_j}(n_2)\,\overline{\psi_{b_j}(n_1{-}m_1)}\,\overline{\varphi_{b_j}(n_2)}$$
$$=\tfrac{1}{d_1}\sum_j p_j\sum_{a_j,b_j}\sqrt{\lambda_{a_j}\lambda_{b_j}}\sum_{m_1\in\mathbb{Z}_{d_1}}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_1}k_1m_1}\langle\varphi_{b_j}|\varphi_{a_j}\rangle\times\psi_{a_j}(n_1{+}m_1)\,\overline{\psi_{b_j}(n_1{-}m_1)}$$
$$=\tfrac{1}{d_1}\sum_j p_j\sum_{a_j}\lambda_{a_j}\sum_{m_1\in\mathbb{Z}_{d_1}}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_1}k_1m_1}\psi_{a_j}(n_1{+}m_1)\,\overline{\psi_{a_j}(n_1{-}m_1)}$$
$$=\tfrac{1}{d_1}\sum_j p_j\sum_{a_j}\lambda_{a_j}\sum_{m_1\in\mathbb{Z}_{d_1}}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_1}k_1m_1}\langle n_1{+}m_1|\psi_{a_j}\rangle\langle\psi_{a_j}|n_1{-}m_1\rangle)$$
$$\Downarrow\ \boxed{B15}$$
$$=\tfrac{1}{d_1}\sum_{m_1\in\mathbb{Z}_{d_1}}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d_1}k_1m_1}\langle n_1{+}m_1|\varrho_1|n_1{-}m_1\rangle$$
$$=\mathcal{W}_{\varrho_1}(n_1,k_1).$$

$\boxed{B17}$

$$\begin{aligned}\widetilde{A\otimes B}(n_1,n_2,k_1,k_2)&=\mathrm{tr}(A{\otimes}B\ \Pi(n_1,n_2,k_1,k_2))\\&=\mathrm{tr}(A{\otimes}B\ \Pi(n_1,k_1){\otimes}\Pi(n_2,k_2))\\&=\mathrm{tr}((A\,\Pi(n_1,k_1)){\otimes}(B\,\Pi(n_2,k_2)))\\&=\mathrm{tr}(A\,\Pi(n_1,k_1))\ \mathrm{tr}(B\,\Pi(n_2,k_2))\\&=\tilde{A}(n_1,k_1)\,\tilde{B}(n_2,k_2).\end{aligned}$$

$\boxed{B18}$

$$\begin{aligned}\mathcal{W}_{A\otimes B}(n_1,n_2,k_1,k_2)&=\tfrac{1}{d_1d_2}\,\widetilde{A\otimes B}(n_1,n_2,k_1,k_2)\\&=\tfrac{1}{d_1d_2}\,\tilde{A}(n_1,k_1)\,\tilde{B}(n_2,k_2)\\&=\mathcal{W}_A(n_1,k_1)\,\mathcal{W}_B(n_2,k_2).\end{aligned}$$

$\boxed{B19}$ *Proof 1.*

$$\begin{aligned}\mathrm{tr}(\varrho(A{\otimes}\mathbb{I}))&=\sum_{a,b}\langle ab|\varrho(A{\otimes}\mathbb{I})|ab\rangle\\&=\sum_{a,b}\sum_{n,k}\langle ab|\varrho|nk\rangle\langle nk|A{\otimes}\mathbb{I}|ab\rangle\\&=\sum_{a,b}\sum_{n,k}\langle ab|\varrho|nk\rangle\langle n|A|a\rangle\langle k|b\rangle\\&=\sum_{a,n}\sum_{b}\langle ab|\varrho|nb\rangle\langle n|A|a\rangle\\&=\sum_{a,n}\langle a|\varrho_1|n\rangle\langle n|A|a\rangle\\&=\sum_a\langle a|\varrho_1A|a\rangle\\&=\mathrm{tr}(\varrho_1\,A).\end{aligned}$$

Proof 2.

$$\mathrm{tr}(\varrho(A{\otimes}\mathbb{I}))=\sum_{n_1,n_2,k_1,k_2}\mathcal{W}_\varrho(n_1,n_2,k_1,k_2)\,\widetilde{A{\otimes}\mathbb{I}}(n_1,n_2,k_1,k_2)$$
$$=\sum_{n_1,n_2,k_1,k_2}\mathcal{W}_\varrho(n_1,n_2,k_1,k_2)\,\widetilde{A{\otimes}\mathbb{I}}(n_1,n_2,k_1,k_2)$$
$$\Downarrow\ \boxed{B17}$$
$$=\sum_{n_1,n_2,k_1,k_2}\mathcal{W}_\varrho(n_1,n_2,k_1,k_2)\,\tilde{A}(n_1,k_1)\,\tilde{\mathbb{I}}(n_2,k_2)$$
$$\Downarrow\ \boxed{A36}$$
$$=\sum_{n_1,n_2,k_1,k_2}\mathcal{W}_\varrho(n_1,n_2,k_1,k_2)\,\tilde{A}(n_1,k_1)$$
$$=\sum_{n_1,k_1}\sum_{n_2,k_2}\mathcal{W}_\varrho(n_1,n_2,k_1,k_2)\,\tilde{A}(n_1,k_1)$$
$$\Downarrow\ \boxed{B16}$$
$$=\sum_{n_1,k_1}\mathcal{W}_{\varrho_1}(n_1,k_1)\,\tilde{A}(n_1,k_1)$$
$$\Downarrow\ \boxed{A34}$$
$$=\mathrm{tr}(\varrho_1\,A).$$

$\boxed{B20}$

$$|\Psi\rangle=\sum\sqrt{\lambda_j}|\varphi_j\rangle\otimes|\varphi_j\rangle$$
$$\Downarrow$$
$$\begin{aligned}|\Psi\rangle\langle\Psi|&=\sum\sqrt{\lambda_j\lambda_k}|\varphi_j\varphi_j\rangle\langle\varphi_k\varphi_k|\\&=\sum\sqrt{\lambda_j\lambda_k}|\varphi_j\rangle\langle\varphi_k|\otimes|\varphi_j\rangle\langle\varphi_k|\end{aligned}$$
$$\Downarrow$$
$$\mathcal{W}_\Psi(n_1,n_2,k_1,k_2)=\sum_{j,k}\sqrt{\lambda_j\lambda_k}\,\mathcal{W}_{|\varphi_j\rangle\langle\varphi_k|}(n_1,k_1)\times\mathcal{W}_{|\varphi_j\rangle\langle\varphi_k|}(n_2,k_2)$$
$$\Downarrow\ \boxed{A33}$$
$$\begin{aligned}\sum_{n_2,k_2}\mathcal{W}_\Psi(n_1,n_2,k_1,k_2)&=\sum_{j,k}\sqrt{\lambda_j\lambda_k}\,\mathcal{W}_{|\varphi_j\rangle\langle\varphi_k|}(n_1,k_1)\times\mathrm{tr}(|\varphi_j\rangle\langle\varphi_k|)\\&=\sum_{j,k}\sqrt{\lambda_j\lambda_k}\,\mathcal{W}_{|\varphi_j\rangle\langle\varphi_k|}(n_1,k_1)\times\langle\varphi_k|\varphi_j\rangle\\&=\sum_{j,k}\sqrt{\lambda_j\lambda_k}\,\mathcal{W}_{|\varphi_j\rangle\langle\varphi_k|}(n_1,k_1)\times\delta_{kj}\\&=\sum_j\lambda_j\,\mathcal{W}_{|\varphi_j\rangle\langle\varphi_j|}(n_1,k_1)\\&\Downarrow\ \varrho=\sum\lambda_j|\varphi_j\rangle\langle\varphi_j|\\&=\mathcal{W}_\varrho(n_1,k_1).\end{aligned}$$

$\boxed{\mathbf{C}}$ Gaussian functions of discrete variable

$\boxed{C5}$ For more details, alternative versions for the discrete coordinate operator, discrete momentum operator and discrete phase space see [5].

$\boxed{C6}$ For more details and an application concerning an alternative definition for the fractional discrete Fourier transform see [7].

$\boxed{C7}$

$$\mathfrak{g}\equiv\frac{1}{\sqrt{\langle g_1|g_1\rangle}}\,g_1$$
$$\Downarrow\ \boxed{A22}$$
$$\langle a|n,k\rangle=\mathrm{e}^{-\frac{\pi\mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}ka}\,\mathfrak{g}(a-n)$$
$$\Downarrow$$
$$\begin{aligned}\left\langle a\left|\frac{1}{d}\sum_{(n,k)\in\mathfrak{S}}|n,k\rangle\langle n,k|\right|b\right\rangle&=\frac{1}{d}\sum_{(n,k)\in\mathfrak{S}}\langle a|n,k\rangle\langle n,k|b\rangle\\&=\frac{1}{d}\sum_{(n,k)\in\mathfrak{S}}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}ka}\,\mathfrak{g}(a-n)\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}kb}\,\mathfrak{g}(b-n)\\&=\frac{1}{d}\sum_{(n,k)\in\mathfrak{S}}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k(a-b)}\,\mathfrak{g}(a-n)\,\mathfrak{g}(b-n)\\&=\sum_{n=-s}^{s}\frac{1}{d}\sum_{k=-s}^{s}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k(a-b)}\,\mathfrak{g}(a-n)\,\mathfrak{g}(b-n)\\&\Downarrow\ \boxed{A15}\\&=\sum_{n=-s}^{s}\delta_0(a-b)\,\mathfrak{g}(a-n)\,\mathfrak{g}(b-n)\\&=\sum_{n=-s}^{s}\delta_{ab}\,\mathfrak{g}(a-n)\,\mathfrak{g}(b-n)\\&=\delta_{ab}\sum_{n=-s}^{s}\mathfrak{g}(a-n)^2\\&=\delta_{ab}\sum_{n=-s}^{s}\mathfrak{g}(n-a)^2\\&\Downarrow\ n-a\mapsto m\\&=\delta_{ab}\sum_{n=-s}^{s}\mathfrak{g}(m)^2\\&=\delta_{ab}\,||\mathfrak{g}||^2\\&=\delta_{ab}\\&=\langle a|\mathbb{I}|b\rangle.\end{aligned}$$

$\boxed{C8}$

The periodic function

$$\mathcal{G}_\kappa:\mathbb{R}\to\mathbb{R},\qquad \mathcal{G}_\kappa(x)=\sum_{\alpha=-\infty}^{\infty}\mathrm{e}^{-\frac{\kappa\pi}{d}(x+\alpha d)^2}$$

admits the Fourier expansion

$$\mathcal{G}_\kappa(x)=\sum_{\ell=-\infty}^{\infty}a_\ell\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}\ell x}$$

with

$$\begin{aligned}a_\ell&=\frac{1}{d}\int_0^d\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}\ell x}\sum_{\alpha=-\infty}^{\infty}\mathrm{e}^{-\frac{\kappa}{2}\left(\sqrt{\frac{2\pi}{d}}\,(x+\alpha d)\right)^2}dx\\&=\frac{1}{d}\sum_{\alpha=-\infty}^{\infty}\int_0^d\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}\ell x}\mathrm{e}^{-\frac{\kappa}{2}\left(\sqrt{\frac{2\pi}{d}}\,(x+\alpha d)\right)^2}dx.\end{aligned}$$

By denoting $t=\sqrt{\frac{2\pi}{d}}\,(x+\alpha d)$ and using the relation

$$\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\xi t}\,\mathrm{e}^{-at^2}\,dt=\sqrt{\frac{\pi}{a}}\,\mathrm{e}^{-\frac{\xi^2}{4a}}$$

we get

$$\begin{aligned}a_\ell&=\frac{1}{\sqrt{2\pi d}}\sum_{\alpha=-\infty}^{\infty}\int_{\alpha\sqrt{2\pi d}}^{(\alpha+1)\sqrt{2\pi d}}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}\ell\left(t\sqrt{\frac{d}{2\pi}}\right)}\mathrm{e}^{-\frac{\kappa}{2}t^2}dt\\&=\frac{1}{\sqrt{2\pi d}}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\ell t\sqrt{\frac{2\pi}{d}}}\mathrm{e}^{-\frac{\kappa}{2}t^2}dt\\&=\frac{1}{\sqrt{\kappa d}}\,\mathrm{e}^{-\frac{\pi}{\kappa d}\ell^2}\end{aligned}$$

whence, we have the relation

$$\mathcal{G}_\kappa(x)=\frac{1}{\sqrt{\kappa d}}\sum_{\ell=-\infty}^{\infty}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}\ell x}\,\mathrm{e}^{-\frac{\pi}{\kappa d}\ell^2}$$

leading to

$$\begin{aligned}g_\kappa(k)&=\frac{1}{\sqrt{\kappa d}}\sum_{\ell=-\infty}^{\infty}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}\ell k}\,\mathrm{e}^{-\frac{\pi}{\kappa d}\ell^2}\\&=\frac{1}{\sqrt{\kappa d}}\sum_{n=-s}^{s}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kn}\sum_{\alpha=-\infty}^{\infty}\mathrm{e}^{-\frac{\pi}{\kappa d}(n+\alpha d)^2}\\&=\frac{1}{\sqrt{\kappa d}}\sum_{n=-s}^{s}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kn}\,g_{\frac{1}{\kappa}}(n)\\&=\frac{1}{\sqrt{\kappa}}\,F^{-1}[g_{\frac{1}{\kappa}}](k).\end{aligned}$$

$\boxed{C9}$

The periodic function

$$\mathcal{G}^+_{2\kappa}:\mathbb{R}\to\mathbb{R},\qquad \mathcal{G}^+_{2\kappa}(x)=\sum_{\alpha=-\infty}^{\infty}\mathrm{e}^{-\frac{2\kappa\pi}{d}\left(x+\left(\alpha+\frac{1}{2}\right)d\right)^2}$$

admits the Fourier expansion

$$\mathcal{G}^+_{2\kappa}(x)=\sum_{\ell=-\infty}^{\infty}c_\ell\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}\ell x}$$

with

$$\begin{aligned}c_\ell&=\frac{1}{d}\int_0^d\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}\ell x}\sum_{\alpha=-\infty}^{\infty}\mathrm{e}^{-\kappa\left(\sqrt{\frac{2\pi}{d}}\,\left(x+\left(\alpha+\frac{1}{2}\right)d\right)\right)^2}dx\\&=\frac{1}{d}\sum_{\alpha=-\infty}^{\infty}\int_0^d\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}\ell x}\mathrm{e}^{-\kappa\left(\sqrt{\frac{2\pi}{d}}\,\left(x+\left(\alpha+\frac{1}{2}\right)d\right)\right)^2}dx.\end{aligned}$$

By denoting $t=\sqrt{\frac{2\pi}{d}}\,\left(x+\left(\alpha+\frac{1}{2}\right)d\right)$ and using the relation

$$\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\xi t}\,\mathrm{e}^{-at^2}\,dt=\sqrt{\frac{\pi}{a}}\,\mathrm{e}^{-\frac{\xi^2}{4a}}$$

we get

$$\begin{aligned}c_\ell&=\frac{1}{\sqrt{2\pi d}}\sum_{\alpha=-\infty}^{\infty}\int_{(\alpha-1/2)\sqrt{2\pi d}}^{(\alpha+1/2)\sqrt{2\pi d}}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}\ell\left(t\sqrt{\frac{d}{2\pi}}-\left(\alpha+\frac{1}{2}\right)d\right)}\mathrm{e}^{-\kappa t^2}\,dt\\&=\frac{(-1)^\ell}{\sqrt{2\pi d}}\sum_{\alpha=-\infty}^{\infty}\int_{(\alpha-1/2)\sqrt{2\pi d}}^{(\alpha+1/2)\sqrt{2\pi d}}\mathrm{e}^{-\mathrm{i}\ell t\sqrt{\frac{2\pi}{d}}}\mathrm{e}^{-\kappa t^2}\,dt\\&=\frac{(-1)^\ell}{\sqrt{2\pi d}}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\ell t\sqrt{\frac{2\pi}{d}}}\mathrm{e}^{-\kappa t^2}\,dt\\&=\frac{(-1)^\ell}{\sqrt{2\kappa d}}\,\mathrm{e}^{-\frac{\pi}{2\kappa d}\ell^2}\end{aligned}$$

whence

$$G^+_{2\kappa}(x) = \frac{1}{\sqrt{2\kappa d}} \sum_{\ell=-\infty}^{\infty} \mathrm{e}^{\frac{2\pi \mathrm{i}}{d}\ell x}\,(-1)^\ell\,\mathrm{e}^{-\frac{\pi}{2\kappa d}\ell^2}.$$

Particularly, we have

$$\begin{aligned}
g^+_{2\kappa}(m) = \mathcal{G}^+_{2\kappa}(m) &= \frac{1}{\sqrt{2\kappa d}} \sum_{\ell=-\infty}^{\infty} \mathrm{e}^{\frac{2\pi \mathrm{i}}{d} m\ell}\,(-1)^\ell\,\mathrm{e}^{-\frac{\pi}{2\kappa d}\ell^2}\\
&= \frac{1}{\sqrt{2\kappa d}} \sum_{n=-s}^{s} \sum_{\alpha=-\infty}^{\infty} \mathrm{e}^{\frac{2\pi \mathrm{i}}{d} m(n+\alpha d)}\,(-1)^{(n+\alpha d)}\mathrm{e}^{-\frac{\pi}{2\kappa d}(n+\alpha d)^2}\\
&= \frac{1}{\sqrt{d}} \sum_{n=-s}^{s} \mathrm{e}^{\frac{2\pi \mathrm{i}}{d} mn}\,\frac{(-1)^n}{\sqrt{2\kappa}} \sum_{\alpha=-\infty}^{\infty} (-1)^\alpha\,\mathrm{e}^{-\frac{\pi}{2\kappa d}(n+\alpha d)^2}
\end{aligned}$$

whence

$$F[g^+_{2\kappa}](n) = \frac{(-1)^n}{\sqrt{2\kappa}} \sum_{\alpha=-\infty}^{\infty} (-1)^\alpha\,\mathrm{e}^{-\frac{\pi}{2\kappa d}(n+\alpha d)^2}$$

and we get

$$\begin{aligned}
F[g^+_{2\kappa}](2k) &= \frac{1}{\sqrt{2\kappa}} \sum_{\alpha=-\infty}^{\infty} (-1)^\alpha\,\mathrm{e}^{-\frac{\pi}{2\kappa d}(2k+\alpha d)^2}\\
&= \frac{1}{\sqrt{2\kappa}} \sum_{\alpha=-\infty}^{\infty} \mathrm{e}^{-\frac{\pi}{2\kappa d}(2k+2\alpha d)^2} - \frac{1}{\sqrt{2\kappa}} \sum_{\alpha=-\infty}^{\infty} \mathrm{e}^{-\frac{\pi}{2\kappa d}(2k+(2\alpha+1)d)^2}\\
&= \frac{1}{\sqrt{2\kappa}} \sum_{\alpha=-\infty}^{\infty} \mathrm{e}^{-\frac{2\pi}{\kappa d}(k+\alpha d)^2} - \frac{1}{\sqrt{2\kappa}} \sum_{\alpha=-\infty}^{\infty} \mathrm{e}^{-\frac{2\pi}{\kappa d}(k+\alpha d+\frac{d}{2})^2}\\
&= \frac{1}{\sqrt{2\kappa}} \left(g_{\frac{2}{\kappa}}(k) - g^+_{\frac{2}{\kappa}}(k) \right).
\end{aligned}$$

$\boxed{C10}$

By using $\boxed{C8}$, $\boxed{C9}$, the relation

$$\begin{aligned}
F[g_{2\kappa}](2k) &= \frac{1}{\sqrt{2\kappa}}\, g_{\frac{1}{2\kappa}}(2k)\\
&= \frac{1}{\sqrt{2\kappa}} \sum_{\alpha=-\infty}^{\infty} \mathrm{e}^{-\frac{\pi}{2\kappa d}(2k+\alpha d)^2}\\
&= \frac{1}{\sqrt{2\kappa}} \sum_{\alpha=-\infty}^{\infty} \mathrm{e}^{-\frac{\pi}{2\kappa d}(2k+2\alpha d)^2} + \frac{1}{\sqrt{2\kappa}} \sum_{\alpha=-\infty}^{\infty} \mathrm{e}^{-\frac{\pi}{2\kappa d}(2k+(2\alpha+1)d)^2}\\
&= \frac{1}{\sqrt{2\kappa}} \sum_{\alpha=-\infty}^{\infty} \mathrm{e}^{-\frac{2\pi}{\kappa d}(k+\alpha d)^2} + \frac{1}{\sqrt{2\kappa}} \sum_{\alpha=-\infty}^{\infty} \mathrm{e}^{-\frac{2\pi}{\kappa d}(k+\alpha d+\frac{d}{2})^2}\\
&= \frac{1}{\sqrt{2\kappa}} (g_{\frac{2}{\kappa}}(k) + g^+_{\frac{2}{\kappa}}(k))
\end{aligned}$$

and the formula

$$\begin{aligned}
\sum_{\alpha,\beta=-\infty}^{\infty} E_{\alpha,\beta} &= \sum_{\substack{\alpha,\,\beta\\ \text{both even}\\ \text{or}\\ \text{both odd}}} E_{\alpha,\beta} + \sum_{\substack{\alpha,\,\beta\\ \text{one even}\\ \text{and}\\ \text{other odd}}} E_{\alpha,\beta}\\
&= \sum_{\mu,\eta=-\infty}^{\infty} E_{\mu+\eta,\mu-\eta} + \sum_{\mu,\eta=-\infty}^{\infty} E_{\mu+\eta+1,\mu-\eta},
\end{aligned}$$

we get (up to a normalizing multiplicative constant)

$$\begin{aligned}
\mathcal{W}_{g_\kappa}(n,k) &= \frac{1}{d} \sum_{m=-s}^{s} \mathrm{e}^{-\frac{4\pi \mathrm{i}}{d} km} \sum_{\alpha,\beta=-\infty}^{\infty} \mathrm{e}^{-\frac{\kappa\pi}{d}(n-m+\alpha d)^2}\mathrm{e}^{-\frac{\kappa\pi}{d}(n+m+\beta d)^2}\\
&= \frac{1}{d} \sum_{m=-s}^{s} \mathrm{e}^{-\frac{4\pi \mathrm{i}}{d} km} \sum_{\mu,\eta=-\infty}^{\infty} \mathrm{e}^{-\frac{\kappa\pi}{d}(n-m+(\mu+\eta)d)^2}\mathrm{e}^{-\frac{\kappa\pi}{d}(n+m+(\mu-\eta)d)^2}\\
&\quad + \frac{1}{d} \sum_{m=-s}^{s} \mathrm{e}^{-\frac{4\pi \mathrm{i}}{d} km} \sum_{\mu,\eta=-\infty}^{\infty} \mathrm{e}^{-\frac{\kappa\pi}{d}(n-m+(\mu+\eta+1)d)^2}\mathrm{e}^{-\frac{\kappa\pi}{d}(n+m+(\mu-\eta)d)^2}\\
&= \frac{1}{d} \sum_{\mu=-\infty}^{\infty} \mathrm{e}^{-\frac{2\kappa\pi}{d}(n+\mu d)^2} \sum_{m=-s}^{s} \mathrm{e}^{-\frac{4\pi \mathrm{i}}{d} km} \sum_{\eta=-\infty}^{\infty} \mathrm{e}^{-\frac{2\kappa\pi}{d}(-m+\eta d)^2}\\
&\quad + \frac{1}{d} \sum_{\mu=-\infty}^{\infty} \mathrm{e}^{-\frac{2\kappa\pi}{d}(n+\frac{d}{2}+\mu d)^2} \sum_{m=-s}^{s} \mathrm{e}^{-\frac{4\pi \mathrm{i}}{d} km} \sum_{\eta=-\infty}^{\infty} \mathrm{e}^{-\frac{2\kappa\pi}{d}(-m+\frac{d}{2}+\eta d)^2}\\
&= \frac{1}{\sqrt{d}} g_{2\kappa}(n)\, F[g_{2\kappa}](2k) + \frac{1}{\sqrt{d}} g^+_{2\kappa}(n)\, F[g^+_{2\kappa}](2k)\\
&= \frac{1}{\sqrt{2\kappa d}}\, g_{2\kappa}(n)\, \left(g_{\frac{2}{\kappa}}(k) + g^+_{\frac{2}{\kappa}}(k) \right) + \frac{1}{\sqrt{2\kappa d}}\, g^+_{2\kappa}(n)\, \left(g_{\frac{2}{\kappa}}(k) - g^+_{\frac{2}{\kappa}}(k) \right).
\end{aligned}$$