Discrete-variable Gaussian states and the canonical commutation relation

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In the case of a quantum system with finite-dimensional Hilbert space, the discrete versions of the position and momentum do not satisfy the discrete version of the usual commutation relation. However, if the dimension of the Hilbert space is large enough, then this relation is approximately satisfied by a significant part of the pure states. We show that among these states there is a Fourier invariant family of discrete-variable Gaussian states depending on a continuous parameter. These states behave similar to their continuous-variable counterpart, and may have useful applications.

The quantum systems with finite-dimensional Hilbert space play an important role in the investigation of quantum mechanics foundations, in Quantum Information, and in other applications. The mathematical description of these discrete-variable systems is obtained by following, as much as possible, the analogy with the continuous-variable quantum systems.

In certain cases, the discrete-variable versions have properties very similar to those concerning the continuous systems, but this does not happen in all the cases. Here, we continue the investigation of the discrete version of the canonical commutation relation, started in [1]. Among the new results, there are the formulas (26), (28), (30), (32), (36), Fig. 2 and Fig. 3.

A spinless particle with mass, in one dimension, is usually described by using the Hilbert space

$$L^2(\mathbb{R})=\overline{\left\{\Psi:\mathbb{R}\to\mathbb{C}\ \middle|\ \int_{-\infty}^{\infty}|\Psi(q)|^2\,dq<\infty\right\}} \tag{1}$$

of all the square integrable functions with

$$\langle\Phi,\Psi\rangle=\int_{-\infty}^{\infty}\overline{\Phi(q)}\,\Psi(q)\,dq. \tag{2}$$

In order to have a simple analogy, we shall consider only the case of a discrete-variable quantum system described by an odd-dimensional ($d=2s+1$) Hilbert space $\mathcal{H}$ regarded as a space of functions, namely

$$\mathcal{H}=\{\psi:\{-s,-s+1,...,s-1,s\}\to\mathbb{C}\,\} \tag{3}$$

with the inner product

$$\langle\varphi|\psi\rangle=\sum_{n=-s}^{s}\overline{\varphi(n)}\,\psi(n). \tag{4}$$

The usual Fourier transform

$$\begin{array}{l} F:L^2(\mathbb{R})\to L^2(\mathbb{R}):\Psi\mapsto F[\Psi],\\ F[\Psi](p)=\frac{1}{\sqrt{h}}\int_{-\infty}^{\infty}\mathrm{e}^{-2\pi\mathrm{i}pq/h}\,\Psi(q)\,dq, \end{array} \tag{5}$$

where h is the Planck constant, is a unitary transform. Its inverse is the adjoint transform

$$\begin{array}{l} F^\dagger:L^2(\mathbb{R})\to L^2(\mathbb{R}):\Psi\mapsto F^\dagger[\Psi],\\ F^\dagger[\Psi](p)=\frac{1}{\sqrt{h}}\int_{-\infty}^{\infty}\mathrm{e}^{2\pi\mathrm{i}pq/h}\,\Psi(q)\,dq, \end{array} \tag{6}$$

The corresponding discrete-variable version [2, 3],

$$\begin{array}{l} \mathfrak{F}:\mathcal{H}\to\mathcal{H}:\psi\mapsto\mathfrak{F}[\psi],\\ \mathfrak{F}[\psi](k)=\frac{1}{\sqrt{d}}\sum\limits_{n=-s}^{s}\mathrm{e}^{-2\pi\mathrm{i}kn/d}\,\psi(n), \end{array} \tag{7}$$

is also a unitary transform. Its inverse is the adjoint

$$\begin{array}{l} \mathfrak{F}^\dagger:\mathcal{H}\to\mathcal{H}:\psi\mapsto\mathfrak{F}^\dagger[\psi],\\ \mathfrak{F}^\dagger[\psi](k)=\frac{1}{\sqrt{d}}\sum\limits_{n=-s}^{s}\mathrm{e}^{2\pi\mathrm{i}kn/d}\,\psi(n). \end{array} \tag{8}$$

The position operator

$$\begin{array}{l} \hat{q}:D_q\subset L^2(\mathbb{R})\to L^2(\mathbb{R}):\Psi\mapsto\hat{q}\,\Psi,\\ (\hat{q}\,\Psi)(q)=q\,\Psi(q), \end{array} \tag{9}$$

whose domain of definition is

$$D_q=\left\{\ \Psi\in L^2(\mathbb{R})\ \middle|\ \int_{-\infty}^{\infty}|q\,\Psi(q)|^2\,dq<\infty\right\}, \tag{10}$$

admits the discrete-variable version [2, 4–6]

$$\hat{\mathfrak{q}}\,:\,\mathcal{H}\to\mathcal{H}:\psi\mapsto\hat{\mathfrak{q}}\psi,\quad(\hat{\mathfrak{q}}\psi)(n)=n\,\psi(n). \tag{11}$$

The momentum operator

$$\begin{array}{l} \hat{p}:D_p\subset L^2(\mathbb{R})\to L^2(\mathbb{R}):\Psi\mapsto\hat{p}\,\Psi,\\ \hat{p}\,\Psi=-\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}q}\Psi, \end{array} \tag{12}$$

whose domain of definition is

$$D_p=\left\{\ \Psi\in L^2(\mathbb{R})\ \middle|\ \int_{-\infty}^{\infty}|\Psi'(q)|^2\,dq<\infty\right\}, \tag{13}$$

satisfies the relation

$$\hat{p}=F^\dagger\hat{q}F. \tag{14}$$

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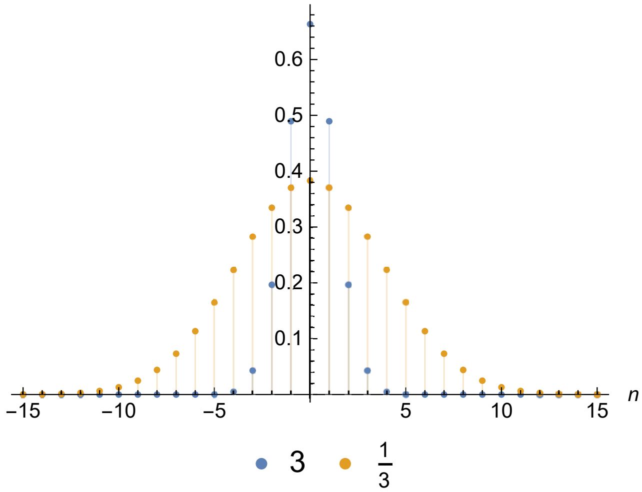

FIG. 1. Gaussian functions $\mathbf{g}_3$ and $\mathbf{g}_{1/3}$ in the case $d\!=\!31$.

The discrete-variable version of momentum is [2, 4–6]

$$\hat{\mathfrak{p}}:\mathcal{H}\to\mathcal{H},\quad \hat{\mathfrak{p}}=\mathfrak{F}^\dagger\hat{\mathfrak{q}}\mathfrak{F}. \tag{15}$$

For any $\kappa\in(0,\infty)$, the Gaussian function $g_\kappa:\mathbb{R}\to\mathbb{R}$,

$$g_\kappa(q)=\mathrm{e}^{-\kappa\pi q^2/h}, \tag{16}$$

satisfies the relation

$$F[g_\kappa]=\tfrac{1}{\sqrt{\kappa}}\,g_{1/\kappa}. \tag{17}$$

The corresponding discrete-variable Gaussian function

$$\begin{array}{l}\mathfrak{g}_\kappa:\{-s,-s+1,\dots,s-1,s\}\to\mathbb{R},\\ \mathfrak{g}_\kappa(n)=\displaystyle\sum_{\alpha=-\infty}^{\infty}\mathrm{e}^{-\kappa\pi(n+\alpha d)^2/d},\end{array} \tag{18}$$

satisfies the relation [2, 7, 8]

$$\mathfrak{F}[\mathfrak{g}_\kappa]=\tfrac{1}{\sqrt{\kappa}}\,\mathfrak{g}_{1/\kappa} \tag{19}$$

similar to (17). It is a particular case of Jacobi θ function [2, 3, 7]. Because $\mathfrak{F}$ is a unitary transform, the corresponding normalized discrete Gaussian functions

$$\mathbf{g}_\kappa=\frac{\mathfrak{g}_\kappa}{||\mathfrak{g}_\kappa||}\quad\text{where}\quad ||\mathfrak{g}_\kappa||=\sqrt{\sum_{n=-s}^{s}(\mathfrak{g}_\kappa(n))^2}, \tag{20}$$

satisfy the relation

$$\mathfrak{F}[\mathbf{g}_\kappa]=\mathbf{g}_{1/\kappa}. \tag{21}$$

In the continuous case, the usual position and momentum satisfy the remarkable relation

$$[\hat{q},\hat{p}]=\mathrm{i}\tfrac{h}{2\pi}, \tag{22}$$

where $[\hat{q},\hat{p}]=\hat{q}\hat{p}-\hat{p}\hat{q}$ is the commutator of $\hat{q}$ and $\hat{p}$. This means that

$$\hat{q}(\hat{p}\Psi)-\hat{p}(\hat{q}\Psi)=\mathrm{i}\tfrac{h}{2\pi}\Psi \tag{23}$$

TABLE I. Eigenvalues of $[\hat{\mathfrak{q}},\hat{\mathfrak{p}}]-\mathrm{i}\frac{d}{2\pi}$ in two cases.

Case $d=11$	Case $d=31$	
	$-6.8034\times10^{-16}\,\mathrm{i}$	$1.44395\times10^{-7}\,\mathrm{i}$
	$-1.28585\times10^{-15}\,\mathrm{i}$	$-1.10354\times10^{-6}\,\mathrm{i}$
$-1.48548\times10^{-8}\,\mathrm{i}$	$2.07899\times10^{-15}\,\mathrm{i}$	$7.76978\times10^{-6}\,\mathrm{i}$
$7.96337\times10^{-7}\,\mathrm{i}$	$-3.04408\times10^{-15}\,\mathrm{i}$	$-0.000050515\,\mathrm{i}$
$2.04883\times10^{-5}\,\mathrm{i}$	$3.57322\times10^{-15}\,\mathrm{i}$	$0.000303805\,\mathrm{i}$
$3.36098\times10^{-4}\,\mathrm{i}$	$5.70417\times10^{-15}\,\mathrm{i}$	$-0.00169225\,\mathrm{i}$
$3.93615\times10^{-3}\,\mathrm{i}$	$-1.12090\times10^{-14}\,\mathrm{i}$	$0.00873614\,\mathrm{i}$
$3.49009\times10^{-2}\,\mathrm{i}$	$1.58666\times10^{-14}\,\mathrm{i}$	$-0.0417956\,\mathrm{i}$
$-0.240939\,\mathrm{i}$	$-2.13350\times10^{-14}\,\mathrm{i}$	$0.185225\,\mathrm{i}$
$1.33714\,\mathrm{i}$	$1.13714\times10^{-13}\,\mathrm{i}$	$-0.757941\,\mathrm{i}$
$-5.45524\,\mathrm{i}$	$-1.41394\times10^{-12}\,\mathrm{i}$	$2.86832\,\mathrm{i}$
$19.9934\,\mathrm{i}$	$1.73886\times10^{-11}\,\mathrm{i}$	$-9.80352\,\mathrm{i}$
$-34.9234\,\mathrm{i}$	$-1.91428\times10^{-10}\,\mathrm{i}$	$31.5302\,\mathrm{i}$
	$1.9093\times10^{-9}\,\mathrm{i}$	$-80.4393\,\mathrm{i}$
	$-1.73552\times10^{-8}\,\mathrm{i}$	$214.181\,\mathrm{i}$
		$-310.677\,\mathrm{i}$

for any Ψ from the subspace

$$\{\,\Psi\in D_q\cap D_p\ \mid\ \hat{q}\Psi\in D_p\ \text{ and }\ \hat{p}\Psi\in D_q\,\}. \tag{24}$$

If we compare the continuous and discrete version of the Fourier transform, then we arrive at the conclusion that, in the discrete case, the role of Planck constant is played by the dimension d of the Hilbert space. Consequently, the discrete version of the relation (22) is

$$[\hat{\mathfrak{q}},\hat{\mathfrak{p}}]=\mathrm{i}\tfrac{d}{2\pi} \tag{25}$$

A direct computation shows that, generally, this relation is not satisfied. By computing the eigenvalues of the operator $[\hat{\mathfrak{q}},\hat{\mathfrak{p}}]-\mathrm{i}\frac{d}{2\pi}$ for different values of d, one arrives at the conclusion that the eigenvalues λ with $|\lambda|<0.001$ represent 36% in the case $d=11$, represent 64% in the case $d=31$, represent 77% in the case $d=61$ and 84% in the case $d\!=\!101$ (see Table I and [1]). So, for d large enough, a significant part of the eigenvalues of $[\hat{\mathfrak{q}},\hat{\mathfrak{p}}]-\mathrm{i}\frac{d}{2\pi}$ are almost null.

By using the relation

$$\begin{aligned}(\mathfrak{F}\hat{\mathfrak{q}}\mathfrak{F}^\dagger)\psi(n)&=\tfrac{1}{d}\sum_{m,k=-s}^{s}\mathrm{e}^{2\pi\mathrm{i}m(k-n)/d}\,m\,\psi(k)\\&=\tfrac{1}{d}\sum_{m,k=-s}^{s}\mathrm{e}^{-2\pi\mathrm{i}m(k-n)/d}\,(-m)\,\psi(k)\\&=-(\mathfrak{F}^\dagger\hat{\mathfrak{q}}\mathfrak{F})\psi(n)=-\hat{\mathfrak{p}}\psi(n)\end{aligned}$$

TABLE II. Case $d\!=\!11$: $\varphi_1,\varphi_2,\dots,\varphi_d$ are eigenstates of $\mathfrak{F}$.

k	$\Vert\,\mathfrak{F}\varphi_k-\varphi_k\,\Vert$	$\Vert\,\mathfrak{F}\varphi_k+\varphi_k\,\Vert$	$\Vert\,\mathfrak{F}\varphi_k-\mathrm{i}\varphi_k\,\Vert$	$\Vert\,\mathfrak{F}\varphi_k+\mathrm{i}\varphi_k\,\Vert$
1	2.0×10^{-10}	2	1.4142	1.4142
2	1.4142	1.4142	2	1.2×10^{-10}
3	2	6.3×10^{-11}	1.4142	1.4142
4	1.4142	1.4142	1.3×10^{-12}	2
5	2.7×10^{-13}	2	1.4142	1.4142
6	1.4142	1.4142	2	2.3×10^{-14}
7	2	1.3×10^{-14}	1.4142	1.4142
8	1.4142	1.4142	3.4×10^{-15}	2
9	1.0×10^{-15}	2	1.4142	1.4142
10	1.4142	1.4142	2	8.3×10^{-16}
11	2	8.4×10^{-16}	1.4142	1.4142

TABLE III. Case $d=11$: for certain κ, the Gaussian $\mathbf{g}_\kappa$ is mainly a linear combination of φ_1, φ_2, φ_3, φ_4, φ_5.

	$\kappa=1$	$\kappa=2$	$\kappa=1/2$	$\kappa=3$	$\kappa=1/3$
$\lvert\langle\varphi_1\lvert\mathbf{g}_\kappa\rangle\rvert$	0.999968	0.970268	0.970268	0.926811	0.926811
$\lvert\langle\varphi_2\lvert\mathbf{g}_\kappa\rangle\rvert$	8.79611×10^{-11}	8.56453×10^{-11}	8.50632×10^{-11}	8.19559×10^{-11}	8.11208×10^{-8}
$\lvert\langle\varphi_3\lvert\mathbf{g}_\kappa\rangle\rvert$	3.14918×10^{-11}	0.228139	0.228139	0.327299	0.327299
$\lvert\langle\varphi_4\lvert\mathbf{g}_\kappa\rangle\rvert$	3.83482×10^{-13}	3.8548×10^{-13}	3.58495×10^{-13}	3.74436×10^{-13}	3.36051×10^{-13}
$\lvert\langle\varphi_5\lvert\mathbf{g}_\kappa\rangle\rvert$	0.00798631	0.0736351	0.0736351	0.151392	0.151392
$\lvert\langle\varphi_6\lvert\mathbf{g}_\kappa\rangle\rvert$	8.61761×10^{-15}	9.0834×10^{-15}	8.25462×10^{-15}	9.15919×10^{-15}	8.20907×10^{-15}
$\lvert\langle\varphi_7\lvert\mathbf{g}_\kappa\rangle\rvert$	2.78121×10^{-15}	0.0288995	0.0288995	0.0830314	0.0830314
$\lvert\langle\varphi_8\lvert\mathbf{g}_\kappa\rangle\rvert$	1.82021×10^{-16}	1.7344×10^{-16}	2.47453×10^{-16}	1.32587×10^{-16}	3.49103×10^{-16}
$\lvert\langle\varphi_9\lvert\mathbf{g}_\kappa\rangle\rvert$	0.000416018	0.0146752	0.0146752	0.0549712	0.0549712
$\lvert\langle\varphi_{10}\lvert\mathbf{g}_\kappa\rangle\rvert$	5.0058×10^{-17}	4.89024×10^{-17}	6.26338×10^{-17}	4.68507×10^{-17}	7.58437×10^{-17}
$\lvert\langle\varphi_{11}\lvert\mathbf{g}_\kappa\rangle\rvert$	1.98563×10^{-16}	0.00777324	0.00777324	0.0325688	0.0325688

we get $\mathfrak{F}\hat{\mathfrak{q}}\mathfrak{F}^\dagger=-\hat{\mathfrak{p}}$ and

$$\begin{aligned}\mathfrak{F}[\hat{\mathfrak{q}},\hat{\mathfrak{p}}] &=\mathfrak{F}\hat{\mathfrak{q}}\,\hat{\mathfrak{p}}-\mathfrak{F}\hat{\mathfrak{p}}\,\hat{\mathfrak{q}}=\mathfrak{F}\hat{\mathfrak{q}}\,\mathfrak{F}^\dagger\hat{\mathfrak{q}}\mathfrak{F}-\mathfrak{F}\mathfrak{F}^\dagger\hat{\mathfrak{q}}\mathfrak{F}\,\hat{\mathfrak{q}}\\ &=-\hat{\mathfrak{p}}\,\hat{\mathfrak{q}}\mathfrak{F}-\hat{\mathfrak{q}}\mathfrak{F}\,\hat{\mathfrak{q}}\mathfrak{F}^\dagger\mathfrak{F}=[\hat{\mathfrak{q}},\hat{\mathfrak{p}}]\mathfrak{F},\end{aligned}$$

that is

$$\mathfrak{F}[\hat{\mathfrak{q}},\hat{\mathfrak{p}}]=[\hat{\mathfrak{q}},\hat{\mathfrak{p}}]\mathfrak{F}. \qquad (26)$$

Let λ_1, λ_2, ..., λ_d be the eigenvalues of $[\hat{\mathfrak{q}},\hat{\mathfrak{p}}]-\mathrm{i}\frac{d}{2\pi}$, considered in the increasing order of their modulus ($|\lambda_1|\leq|\lambda_2|\leq...\leq|\lambda_d|$), and let φ_1, φ_2, ... , φ_d be the corresponding eigenfunctions, that is

$$([\hat{\mathfrak{q}},\hat{\mathfrak{p}}]-\mathrm{i}\tfrac{d}{2\pi})\,\varphi_k=\lambda_k\ \varphi_\kappa. \qquad (27)$$

The relation

$$([\hat{\mathfrak{q}},\hat{\mathfrak{p}}]-\mathrm{i}\tfrac{d}{2\pi})\,\varphi_k=\lambda_k\ \varphi_\kappa\ \Rightarrow\ ([\hat{\mathfrak{q}},\hat{\mathfrak{p}}]-\mathrm{i}\tfrac{d}{2\pi})\,\mathfrak{F}\varphi_k=\lambda_k\ \mathfrak{F}\varphi_\kappa$$

and the numerical data (see Table II) suggest that the states φ_1, φ_2, ... , φ_d are also eigenstates of $\mathfrak{F}$, namely

$$\mathfrak{F}\varphi_n=\theta_n\ \varphi_n,\quad \text{where}\quad \theta_n\in\{1,-1,\mathrm{i},-\mathrm{i}\}. \qquad (28)$$

For d large enough and small $\varepsilon>0$, there exists $\delta>0$ such that, for $\kappa\in(\frac{1}{\delta},\delta)$, the most significant coordinates of $\mathbf{g}_\kappa$ in the eigenbasis $\{\varphi_1,\varphi_2,...,\varphi_d\}$ of $[\hat{\mathfrak{q}},\hat{\mathfrak{p}}]-\mathrm{i}\frac{d}{2\pi}$ are those corresponding to the functions φ_k with $|\lambda_k|<\varepsilon$ (see Table III). We can consider that

$$[\hat{\mathfrak{q}},\hat{\mathfrak{p}}]\approx\mathrm{i}\tfrac{d}{2\pi}\quad\text{for }\mathbf{g}_\kappa\text{ with }\ \kappa\in(\tfrac{1}{\delta},\delta) \qquad (29)$$

(see Tables III, IV and Figure 2). From (28), we get

$$|\langle\varphi_n|\mathbf{g}_\kappa\rangle|=|\langle\theta_n\varphi_n|\mathbf{g}_\kappa\rangle|=|\langle\mathfrak{F}\varphi_n|\mathbf{g}_\kappa\rangle|=|\langle\varphi_n|\mathbf{g}_{1/\kappa}\rangle|, \qquad (30)$$

in agreement with the numerical data from Table III. Since $\mathfrak{F}$ is a unitary transform, we get

$$\begin{aligned}||[\hat{\mathfrak{q}},\hat{\mathfrak{p}}]\psi-\mathrm{i}\tfrac{d}{2\pi}\psi|| &=||\mathfrak{F}[\hat{\mathfrak{q}},\hat{\mathfrak{p}}]\psi-\mathrm{i}\tfrac{d}{2\pi}\mathfrak{F}\psi||\\ &=||[\hat{\mathfrak{q}},\hat{\mathfrak{p}}]\mathfrak{F}\psi-\mathrm{i}\tfrac{d}{2\pi}\mathfrak{F}\psi||,\end{aligned} \qquad (31)$$

for any $\psi\in\mathcal{H}$. Particularly, we have

$$\|\,([\hat{\mathfrak{q}},\hat{\mathfrak{p}}]-\mathrm{i}\tfrac{d}{2\pi})\mathbf{g}_\kappa\,\|=\|\,([\hat{\mathfrak{q}},\hat{\mathfrak{p}}]-\mathrm{i}\tfrac{d}{2\pi})\mathbf{g}_{1/\kappa}\,\|, \qquad (32)$$

TABLE IV. $[\hat{\mathfrak{q}},\hat{\mathfrak{p}}]\mathbf{g}_\kappa\approx\mathrm{i}\frac{d}{2\pi}\mathbf{g}_\kappa$ is satisfied for certain $\mathbf{g}_\kappa$.

κ	$\lvert\lvert\,([\hat{\mathfrak{q}},\hat{\mathfrak{p}}]-\mathrm{i}\frac{d}{2\pi})\,\mathbf{g}_\kappa\rvert\rvert$ Case $d=11$	Case $d=31$
1	0.0022697	2.68152×10^{-9}
2	0.283112	0.000670943
1/2	0.283112	0.000670943
3		0.0383181
1/3		0.0383181

in agreement with the numerical data from Table IV. A direct consequence of (29) is

$$|\langle\mathbf{g}_\kappa|[\hat{\mathfrak{q}},\hat{\mathfrak{p}}]|\mathbf{g}_\kappa\rangle|\approx\tfrac{d}{2\pi}\quad\text{for any}\quad\kappa\in(\tfrac{1}{\delta},\delta) \qquad (33)$$

(see Table V). From the Robertson-Schrödinger uncertainty relation, it follows that the inequality

$$\Delta\hat{\mathfrak{q}}\ \Delta\hat{\mathfrak{p}}\geq\frac{|\langle\mathbf{g}_\kappa|[\hat{\mathfrak{q}},\hat{\mathfrak{p}}]|\mathbf{g}_\kappa\rangle|}{2}\quad\text{for }\kappa\in(\tfrac{1}{\delta},\delta) \qquad (34)$$

that is

$$\Delta\hat{\mathfrak{q}}\ \Delta\hat{\mathfrak{p}}\geq\tfrac{d}{4\pi}\quad\text{for }\mathbf{g}_\kappa\text{with }\kappa\in(\tfrac{1}{\delta},\delta) \qquad (35)$$

is approximately satisfied (see Table V and Figure 3). By using (21) and (26) we get

$$\langle\mathbf{g}_{1/\kappa}|[\hat{\mathfrak{q}},\hat{\mathfrak{p}}]|\mathbf{g}_{1/\kappa}\rangle=\langle\mathfrak{F}\mathbf{g}_\kappa|\mathfrak{F}[\hat{\mathfrak{q}},\hat{\mathfrak{p}}]|\mathbf{g}_\kappa\rangle=\langle\mathbf{g}_\kappa|[\hat{\mathfrak{q}},\hat{\mathfrak{p}}]|\mathbf{g}_\kappa\rangle, \qquad (36)$$

in agreement with the numerical data from Table V.

In conclusion, if the dimension d of the Hilbert space describing the quantum system (qudit) is large enough, there exists a family $\mathcal{G}_\delta=\{\,\mathbf{g}_\kappa\mid\kappa\in(\frac{1}{\delta},\delta)\,\}$ of discrete-variable Gaussian states $\mathbf{g}_\kappa$ depending on a continuous parameter $\kappa\in(\frac{1}{\delta},\delta)$ such that the relations

$$[\hat{\mathfrak{q}},\hat{\mathfrak{p}}]=\mathrm{i}\tfrac{d}{2\pi}\quad\text{and}\quad\Delta\hat{\mathfrak{q}}\ \Delta\hat{\mathfrak{p}}\geq\tfrac{d}{4\pi} \qquad (37)$$

are approximately satisfied for any $\mathbf{g}_\kappa\in\mathcal{G}_\delta$. Any desired precision can be obtained by increasing the dimension d of the Hilbert space and use of an adequate interval $(\frac{1}{\delta},\delta)$. In addition, the family $\mathcal{G}_\delta$ is Fourier invariant:

$$\mathbf{g}_\kappa\in\mathcal{G}_\delta\quad\Rightarrow\quad\mathfrak{F}[\mathbf{g}_\kappa]=\mathbf{g}_{1/\kappa}\in\mathcal{G}_\delta. \qquad (38)$$

The relations (37) are approximately satisfied in a space larger than $\mathcal{G}_\delta$, containing certain linear combinations of

TABLE V. $|\langle \mathbf{g}_\kappa|[\hat{\mathfrak{q}},\hat{\mathfrak{p}}]|\mathbf{g}_\kappa\rangle| \approx \frac{d}{2\pi}$ is satisfied for certain $\mathbf{g}_\kappa$.

κ	$\mid \lvert\langle \mathbf{g}_\kappa\lvert[\hat{\mathfrak{q}},\hat{\mathfrak{p}}]\rvert\mathbf{g}_\kappa\rangle\rvert - \frac{d}{2\pi} \mid$	
	Case $d=11$	Case $d=31$
1	1.21005×10^{-6}	8.88178×10^{-16}
2	0.00350868	2.47852×10^{-9}
1/2	0.00350868	2.47852×10^{-9}
3	0.0552824	7.39351×10^{-6}
1/3	0.0552824	7.39351×10^{-6}
4	0.204735	0.000383435
1/4	0.204735	0.000383435
5		0.00397916
1/5		0.00397916
6		0.0185731
1/6		0.0185731
7		0.0550503
1/7		0.0550503
8		0.122973
1/8		0.122973

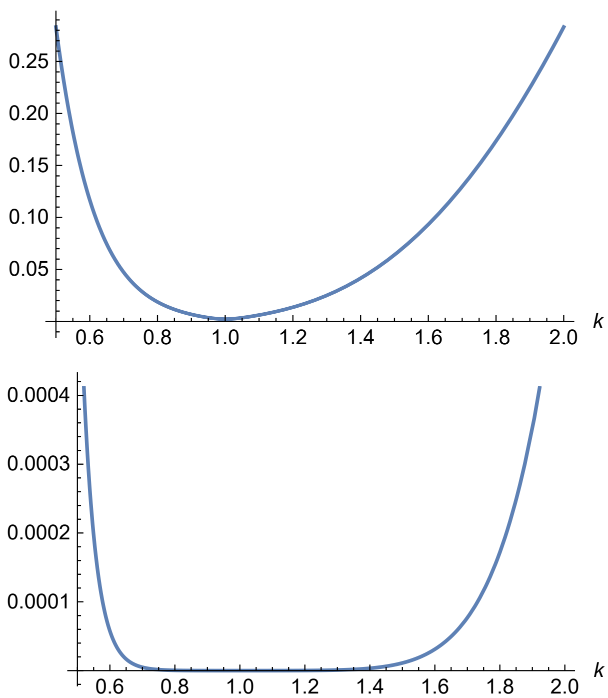

FIG. 2. $\|\,([\hat{\mathfrak{q}},\hat{\mathfrak{p}}]-\mathrm{i}\frac{d}{2\pi})\mathbf{g}_k\|$ in the cases $d\!=\!11$ and $d\!=\!31$.

$\mathbf{g}_\kappa$ from $\mathcal{G}_\delta$. Any function $\mathfrak{g}_\kappa$ admits the explicit definition (18) as the sum of a convergent series. The discrete-variable Wigner function [2, 9] of any function $\mathfrak{g}_\kappa \in \mathcal{G}_\delta$ can be explicitly represented as the sum of a convergent double series [1, 8]

$$\mathfrak{W}_{\mathfrak{g}_\kappa}(n,k)=C\sum_{\alpha,\beta=-\infty}^{\infty}(-1)^{\alpha\beta}\mathcal{W}_{g_\kappa}\Big((n+\alpha\tfrac{d}{2})\sqrt{\tfrac{h}{d}},\,(k+\beta\tfrac{d}{2})\sqrt{\tfrac{h}{d}}\Big) \tag{39}$$

involving the continuous-variable Wigner function

$$\mathcal{W}_{g_\kappa}(q,p)=\sqrt{\frac{2}{\kappa h}}\,\exp\left\{-\frac{2\pi}{h}(\kappa q^2+\frac{1}{\kappa}p^2)\right\} \tag{40}$$

of the corresponding Gaussian state $g_\kappa \in \mathrm{L}^2(\mathbb{R})$ and a normalizing constant C. The existence of these explicit definitions opens a way in the direction of mathematical investigation of properties and creation of some useful mathematical formalisms.

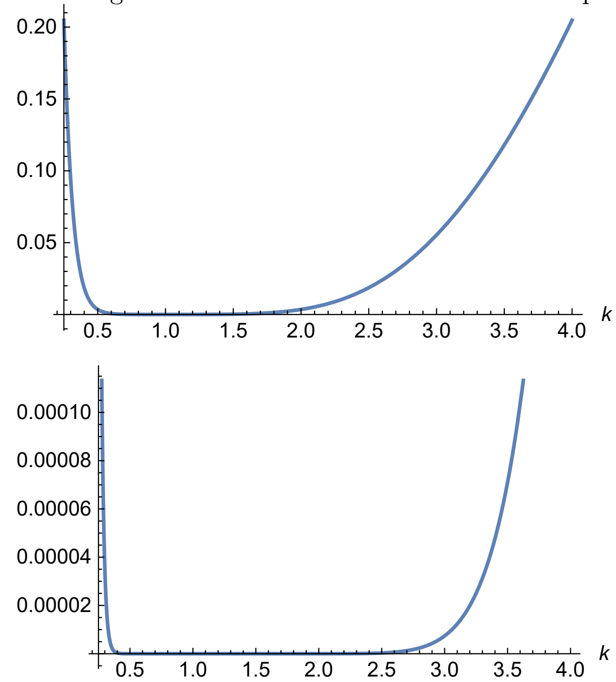

FIG. 3. $\mid |\langle \mathbf{g}_k|[\hat{\mathfrak{q}},\hat{\mathfrak{p}}]|\mathbf{g}_k\rangle| - \frac{d}{2\pi} \mid$ in the cases $d\!=\!11$ and $d\!=\!31$.

The operators

$$\begin{array}{lcl} \hat{\mathfrak{a}}=\sqrt{\frac{\pi}{d}}(\hat{\mathfrak{q}}+\mathrm{i}\hat{\mathfrak{p}}), & & \hat{a}=\sqrt{\frac{\pi}{h}}(\hat{q}+\mathrm{i}\hat{p}),\\ & \text{similar to} & \\ \hat{\mathfrak{a}}^\dagger=\sqrt{\frac{\pi}{d}}(\hat{\mathfrak{q}}-\mathrm{i}\hat{\mathfrak{p}}), & & \hat{a}^\dagger=\sqrt{\frac{\pi}{h}}(\hat{q}-\mathrm{i}\hat{p}), \end{array} \tag{41}$$

also approximately satisfy

$$[\hat{\mathfrak{a}},\hat{\mathfrak{a}}^\dagger]=1 \quad \text{for any} \quad \mathbf{g}_\kappa\in\mathcal{G}_\delta. \tag{42}$$

In this short note, we have considered only the case of pure discrete-variable Gaussian states, but the numerical data show [1] that, in the basis $\{\varphi_1, \varphi_2, ..., \varphi_d\}$, the most significant elements of the density matrix of certain mixed discrete-variable Gaussian states are also those corresponding to small $|\lambda_k|$. More similitude with the continuous-variable Gaussian states can be discovered [1] by investigating the behavior with respect to a discrete version of the unitary Gaussian transforms.

Quantum Information uses a continuous-variable description, but it is mainly limited to a very particular class of states (including the Gaussian states) and a very particular class of unitary transforms (including the Gaussian unitaries). The discrete-variable version of the commutation relation of the discrete-variable version of position and momentum operator is acceptably satisfied by a particular class of states (including certain discrete-variable Gaussian states). We think that, some useful models involving only such particular states, can be built in the discrete-variable case by following the analogy with the continuous-variable case.

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