

# LINEAR ALGEBRA - Solved Exercises 6

(Real and complex finite-dimensional vector spaces)

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## A PARTIAL EVALUATION A

### Exercise 1

A1 Find a basis  $\mathcal{B}_w$  in the subspace

$$\mathcal{W} = \{x = (x_1, x_2, x_3) \mid x_1 + ix_2 - x_3 = 0\}$$

of  $\mathbb{C}^3$  and indicate the dimension  $\dim_{\mathbb{C}} \mathcal{W}$ .

A2 Extend  $\mathcal{B}_w$  up to a basis  $\mathcal{B}$  of  $\mathbb{C}^3$  and find the coordinates of  $x = (3, -i, 2)$  in the basis  $\mathcal{B}$ .

### Exercise 2

A3 Prove that  $\mathcal{B} = \{v_1 = (1, 1), v_2 = (1, -2)\}$  is a basis of  $\mathbb{R}^2$ .

A4 Find a basis  $\mathcal{B}' = \{v'_1, v'_2\}$  of  $\mathbb{R}^2$  such that the change of basis matrix from  $\mathcal{B}$  to  $\mathcal{B}'$  to be

$$S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

### Exercise 3

Let  $A : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ ,  $A(x_1, x_2) = (x_2, -x_1)$ .

A5 Find the eigenvalues of  $A$ .

A6 Describe the corresponding eigenspaces.

A7 Indicate the diagonal form of  $A$  and the used eigenbasis.

### Exercise 4

Consider the vector space of polynomials

$$\mathcal{V} = \{p = a_0 + a_1X + a_2X^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$$

and the linear operator (derivative with respect to  $X$ )

$$D : \mathcal{V} \rightarrow \mathcal{V}, \quad D(a_0 + a_1X + a_2X^2) = a_1 + 2a_2X.$$

A8 Describe the subspaces  $\text{Ker } D$  and  $\text{Im } D$ .

A9 Find the eigenvalues and the corresponding eigenspaces.

### Exercise 5

A10 Prove that

$$B : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad B(x_1, x_2) = (x_1 + x_2, x_2^2)$$

is not a linear operator.

A11 Find the linear operator  $C : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  with the eigenvalues  $\lambda_1 = 6$ ,  $\lambda_2 = 9$  and the eigenspaces

$$\begin{aligned} \{(x_1, x_2) \mid C(x_1, x_2) = 6(x_1, x_2)\} &= \{\alpha(1, 2) \mid \alpha \in \mathbb{R}\} \\ \{(x_1, x_2) \mid C(x_1, x_2) = 9(x_1, x_2)\} &= \{\alpha(1, -1) \mid \alpha \in \mathbb{R}\} \end{aligned}$$

A12 In  $\mathbb{R}^4$ , consider the subspaces

$$\begin{aligned} \mathcal{W}_1 &= \{(x_1, x_2, x_3, x_4) \mid 2x_1 - x_2 = 0, x_3 + x_4 = 0\}, \\ \mathcal{W}_2 &= \text{span}\{w_1 = (1, 1, 1, 1), w_2 = (0, 1, -2, 0), w_3 = (2, 3, 0, 2)\}. \end{aligned}$$

Find a basis in  $\mathcal{W}_1 \cap \mathcal{W}_2$  and a basis in  $\mathcal{W}_1 + \mathcal{W}_2$ .

## Some definitions, notations and results

1 **Definition.** In a vector space over the field  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ ,

$$\text{span}\{v_1, v_2, \dots, v_n\} \stackrel{\text{def}}{=} \{\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \mid \alpha_k \in \mathbb{K}\}$$

2 **Definition.** In the case of vector space over  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ ,  $\lambda$  is an *eigenvalue* of a linear operator  $A : \mathcal{V} \rightarrow \mathcal{V}$  if  $\lambda \in \mathbb{K}$  and there exists a non-null vector  $x \in \mathcal{V}$  such that  $Ax = \lambda x$ . The *eigenspace* corresponding to  $\lambda$  is  $\{x \mid Ax = \lambda x\}$ .

3 The *complex-conjugate transpose* (also called the *adjoint*) of

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \text{ is } A^\dagger = \bar{A}^T = \begin{pmatrix} \bar{a}_{11} & \bar{a}_{21} & \cdots & \bar{a}_{n1} \\ \bar{a}_{12} & \bar{a}_{22} & \cdots & \bar{a}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{1n} & \bar{a}_{2n} & \cdots & \bar{a}_{nn} \end{pmatrix}.$$

4 The *usual inner product* (also called scalar product) :

$$\begin{aligned} \text{In } \mathbb{R}^3 : & \quad \langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 \\ \text{In } \mathbb{C}^2 : & \quad \langle (x_1, x_2), (y_1, y_2) \rangle = \bar{x}_1 y_1 + \bar{x}_2 y_2. \end{aligned}$$

5 *Orthogonal projection* of  $x$  on  $w \neq 0$  is  $P_w x = \frac{\langle w, x \rangle}{\langle w, w \rangle} w$ .

## B PARTIAL EVALUATION B

### Exercise 1

Find orthonormal bases in the Hilbert spaces:

B1  $\mathcal{H}_1 = \{x = (x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 0\} \subset \mathbb{R}^3$ .

B2  $\mathcal{H}_2 = \{x = (x_1, x_2, x_3) \mid x_1 - x_2 - ix_3 = 0\} \subset \mathbb{C}^3$ .

B3  $\mathcal{H}_3 = \left\{ A = \begin{pmatrix} 3x_0 & x_1 + ix_2 \\ x_1 - ix_2 & 4x_0 \end{pmatrix} \mid x_0, x_1, x_2 \in \mathbb{R} \right\}$

with  $\langle A, B \rangle = \text{tr}(AB)$ .

### Exercise 2

The Hadamard operator is  $H : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ ,

$$H(x_0, x_1) = \left( \frac{1}{\sqrt{2}}x_0 + \frac{1}{\sqrt{2}}x_1, \frac{1}{\sqrt{2}}x_0 - \frac{1}{\sqrt{2}}x_1 \right).$$

B4 Prove that  $H$  is a self-adjoint and unitary operator.

B5 Compute  $H^2$ .

### Exercise 3

B6 Find  $\alpha$  and  $\beta$  such that  $A : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ ,

$$A(x_1, x_2) = (x_1 + (\alpha + 3i)x_2, (2 + \beta i)x_1 + x_2)$$

to be a self-adjoint operator.

### Exercise 4

Let  $\mathcal{H}$  be the complex Hilbert space of all the functions

$$\psi : \{0, 1, 2\} \rightarrow \mathbb{C}$$

with the scalar product  $\langle \psi, \varphi \rangle = \sum_{k=0}^2 \overline{\psi(k)} \varphi(k)$ ,

and let  $\Pi : \mathcal{H} \rightarrow \mathcal{H} : \psi \mapsto \Pi\psi$ , be the linear operator  $(\Pi\psi)(k) = \psi(2-k)$ .

B7 Prove that  $\Pi$  is a self-adjoint and unitary operator.

B8 Find the eigenvalues of  $\Pi$ .

B9 Find the corresponding eigenspaces of  $\Pi$ .

### Exercise 5

B10 Prove that

$$\left\{ |e_0\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, |e_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, |e_2\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \right\}$$

is an orthonormal basis of  $\mathbb{C}^3$ .

B11 Find the eigenvalues of the operator  $U : \mathbb{C}^3 \rightarrow \mathbb{C}^3$ ,

$$U = |e_0\rangle\langle e_0| + i|e_1\rangle\langle e_1| - |e_2\rangle\langle e_2|.$$

B12 Prove that  $U$  is a unitary operator.

6 **Definition** Norm of a vector  $\|x\| = \sqrt{\langle x, x \rangle}$

7 **Theorem** In a Hilbert space  $\mathcal{H}$ , if  $v_1, v_2$  are linearly indep., then  $w_1 = v_1$  and  $w_2 = v_2 - \frac{\langle w_1, v_2 \rangle}{\langle w_1, w_1 \rangle} w_1$  are orthogonal.

8 **Theorem** In a complex Hilbert space  $\mathcal{H}$  :

$A : \mathcal{H} \rightarrow \mathcal{H}$  is *self-adjoint*  $\Leftrightarrow$  its matrix in an orthonormal basis is Hermitian ( $A^\dagger = A$ ).

$A : \mathcal{H} \rightarrow \mathcal{H}$  is *unitary*  $\Leftrightarrow$  its matrix in an orthonormal basis is unitary ( $A^\dagger A = \mathbb{I}$ ).

9 **Dirac's notation** in  $\mathbb{C}^2$ . If to each  $x = (x_1, x_2) \in \mathbb{C}^2$ , we

$$\text{associate the matrices } |x\rangle = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \langle x| = (\bar{x}_1 \quad \bar{x}_2),$$

$$\text{then } \langle x|y\rangle = (\bar{x}_1 \quad \bar{x}_2) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \bar{x}_1 y_1 + \bar{x}_2 y_2 = \langle x, y \rangle.$$

For  $a = (a_1, a_2)$  and  $b = (b_1, b_2)$ , the formula  $A = |a\rangle\langle b|$  defines the operator  $A : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ ,  $A|x\rangle = |a\rangle\langle b|x\rangle$  with the matrix  $A = |a\rangle\langle b| = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} (\bar{b}_1 \quad \bar{b}_2) = \begin{pmatrix} a_1 \bar{b}_1 & a_1 \bar{b}_2 \\ a_2 \bar{b}_1 & a_2 \bar{b}_2 \end{pmatrix}.$

## SOLUTIONS for PARTIAL EVALUATION A

**A1** We have  $\mathcal{W} = \{ (x_1, x_2, x_3) \mid x_1 + ix_2 - x_3 = 0 \}$   
 $= \{ (x_1, x_2, x_1 + ix_2) \mid x_1, x_2 \in \mathbb{R} \}$   
 $= \{ x_1(1, 0, 1) + x_2(0, 1, i) \mid x_1, x_2 \in \mathbb{R} \}$   
 $= \text{span}\{ (1, 0, 1), (0, 1, i) \}$

But,  $\alpha(1, 0, 1) + \beta(0, 1, i) = (0, 0, 0) \Rightarrow \alpha = \beta = 0$ .  
 Consequently,  $\{(1, 0, 1), (0, 1, i)\}$  is a basis and  $\dim_{\mathbb{C}} \mathcal{W} = 2$ .

**A2** We have to add a vector which does not belong to  $\mathcal{W}$ .  
 For example,  $\{(1, 0, 1), (0, 1, i), (1, 0, 0)\}$  is a basis and  
 $\alpha(1, 0, 1) + \beta(0, 1, i) + \gamma(1, 0, 0) = (3, -i, 2)$   
 $\Rightarrow \alpha = 1, \beta = -i, \gamma = 2$ . The coordinates of  $x$  are 1,  $-i$ , 2.

**A3** Any  $(x_1, x_2) \in \mathbb{R}^2$  is a linear combination of  $v_1$  and  $v_2$ :  
 $(x_1, x_2) = \alpha(1, 1) + \beta(1, -2)$  for  $\alpha = \frac{2x_1 + x_2}{3}, \beta = \frac{x_1 - x_2}{3}$ .  
 The vectors  $v_1$  and  $v_2$  are linearly independent:  
 $\alpha(1, 1) + \beta(1, -2) = (0, 0) \Rightarrow \alpha = \beta = 0$ .

**A4** By definition,  $S$  is a matrix  

$$S = \begin{pmatrix} \alpha_1^1 & \alpha_2^1 \\ \alpha_1^2 & \alpha_2^2 \end{pmatrix}$$
 such that  $v_1' = \alpha_1^1 v_1 + \alpha_2^1 v_2$   
 $v_2' = \alpha_1^2 v_1 + \alpha_2^2 v_2$ .  
 In our case:  $v_1' = v_1 + v_2 = (2, -1), v_2' = v_1 - v_2 = (0, 3)$ .

**A5** In the canonical basis  $\{(1,0), (0,1)\}$ , the matrix of  $A$  is  

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{vmatrix} 0-\lambda & 1 \\ -1 & 0-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 1 = 0 \Rightarrow \begin{matrix} \lambda_1 = i, \\ \lambda_2 = -i. \end{matrix}$$

**A6** The corresponding eigenspaces are:  
 $\{ (x_1, x_2) \mid A(x_1, x_2) = i(x_1, x_2) \} = \{ \alpha(1, i) \mid \alpha \in \mathbb{R} \}$   
 $\{ (x_1, x_2) \mid A(x_1, x_2) = -i(x_1, x_2) \} = \{ \alpha(1, -i) \mid \alpha \in \mathbb{R} \}$ .

**A7** The linear operator has the diagonal matrix  
 $A' = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$  in the eigenbasis  $\mathcal{B}' = \{v_1' = (1, i), v_2' = (1, -i)\}$ .

**A8**  $\text{Ker } D = \{ p \mid Dp = 0 \} = \{ a_0 + a_1X + a_2X^2 \mid a_1 + 2a_2X = 0 \}$   
 $= \{ a_0 \mid a_0 \in \mathbb{R} \} = \text{space of constant polynomials}$ .  
 $\text{Im } D = \{ Dp \mid p \in \mathcal{V} \} = \{ a_1 + 2a_2X \mid a_1, a_2 \in \mathbb{R} \}$   
 $= \text{space of polynomials of degree less than or equal to 1}$ .

**A9** In the canonical basis  $\{1, X, X^2\}$ , the matrix of  $D$  is  

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 2 \\ 0 & 0 & -\lambda \end{vmatrix} = \lambda^3 \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$$
.

The eigenspace corresponding to the eigenvalue 0 is  
 $\{ p \mid Dp = 0 \} = \{ a_0 + a_1X + a_2X^2 \mid a_1 + 2a_2X = 0 \}$   
 $= \{ a_0 \mid a_0 \in \mathbb{R} \} = \{ a_0 \mid a_0 \in \mathbb{R} \}$ . A basis is  $\{1\}$ .

**A10** Since  $B(\alpha(x_1, x_2)) = B(\alpha x_1, \alpha x_2) = (\alpha x_1 + \alpha x_2, \alpha^2 x_2^2)$ ,  
 $\alpha B(x_1, x_2) = \alpha(x_1 + x_2, x_2^2) = (\alpha x_1 + \alpha x_2, \alpha x_2^2)$ ,  
 the relation  $B(\alpha(x_1, x_2)) = \alpha B(x_1, x_2)$  is not satisfied.

**A11** Any linear operator  $C: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  has the form  
 $C(x_1, x_2) = (a_{11}x_1 + a_{12}x_2, a_{21}x_1 + a_{22}x_2)$ .  
 But,  $C(1, 2) = 6(1, 2) \Rightarrow (a_{11} + 2a_{12}, a_{21} + 2a_{22}) = (6, 12)$   
 $C(1, -1) = 9(1, -1) \Rightarrow (a_{11} - a_{12}, a_{21} - a_{22}) = (9, -9)$ ,  
 and 
$$\begin{cases} a_{11} + 2a_{12} = 6 \\ a_{21} + 2a_{22} = 12 \\ a_{11} - a_{12} = 9 \\ a_{21} - a_{22} = -9 \end{cases} \Rightarrow \begin{cases} a_{11} = 8 \\ a_{12} = -1 \\ a_{21} = -2 \\ a_{22} = 7. \end{cases}$$

Consequently,  $C(x_1, x_2) = (8x_1 - x_2, -2x_1 + 7x_2)$ .

**A12** Since  $\mathcal{W}_1 = \{ (x_1, 2x_1, x_3, -x_3) \mid x_1, x_3 \in \mathbb{R} \}$ ,  
 the set  $\{v_1 = (1, 2, 0, 0), v_2 = (0, 0, 1, -1)\}$  is a basis of  $\mathcal{W}_1$ .  
 Since  $w_3 = 2w_1 + w_2$ , we have  $\mathcal{W}_2 = \text{span}\{w_1, w_2\}$ , and  
 the set  $\{w_1 = (1, 1, 1, 1), w_2 = (0, 1, -2, 0)\}$  is a basis of  $\mathcal{W}_2$ .  
 The vector  $x = \alpha w_1 + \beta w_2 = (\alpha, \alpha + \beta, \alpha - 2\beta, \alpha)$  from  $\mathcal{W}_2$   
 belongs to  $\mathcal{W}_1 \Leftrightarrow \begin{cases} 2\alpha - (\alpha + \beta) = 0 \\ (\alpha - 2\beta) + \alpha = 0 \end{cases} \Leftrightarrow \alpha = \beta$ . We get  
 $\mathcal{W}_1 \cap \mathcal{W}_2 = \{ \alpha(1, 2, -1, 1) \mid \alpha \in \mathbb{R} \}$  with the basis  $\{(1, 2, -1, 1)\}$ .  
 $\mathcal{W}_1 + \mathcal{W}_2 = \{ (\alpha v_1 + \beta v_2) + (\gamma w_1 + \delta w_2) \mid \alpha, \beta, \gamma, \delta \in \mathbb{R} \}$ .  
 Since  $w_2 = v_1 - v_2 - w_1$  we get  $\mathcal{W}_1 + \mathcal{W}_2 = \text{span}\{v_1, v_2, w_1\}$   
 A basis of  $\mathcal{W}_1 + \mathcal{W}_2$  is  $\{v_1, v_2, w_1\}$ .

## SOLUTIONS for PARTIAL EVALUATION B

**B1**  $\mathcal{H}_1 = \{ (x_1, x_2, -x_1 - x_2) \mid x_1, x_2 \in \mathbb{R} \}$   
 $= \{ x_1(1, 0, -1) + x_2(0, 1, -1) \mid x_1, x_2 \in \mathbb{R} \} \Rightarrow$   
 $\{v_1 = (1, 0, -1), v_2 = (0, 1, -1)\}$  is a basis of  $\mathcal{H}_1$ .

By using [7] (particular case of Gram-Schmidt), we get  
 the orthogonal basis  $\{w_1, w_2\}$ , where  $w_1 = (1, 0, -1)$ , and  
 $w_2 = (0, 1, -1) - \frac{1}{2}(1, 0, -1) = (-\frac{1}{2}, 1, -\frac{1}{2})$ . Consequently,  
 $\left\{ u_1 = \frac{w_1}{\|w_1\|} = \left( \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right), u_2 = \frac{w_2}{\|w_2\|} = \left( -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \right) \right\}$   
 is an orthonormal basis of  $\mathcal{H}_1$ .

**B2** The method used in the case of  $\mathcal{H}_1$  leads to  
 $\mathcal{H}_2 = \{ x_1(1, 1, 0) + x_3(0, -i, 1) \mid x_1, x_3 \in \mathbb{R} \}$ ,  
 $\{v_1 = (1, 1, 0), v_2 = (0, -i, 1)\}$  is a basis of  $\mathcal{H}_2$ ,  
 $\{w_1 = (1, 1, 0), w_2 = (\frac{i}{2}, -\frac{i}{2}, 1)\}$  an orthogonal basis of  $\mathcal{H}_2$ ,  
 $\left\{ u_1 = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right), u_2 = \left( \frac{i}{\sqrt{6}}, -\frac{i}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right) \right\}$  orthonormal basis.

**B3** The method used in the case of  $\mathcal{H}_1$  leads to  
 $\mathcal{H}_3 = \left\{ x_0 \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} + x_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \mid x_0, x_1, x_2 \in \mathbb{R} \right\}$ ,  
 $\left\{ \begin{pmatrix} \frac{3}{5} & 0 \\ 0 & \frac{4}{5} \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & 0 \end{pmatrix} \right\}$  is orthonormal basis.

**B4** The matrix of  $H$  in the orthonormal basis  $\{(1,0), (0,1)\}$  is  
 $H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$  and  $H^\dagger = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$ . We use [8].  
 $H$  is self-adjoint and unitary because  $H = H^\dagger$  and  $HH^\dagger = \mathbb{I}$ .

**B5** Matrix of  $H^2$  is  $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{I}$ .  
 Consequently  $H^2(x_0, x_1) = (x_0, x_1)$ .

**B6** The matrix of  $A$  in the orthonormal basis  $\{(1,0), (0,1)\}$  is  
 $A = \begin{pmatrix} 1 & \alpha + 3i \\ 2 + \beta i & 1 \end{pmatrix}$  and  $A^\dagger = \begin{pmatrix} 1 & 2 - \beta i \\ \alpha - 3i & 1 \end{pmatrix}$ . We use [8].  
 We have  $A = A^\dagger$  if and only if  $\alpha = 2$  and  $\beta = -3$ .

**B7** The matrix of  $\Pi$  in the orthonormal basis  $\{\delta_0, \delta_1, \delta_2\}$ ,  
 $\delta_0(0) = 1, \delta_1(0) = 0, \delta_2(0) = 0$   
 $\delta_0(1) = 0, \delta_1(1) = 1, \delta_2(1) = 0$  is  $\Pi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$   
 $\delta_0(2) = 0, \delta_1(2) = 0, \delta_2(2) = 1$   
 $\Pi \delta_0(0) = \delta_0(2) = 0 \Rightarrow \Pi \delta_0 = 0 \delta_0 + 0 \delta_1 + 1 \delta_2$ , etc.  
 because  $\Pi \delta_0(1) = \delta_0(1) = 0$   
 $\Pi \delta_0(2) = \delta_0(0) = 1$  We use [8].

$\Pi$  is self-adjoint and unitary because  $\Pi = \Pi^\dagger$  and  $\Pi \Pi^\dagger = \mathbb{I}$ .

**B8** Eigenvalues:  $\begin{vmatrix} 0-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 0-\lambda \end{vmatrix} = 0 \Rightarrow \begin{matrix} \lambda_0 = -1, \\ \lambda_1 = 1 \\ \lambda_2 = 1. \end{matrix}$

**B9** Eigenspaces:  $\{ \psi \mid \Pi \psi = (-1)\psi \} = \text{span}\{ \psi_0 \}$ ,  
 $\{ \psi \mid \Pi \psi = 1 \psi \} = \text{span}\{ \psi_1, \psi_2 \}$  where  
 $\psi_0(0) = \frac{1}{\sqrt{2}}, \psi_1(0) = \frac{1}{\sqrt{2}}, \psi_2(0) = 0$   
 $\psi_0(1) = 0, \psi_1(1) = 0, \psi_2(1) = 1$   
 $\psi_0(2) = -\frac{1}{\sqrt{2}}, \psi_1(2) = \frac{1}{\sqrt{2}}, \psi_2(2) = 0$ .

**B10** We have:  $\langle e_0 | e_0 \rangle = \langle e_1 | e_1 \rangle = \langle e_2 | e_2 \rangle = 1$  that is  $\langle e_j | e_k \rangle = \delta_{jk}$ .  
 $\langle e_0 | e_1 \rangle = \langle e_0 | e_2 \rangle = \langle e_1 | e_2 \rangle = 0$ ,

$U|e_0\rangle = |e_0\rangle \langle e_0 | e_0 \rangle + i|e_1\rangle \langle e_1 | e_0 \rangle - |e_2\rangle \langle e_2 | e_0 \rangle = |e_0\rangle$   
 $U|e_1\rangle = |e_0\rangle \langle e_0 | e_1 \rangle + i|e_1\rangle \langle e_1 | e_1 \rangle - |e_2\rangle \langle e_2 | e_1 \rangle = i|e_1\rangle$   
 $U|e_2\rangle = |e_0\rangle \langle e_0 | e_2 \rangle + i|e_1\rangle \langle e_1 | e_2 \rangle - |e_2\rangle \langle e_2 | e_2 \rangle = -|e_2\rangle$

it follows that  
 $\lambda_0 = 1$  is an eigenvalue of  $U$  with  $|e_0\rangle$  an eigenvector,  
 $\lambda_1 = i$  is an eigenvalue of  $U$  with  $|e_1\rangle$  an eigenvector,  
 $\lambda_2 = -1$  is an eigenvalue of  $U$  with  $|e_2\rangle$  an eigenvector.

**B12** The matrix  $U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -1 \end{pmatrix}$  of  $U$  in the orthonormal  
 basis  $\{|e_0\rangle, |e_1\rangle, |e_2\rangle\}$  is a unitary matrix because  $UU^\dagger = \mathbb{I}$ .