An overview on $\boxed{\textbf{LINEAR DIFFERENTIAL EQUATIONS}}$

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FIRST-ORDER DIFFERENTIAL EQUATIONS

- A *solution* of $\boxed{F(x,y,y')=0}$ is any function $\varphi:(a,b)\to\mathbb{R}$ satisfying $F(x,\varphi(x),\varphi'(x))=0$, for any $x\in(a,b)$.
- A *solution* of $\boxed{y'=f(x,y)}$ is any function $\varphi:(a,b)\to\mathbb{R}$ satisfying $\varphi'(x)=f(x,\varphi(x))$, for any $x\in(a,b)$.
- The *primitives* of a continuous function $f:(a,b)\to\mathbb{R}$ *are* all the functions $F:(a,b)\to\mathbb{R}$, $\boxed{F(x)=\int_{x_0}^{x} f(t)\,dt}$, $x_0\in(a,b)$ is fixed. We have $F'(x)=f(x)$, that is $\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{x_0}^{x} f(t)\,dt\right)=f(x)}$ for any $x\in(a,b)$.

- **Separable equations**.

The solution $y(x)$ of $\boxed{y'=f(x)\,g(y)}$ is defined by $\boxed{\int_{y_0}^{y}\frac{1}{g(u)}du=\int_{x_0}^{x} f(t)\,dt+C}$ x_0, y_0 constants, $g(y_0)\neq 0$.

Proof. $\int_{y_0}^{y}\frac{1}{g(u)}du=\int_{x_0}^{x}f(t)\,dt+C \overset{\frac{\mathrm{d}}{\mathrm{d}x}}{\Longrightarrow} \frac{\mathrm{d}}{\mathrm{d}y}\left(\int_{y_0}^{y}\frac{1}{g(u)}du\right)\frac{\mathrm{d}y}{\mathrm{d}x}=f(x) \Rightarrow \frac{1}{g(y)}y'=f(x)$.

- **Homogeneous equations**.

$\boxed{y'=f(\frac{y}{x})} \xRightarrow[y(x)\,=\,x\,z(x)]{\text{change of function}}$ separable equation.

Proof. $y(x)=x\,z(x) \overset{\frac{\mathrm{d}}{\mathrm{d}x}}{\Longrightarrow} y'(x)=z(x)+x\,z'(x) \;\Rightarrow\; z'=\frac{1}{x}(f(z)-z)$.

- **Linear equation**.

$\boxed{y'=f(x)y}$ has the general solution $\boxed{y(x)=C\,\mathrm{e}^{\int_{x_0}^{x}f(t)dt}}$

Proof. $\frac{y'}{y}=f(x) \;\mapsto\; (\ln y)'=f(x) \;\mapsto\; \ln y(x)=\int_{x_0}^{x}f(t)\,dt+\ln C$.

- **Method of the variation of parameter**.

$\boxed{y'=f(x)y+g(x)}$ admits a particular solution of the form $y_p(x)=C(x)\,\mathrm{e}^{\int_{x_0}^{x}f(t)dt}$.

Proof. $y_p'=f(x)y_p+g(x)\Rightarrow C'(x)=g(x)\,\mathrm{e}^{-\int_{x_0}^{x}f(t)dt}\Rightarrow C(x)=\dots$

- **Linear non-homogeneous equation**.

$\boxed{\begin{array}{c}\text{general solution of}\\ y'=f(x)y+g(x)\end{array}}=\boxed{\begin{array}{c}\text{general solution of}\\ y'=f(x)y\end{array}}+\boxed{\begin{array}{c}\text{a particular solution of}\\ y'=f(x)y+g(x)\end{array}}$

Proof. $\left.\begin{array}{l}y'=f(x)y\\ y_p'=f(x)y_p+g(x)\end{array}\right\}\Rightarrow (y+y_p)'=f(x)(y+y_p)+g(x)$.

- **Bernoulli's equation**.

$\boxed{y'=f(x)y+g(x)y^{\alpha}} \xRightarrow[z(x)=(y(x))^{1-\alpha}]{\text{change of function}}$ linear equation.

Proof. $y^{-\alpha}y'=f(x)y^{1-\alpha}+g(x) \;\Rightarrow\; \frac{1}{1-\alpha}(y^{1-\alpha})'=f(x)y^{1-\alpha}+g(x)$.

- **Riccati's equation**. If we know a particular solution y_p, then

$\boxed{y'=f(x)y^2+g(x)y+h(x)} \xRightarrow[y(x)=y_p(x)+\frac{1}{z(x)}]{\text{change of function}}$ linear equation.

Proof. $y'=f(x)y^2+g(x)y+h(x) \;\Rightarrow\; z'=-(2f(x)\,y_p(x)-g(x))z-f(x)$.

- $\boxed{y'=f(x)}$ can also be written as $\boxed{\frac{\mathrm{d}y}{\mathrm{d}x}=f(x)}$ or $\boxed{f(x)dx-dy=0.}$

- **Symmetric equations**.

Definition: $\begin{array}{l}x=\varphi(t)\\ y=\psi(t)\end{array}$ is a solution of $\boxed{P(x,y)dx+Q(x,y)dy=0}$ if $P(\varphi(t),\psi(t))\varphi'(t)+Q(\varphi(t),\psi(t))\psi'(t)=0$, for any t.

- **Exact equations**. In a suitable domain of definition,

$\boxed{\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}}\Rightarrow\boxed{\begin{array}{c}F(x,y)=\int_{(x_0,y_0)}^{(x,y)}Pdx+Qdy \text{ satisfies}\\ P(x,y)dx+Q(x,y)dy=dF\end{array}}$ Solution of $\boxed{dF=0}$ is $F(x,y)=const.$

- **Method of integrating factor** $\mu(x,y)$.

For $\boxed{P(x,y)dx+Q(x,y)dy=0}$ we look for $\mu(x,y)$ such that $\boxed{\frac{\partial(\mu Q)}{\partial x}=\frac{\partial(\mu P)}{\partial y}}$

- **Method of power series**.

We look for solutions having the form of a convergent power series $\boxed{y(x)=\sum_{n=0}^{\infty}a_n\,x^n}$

LINEAR DIFFERENTIAL EQUATIONS

- For any $a_0\neq 0, a_1,\dots,a_n\in\mathbb{R}$, by denoting $D^k=\frac{\mathrm{d}^k}{\mathrm{d}x^k}$ we get

$\boxed{\begin{array}{l}Ly=a_0y^{(n)}+\cdots+a_{n-1}y'+a_ny\\ \quad=(a_0D^n+\cdots+a_{n-1}D+a_n)y=P(D)y\end{array}}$ $P(\lambda)=a_0\lambda^n+\cdots+a_n$ is the characteristic polynomial

- **Linear equation with constant coefficients**

Theorem: Solutions $y:\mathbb{R}\to\mathbb{R}$ of $\boxed{Ly=0}$ form a vector space of dimension n.

- **General solution of** $Ly=0$.

$y=c_1\,y_1+\dots+c_n\,y_n$ is the *general solution* $\Leftrightarrow$ $\{y_1,\dots,y_n\}$ is a basis in the vector space of solutions

- **Particular solutions of** $Ly=0$.

$\boxed{y(x)=\mathrm{e}^{\lambda x} \text{ is a solution of } Ly=0}\Leftrightarrow\boxed{P(\lambda)=0}$

- In the case $\lambda=\alpha+\beta\mathrm{i}$, $\boxed{\mathrm{e}^{(\alpha+\beta\mathrm{i})x}\overset{\mathrm{def}}{=}\mathrm{e}^{\alpha x}\cos\beta x+\mathrm{i}\,\mathrm{e}^{\alpha x}\sin\beta x}$

- $P(\alpha+\beta\mathrm{i})=0\Rightarrow\begin{cases}y_1(x)=\mathrm{e}^{\alpha x}\cos\beta x\\ y_2(x)=\mathrm{e}^{\alpha x}\sin\beta x\end{cases}$ linearly independent solutions of $Ly=0$.

- λ root of P of multiplicity k $\Rightarrow\begin{cases}y_1(x)=\mathrm{e}^{\lambda x}\\ y_2(x)=x\,\mathrm{e}^{\lambda x}\\ \dots\dots\dots\dots\\ y_k(x)=x^{k-1}\,\mathrm{e}^{\lambda x}\end{cases}$ are linearly independent solutions of $Ly=0$.

- **Method of the variation of parameters**.

$\boxed{\begin{array}{c}y=c_1\,y_1+\dots+c_n\,y_n\\ \text{general solution}\\ \text{of}\quad Ly=0\end{array}}\Rightarrow\boxed{\begin{array}{c}Ly=f \text{ admits a particular}\\ \text{solution of the form}\\ y(x)=c_1(x)\,y_1(x)+\dots+c_n(x)\,y_n(x)\end{array}}$

The functions $c_1(x),\dots,c_n(x)$ can be obtained by solving the system
$$\begin{cases}c_1'(x)\,y_1(x)+\dots+c_n'(x)\,y_n(x)=0\\ c_1'(x)\,y_1'(x)+\dots+c_n'(x)\,y_n'(x)=0\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ c_1'(x)\,y_1^{(n-2)}(x)+\dots+c_n'(x)\,y_n^{(n-2)}(x)=0\\ c_1'(x)\,y_1^{(n-1)}(x)+\dots+c_n'(x)\,y_n^{(n-1)}(x)=\frac{f(x)}{a_0}.\end{cases}$$

- **Linear non-homogeneous equation**.

$\boxed{\begin{array}{c}\text{general solution}\\ \text{of}\;\; Ly=f\end{array}}=\boxed{\begin{array}{c}\text{general solution}\\ \text{of}\;\; Ly=0\end{array}}+\boxed{\begin{array}{c}\text{a particular solution}\\ \text{of}\;\; Ly=f.\end{array}}$

- **Euler's equation.**

$\boxed{a_0x^ny^{(n)}+\dots+a_{n-1}xy'+a_ny=0}\xRightarrow[x=\mathrm{e}^t]{\text{change}}$ linear equation with constant coefficients

SYSTEMS OF LINEAR EQUATIONS

- **Systems of linear equations with constant coefficients.**

$\boxed{Y'=AY+F}$ $Y=\begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix}$, $F=\begin{pmatrix}f_1\\ \vdots\\ f_n\end{pmatrix}$, $A=\begin{pmatrix}a_{11}&\dots&a_{1n}\\ \vdots&&\vdots\\ a_{n1}&\dots&a_{nn}\end{pmatrix}$.

- *Theorem:* Solutions $Y:\mathbb{R}\to\mathbb{R}^n$ of $\boxed{Y'=AY}$ form a vector space of dimension n.

- **Wronski matrix**

$Y_1=\begin{pmatrix}y_{11}\\ \vdots\\ y_{n1}\end{pmatrix},\dots,Y_n=\begin{pmatrix}y_{1n}\\ \vdots\\ y_{nn}\end{pmatrix}$ form a basis $\Leftrightarrow W=\begin{pmatrix}y_{11}&\cdots&y_{1n}\\ \vdots&&\vdots\\ y_{n1}&\dots&y_{nn}\end{pmatrix}$ has $\det W\neq 0$

- **Particular solutions of** $Y'=AY$.

$Y=w\mathrm{e}^{\lambda x}=\begin{pmatrix}w_1\\ \vdots\\ w_n\end{pmatrix}\mathrm{e}^{\lambda x}$ is a solution of $Y'=AY$ $\Leftrightarrow$ $\boxed{A\begin{pmatrix}w_1\\ \vdots\\ w_n\end{pmatrix}=\lambda\begin{pmatrix}w_1\\ \vdots\\ w_n\end{pmatrix}}$

- $\lambda=\alpha+\beta\mathrm{i}\;\Rightarrow\; Y_1(x)=\mathfrak{Re}(w\mathrm{e}^{\lambda x}),\; Y_2(x)=\mathfrak{Im}(w\mathrm{e}^{\lambda x})$ solutions.

- **General solution** of $Y'=AY$ can be written as

$Y=c_1Y_1+\cdots+c_nY_n=W\cdot C$, where $C=\begin{pmatrix}c_1\\ \vdots\\ c_n\end{pmatrix}$.

- **Method of the variation of parameters**.

$\boxed{\begin{array}{c}Y=W\cdot C \text{ general}\\ \text{solution of}\quad Y'=AY\end{array}}\Rightarrow\boxed{\begin{array}{c}Y'=AY+F \text{ admits a}\\ \text{solution } Y(x)=W\cdot C(x)\end{array}}$ with $C'=W^{-1}F$

DIFFERENTIAL EQUATIONS - Exercises

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A Exercises

1 $\boxed{y''-5y'+6y=0}$

Solution : $\lambda^2-5\lambda+6=0 \Rightarrow \begin{array}{l}\lambda_1=2\\ \lambda_2=3\end{array} \Rightarrow y(x)=C_1\,e^{2x}+C_2\,e^{3x}.$

2 $\boxed{y''-4y'+4y=0}$

Solution : $\lambda^2-4\lambda+4=0 \Rightarrow \lambda_1=\lambda_2=2 \Rightarrow y(x)=C_1\,e^{2x}+C_2\,x\,e^{2x}.$

3 $\boxed{y''-y'+y=0}$

Solution : $\lambda^2-\lambda+1=0 \Rightarrow \lambda_{1,2}=\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

$\Rightarrow y(x)=C_1\,e^{\frac{1}{2}x}\cos\frac{\sqrt{3}}{2}x+C_2\,e^{\frac{1}{2}x}sin\,\frac{\sqrt{3}}{2}x.$

4 $\boxed{y'''-y=0}$ *Solution* : $\lambda^3-1=0 \Rightarrow \lambda_1=1,\quad \lambda_{2,3}=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

$y(x)=C_1\,e^{x}+C_2\,e^{-\frac{1}{2}x}\cos\frac{\sqrt{3}}{2}x+C_3\,e^{-\frac{1}{2}x}\sin\frac{\sqrt{3}}{2}x,$

5 $\boxed{y'=x^3\,y^2}$ *Solution* : $y(x)=0$ is a solution of the equation.

For $y\neq 0$: $\frac{y'}{y^2}=x^3 \Rightarrow \left(-\frac{1}{y}\right)'=x^3 \Rightarrow \frac{1}{y}=-\frac{x^4}{4}+C \Rightarrow y(x)=\frac{4}{4C-x^4}.$

6 $\boxed{y'=-\frac{1}{x}y-1}$ *Solution* : $y'=-\frac{1}{x}y \Rightarrow y(x)=\frac{C}{x}.$ We use the method of the variation of parameter.

$y_p(x)=\frac{C(x)}{x}$ satisfies $y'=-\frac{1}{x}y-1$ if $C'(x)=-x$.

The general solution of $y'=-\frac{1}{x}y-1$ is $y(x)=\frac{C}{x}+y_p(x)=\frac{C}{x}-\frac{x}{2}.$

7 $\boxed{\begin{cases} y_1'=y_2\\ y_2'=y_3\\ y_3'=y_1\end{cases}}$ *Solution* : After two substitutions, we get $y_1'''-y_1=0$ with the solution

$y_1(x)=C_1\,e^{x}+C_2\,e^{-\frac{1}{2}x}\cos\frac{\sqrt{3}}{2}x+C_3\,e^{-\frac{1}{2}x}\sin\frac{\sqrt{3}}{2}x,$

and $y_2(x)=y_1'(x),\quad y_3(x)=y_1''(x).$

B Exercises

1 $\boxed{y'=\frac{y}{x}+e^{\frac{y}{x}}}$ *Solution*: After the change of function $z(x)=\frac{y(x)}{x}$ the equation becomes $e^{-z}z'=\frac{1}{x}$ with solution $z(x)=-\ln(-\ln(Cx))$. Solution of $y'=\frac{y}{x}+e^{\frac{y}{x}}$ is $y(x)=-x\ln(-\ln(Cx))$.

2 $\boxed{\begin{cases} y_1'=y_2+y_3\\ y_2'=y_1+y_3\\ y_3'=y_1+y_2\end{cases}}$ *Solution* :

$$\begin{vmatrix} -\lambda & 1 & 1\\ 1 & -\lambda & 1\\ 1 & 1 & -\lambda\end{vmatrix}=0 \Rightarrow \begin{array}{l}\lambda_1=2\\ \lambda_2=\lambda_3=-1\end{array}$$

By choosing linearly independent eigenvectors, we get

$$\begin{pmatrix} y_1(x)\\ y_2(x)\\ y_3(x)\end{pmatrix}=C_1\begin{pmatrix}1\\1\\1\end{pmatrix}e^{2x}+C_2\begin{pmatrix}1\\0\\-1\end{pmatrix}e^{-x}+C_3\begin{pmatrix}0\\1\\-1\end{pmatrix}e^{-x}.$$

3 $\boxed{y''-4y'+4y=x^2e^{2x}}$ (method of the variation of parameters).

Solution : $y''-4y'+4y=0 \Rightarrow y(x)=C_1\,e^{2x}+C_2\,xe^{2x}.$

$y_p(x)=C_1(x)e^{2x}+C_2(x)xe^{2x}$ satisfies $y''-4y'+4y=x^2e^{2x}$ if $\begin{cases} C_1'(x)\,e^{2x}+C_2'(x)\,xe^{2x}=0\\ C_1'(x)(e^{2x})'+C_2'(x)(xe^{2x})'=x^2e^{2x}\end{cases}$

General solution of $y''-4y'+4y=x^2e^{2x}$ is $y(x)=C_1\,e^{2x}+C_2\,xe^{2x}+y_p(x)=(C_1+C_2x+\frac{x^4}{12})e^{2x}.$

C Exercises

1 $\boxed{(x+y)dx+xdy=0}$ *Solution 1* : $\frac{\partial(x+y)}{\partial y}=\frac{\partial x}{\partial x} \Rightarrow$ eq. is exact.

By integrating $(x+y)dx+xdy$ along the union of paths $\gamma_1:[0,1]\to\mathbb{R}^2,\ \gamma_1(t)=(tx,0)$, $\gamma_2:[0,1]\to\mathbb{R}^2,\ \gamma_2(t)=(x,ty)$,

we get $F(x,y)=\int_{(0,0)}^{(x,y)}(x+y)dx+xdy=\frac{x^2}{2}+xy$. The equation can be written as $d(\frac{x^2}{2}+xy)=0 \Rightarrow$ solution is defined by $\frac{x^2}{2}+xy=C$.

Solution 2 : Written as $y'=-\frac{1}{x}y-1$, the equation coincides to A6.

2 $\boxed{(1+\frac{y}{x})dx+dy=0}$ *Solution* : The equation can be reduced to the previous one by using the integrating factor $\mu(x,y)=x$.

3 $\boxed{\frac{dy}{dx}=\frac{2y^4}{x(x^2+2y^3)}}$ *Hint* : By changing the role of variables, we get the Bernoulli's equation $\frac{dx}{dy}=\frac{1}{y}x+\frac{1}{2y^4}x^3$.

4 $\boxed{y'=\frac{y^2}{x^3}+\frac{y}{x}-\frac{1}{x}}$ *Hint* : This Riccati equation admits the solution $y(x)=x$. The change of function $y(x)=x+\frac{1}{z(x)}$ reduces it to $z'=-(\frac{1}{x}+\frac{2}{x^2})z-\frac{1}{x^3}$.

5 $\boxed{\begin{cases} y_1'=y_1+y_2\\ y_2'=-y_1+y_2\end{cases}}$ that is $\boxed{\frac{d}{dx}\begin{pmatrix}y_1\\y_2\end{pmatrix}=\begin{pmatrix}1&1\\-1&1\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}}$

Solution : $\begin{vmatrix}1-\lambda & 1\\ -1 & 1-\lambda\end{vmatrix}=0 \Rightarrow \lambda_{1,2}=1\pm i.$

A solution of $\begin{pmatrix}1&1\\-1&1\end{pmatrix}\begin{pmatrix}w_1\\w_2\end{pmatrix}=(1+i)\begin{pmatrix}w_1\\w_2\end{pmatrix}$ is $\begin{pmatrix}w_1\\w_2\end{pmatrix}=\begin{pmatrix}1\\i\end{pmatrix}$.

General solution of the system

$$\begin{pmatrix}y_1(x)\\y_2(x)\end{pmatrix}=C_1\,\mathfrak{Re}\left\{\begin{pmatrix}1\\i\end{pmatrix}e^{(1+i)x}\right\}+C_2\,\mathfrak{Im}\left\{\begin{pmatrix}1\\i\end{pmatrix}e^{(1+i)x}\right\}$$
$$=C_1\begin{pmatrix}\cos x\\-\sin x\end{pmatrix}e^{x}+C_2\begin{pmatrix}\sin x\\\cos x\end{pmatrix}e^{x}.$$

6 $\boxed{\begin{cases} y_1'=-y_1+y_2\\ y_2'=-y_2+4y_3\\ y_3'=\ y_1-4y_3\end{cases}}$ *Solution* : The eigenvalues are

$\lambda_1=0$ with eigenspace $\{\alpha(4,4,1)\ |\ \alpha\in\mathbb{R}\}$

$\lambda_{2,3}=-3$ with eigenspace $\{\alpha(1,-2,1)\ |\ \alpha\in\mathbb{R}\}$

For $\lambda_{2,3}=-3$, by looking for solutions of the form $\begin{pmatrix}\alpha_1x+\beta_1\\ \alpha_2x+\beta_2\\ \alpha_3x+\beta_3\end{pmatrix}e^{-3x}$,

we get $$\begin{pmatrix}y_1(x)\\y_2(x)\\y_3(x)\end{pmatrix}=C_1\begin{pmatrix}4\\4\\1\end{pmatrix}+C_2\begin{pmatrix}1\\-2\\1\end{pmatrix}e^{-3x}+C_3\begin{pmatrix}x\\-2x+1\\x-1\end{pmatrix}e^{-3x}.$$

7 $\boxed{\begin{cases} y_1'=-y_1-2y_2+\cos x+\sin x+e^{-x}\\ y_2'=2y_1-y_2-\cos x+\sin x\end{cases}}$ $\quad Y'=AY+F$

Solution :

$Y'=AY \Rightarrow Y(x)=C_1\begin{pmatrix}\cos 2x\\-\sin 2x\end{pmatrix}e^{-x}+C_2\begin{pmatrix}\sin 2x\\\cos 2x\end{pmatrix}e^{-x}=W(x)\,C,$

where $W(x)=\begin{pmatrix}e^{-x}\cos 2x & e^{-x}\sin 2x\\ -e^{-x}\sin 2x & e^{-x}\cos 2x\end{pmatrix},\quad C=\begin{pmatrix}C_1\\C_2\end{pmatrix}.$

$Y_p(x)=W(x)\,C(x)$ satisfies $Y'=AY+F$ if $C'(x)=W(x)^{-1}F$.

General solution of $Y'=AY+F$ is $Y(x)=W(x)\,C+Y_p(x)$

$$=C_1\begin{pmatrix}\cos 2x\\-\sin 2x\end{pmatrix}e^{-x}+C_2\begin{pmatrix}\sin 2x\\\cos 2x\end{pmatrix}e^{-x}+\frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}e^{-x}-\frac{1}{3}\begin{pmatrix}\cos x-\sin x\\\cos x+\sin x\end{pmatrix}.$$

- **a** $\boxed{y''-3y'+2y=0}$
- **b** $\boxed{y''+2y'+y=0}$
- **c** $\boxed{y''-y'+2y=0}$
- **d** $\boxed{y'=\frac{x^2+y^2}{xy}}$
- **e** $\boxed{y'+2xy=x^3}$
- **f** $\boxed{y'=\frac{x^2+y^2}{xy}}$
- **g** $\boxed{y'''-y''-y'+y=0}$
- **h** $\boxed{y''=x+\sin x}$
- **k** $\boxed{y'-\frac{2}{x}y=x^3}$
- **l** $\boxed{y'=\frac{2x-2y-1}{x-y+1}}$
- **m** $\boxed{1+y^2+xyy'=0}$
- **n** $\boxed{(1+e^x)yy'=e^x}$
- **o** $\boxed{2xydx+dy=0}$
- **p** $\boxed{y'''+y''=x}$
- **q** $\boxed{xy''=y'}$
- **r** $\boxed{ydx+(x+y)dy=0}$
- **s** $\boxed{\begin{cases} y_1'=y_1+4y_2\\ y_2'=y_1+y_2\end{cases}}$
- **t** $\boxed{y'''-8y=x^2}$
- **u** $\boxed{y''+y=xe^{-x}}$
- **v** $\boxed{y'+y^2=1+x^2}$
- **w** $\boxed{(x-y)ydx-x^2dy=0}$
- **x** $\boxed{x^2y''-2xy'+2y=0}$
- **y** $\boxed{x^2y''+4xy'+2y=\cos x}$
- **z** $\boxed{\begin{cases} y_1'=y_2\\ y_2'=-4y_1+4y_2\\ y_3'=-2y_1+y_2+2y_3\end{cases}}$
- **$** $\boxed{\begin{cases} y_1'=y_1-3y_2+3y_3\\ y_2'=-2y_1-6y_2+13y_3\\ y_3'=-y_1-4y_2+8y_3\end{cases}}$
- **#** $\boxed{\begin{cases} y_1'=-y_2+x^2+6x+1\\ y_2'=y_1-3x^2+3x+1\end{cases}}$