An overview on GAUSSIAN FUNCTIONS OF DISCRETE VARIABLE

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GAUSSIAN FUNCTIONS OF CONTINUOUS VARIABLE

Gaussian functions of one continuous variable

■ For $\kappa\in(0,\infty)$, we define $\boxed{g_\kappa(q)\overset{\text{def}}{=}\mathrm{e}^{-\frac{\kappa}{2}q^2}}$

■ *Fourier transform.* By using the definition

$$\mathcal{F}[\psi](p)\overset{\text{def}}{=}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}pq}\,\psi(q)\,dq \quad \text{we get} \quad \boxed{\mathcal{F}[g_\kappa]=\frac{1}{\sqrt{\kappa}}\,g_{\kappa^{-1}}}$$

■ *Wigner function.* By using the definition

$$\mathcal{W}_\psi(q,p)\overset{\text{def}}{=}\frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-2\mathrm{i}px}\,\overline{\psi(q-x)}\;\psi(q+x)\,dx$$

we get $\boxed{\mathcal{W}_{g_\kappa}(q,p)=\frac{1}{\sqrt{\kappa\pi}}\,g_{2\kappa}(q)\,g_{2\kappa^{-1}}(p)}$

Gaussian functions of two continuous variables

■ For $\sigma=\begin{pmatrix}a&b\\b&c\end{pmatrix}>0$, $\boxed{g_\sigma(q_1,q_2)\overset{\text{def}}{=}\mathrm{e}^{-\frac{1}{2}(q_1\ \ q_2)\begin{pmatrix}a&b\\b&c\end{pmatrix}\begin{pmatrix}q_1\\q_2\end{pmatrix}}}$

that is

$$g_\sigma(q_1,q_2)=\mathrm{e}^{-\frac{1}{2}(aq_1^2+2bq_1\,q_2+cq_2^2)}=\mathrm{e}^{-\frac{a}{2}q_1^2}\ \mathrm{e}^{-bq_1q_2}\ \mathrm{e}^{-\frac{c}{2}q_2^2}.$$

■ *Fourier transform.* By using the definition

$$\mathcal{F}[\psi](p_1,p_2)\overset{\text{def}}{=}\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}(p_1q_1+p_2q_2)}\,\psi(q_1,q_2)\,dq_1dq_2$$

we get $\boxed{\mathcal{F}[g_\sigma]=\frac{1}{\sqrt{\det\sigma}}\,g_{\sigma^{-1}}}$

■ *Wigner function.* By using the definition

$$\mathcal{W}_\psi(q_1,q_2,p_1,p_2)\overset{\text{def}}{=}\frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-2\mathrm{i}(p_1x_1+p_2x_2)}$$
$$\times\overline{\psi(q_1-x_1,q_2-x_2)}\,\psi(q_1+x_1,q_2+x_2)\,dx_1dx_2$$

we get

$$\boxed{\mathcal{W}_{g_\sigma}(q_1,q_2,p_1,p_2)=\frac{1}{\pi\sqrt{\det\sigma}}\,g_{2\sigma}(q_1,q_2)\,g_{2\sigma^{-1}}(p_1,p_2)}$$

GAUSSIAN FUNCTIONS OF DISCRETE VARIABLE

($d=2j+1$ is a positive odd integer)

Gaussian functions of one discrete variable

■ For $\kappa\in(0,\infty)$, we define $\boxed{\mathbf{g}_\kappa(n)\overset{\text{def}}{=}\sum_{\alpha=-\infty}^{\infty}\mathrm{e}^{-\frac{\kappa\pi}{d}(n+\alpha d)^2}}$

and $\quad\mathbf{g}_\kappa^+(n)\overset{\text{def}}{=}\sum_{\alpha=-\infty}^{\infty}\mathrm{e}^{-\frac{\kappa\pi}{d}(n+(\alpha+\frac{1}{2})d)^2}$

■ *Discrete Fourier transform.* By using the definition

$$\mathbf{F}[\psi](k)\overset{\text{def}}{=}\frac{1}{\sqrt{d}}\sum_{n=-j}^{j}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}kn}\,\psi(n) \quad \text{we get} \quad \boxed{\mathbf{F}[\mathbf{g}_\kappa]=\frac{1}{\sqrt{\kappa}}\,\mathbf{g}_{\kappa^{-1}}}$$

■ *Discrete Wigner function.* By using the definition

$$\mathbf{W}_\psi(n,k)\overset{\text{def}}{=}\frac{1}{d}\sum_{m=-j}^{j}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\,\overline{\psi(n-m)}\;\psi(n+m)$$

we get [arXiv:1205.6302]

$$\boxed{\begin{aligned}\mathbf{W}_{\mathbf{g}_\kappa}(n,k)=&\frac{1}{\sqrt{2\kappa d}}\,\mathbf{g}_{2\kappa}(n)\ \left[\mathbf{g}_{2\kappa^{-1}}(k)+\mathbf{g}^+_{2\kappa^{-1}}(k)\right]\\&+\frac{1}{\sqrt{2\kappa d}}\,\mathbf{g}^+_{2\kappa}(n)\ \left[\mathbf{g}_{2\kappa^{-1}}(k)-\mathbf{g}^+_{2\kappa^{-1}}(k)\right]\end{aligned}}$$

Gaussian functions of two discrete variables

■ $\boxed{\mathbf{g}_\sigma(n_1,n_2)\overset{\text{def}}{=}\sum_{\alpha_1,\alpha_2=-\infty}^{\infty}\mathrm{e}^{-\frac{\pi}{d}(n_1+\alpha_1d\ \ n_2+\alpha_2d)\begin{pmatrix}a&b\\b&c\end{pmatrix}\begin{pmatrix}n_1+\alpha_1d\\n_2+\alpha_2d\end{pmatrix}}}$ and

$$\mathbf{g}_\sigma^{+0}(n_1,n_2)\overset{\text{def}}{=}\sum_{\alpha_1,\alpha_2=-\infty}^{\infty}\mathrm{e}^{-\frac{\pi}{d}(n_1+(\alpha_1+\frac{1}{2})d\ \ n_2+\alpha_2d)\begin{pmatrix}a&b\\b&c\end{pmatrix}\begin{pmatrix}n_1+(\alpha_1+\frac{1}{2})d\\n_2+\alpha_2d\end{pmatrix}},$$

$$\mathbf{g}_\sigma^{0+}(n_1,n_2)\overset{\text{def}}{=}\sum_{\alpha_1,\alpha_2=-\infty}^{\infty}\mathrm{e}^{-\frac{\pi}{d}(n_1+\alpha_1d\ \ n_2+(\alpha_2+\frac{1}{2})d)\begin{pmatrix}a&b\\b&c\end{pmatrix}\begin{pmatrix}n_1+\alpha_1d\\n_2+(\alpha_2+\frac{1}{2})d\end{pmatrix}},$$

$$\mathbf{g}_\sigma^{++}(n_1,n_2)\overset{\text{def}}{=}\sum_{\alpha_1,\alpha_2=-\infty}^{\infty}\mathrm{e}^{-\frac{\pi}{d}(n_1+(\alpha_1+\frac{1}{2})d\ \ n_2+(\alpha_2+\frac{1}{2})d)\begin{pmatrix}a&b\\b&c\end{pmatrix}\begin{pmatrix}n_1+(\alpha_1+\frac{1}{2})d\\n_2+(\alpha_2+\frac{1}{2})d\end{pmatrix}}.$$

■ *Discrete Fourier transform.* By using the definition

$$\mathbf{F}[\psi](k_1,k_2)\overset{\text{def}}{=}\frac{1}{d}\sum_{n_1=-j}^{j}\sum_{n_2=-j}^{j}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}(k_1n_1+k_2n_2)}\,\psi(n_1,n_2)$$

we get $\boxed{\mathbf{F}[\mathbf{g}_\sigma]=\frac{1}{\sqrt{\det\sigma}}\,\mathbf{g}_{\sigma^{-1}}}$

■ *Discrete Wigner function.* By using the definition

$$\mathbf{W}_\psi(n_1,n_2,k_1,k_2)\overset{\text{def}}{=}\frac{1}{d^2}\sum_{m_1=-j}^{j}\sum_{m_2=-j}^{j}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}(k_1m_1+k_2m_2)}$$
$$\times\overline{\psi(n_1-m_1,n_2-m_2)}\,\psi(n_1+m_1,n_2+m_2)$$

we get [arXiv:1912.01998]

$$\boxed{\begin{aligned}&\mathbf{W}_{\mathbf{g}_\sigma}(n_1,n_2,k_1,k_2)\\&=\frac{1}{2d\sqrt{\det\sigma}}\ \mathbf{g}_{2\sigma}(n_1,n_2)\left[\mathbf{g}_{2\sigma^{-1}}(k_1,k_2)+\mathbf{g}^{+0}_{2\sigma^{-1}}(k_1,k_2)+\mathbf{g}^{0+}_{2\sigma^{-1}}(k_1,k_2)+\mathbf{g}^{++}_{2\sigma^{-1}}(k_1,k_2)\right]\\&+\frac{1}{2d\sqrt{\det\sigma}}\ \mathbf{g}^{+0}_{2\sigma}(n_1,n_2)\left[\mathbf{g}_{2\sigma^{-1}}(k_1,k_2)-\mathbf{g}^{+0}_{2\sigma^{-1}}(k_1,k_2)+\mathbf{g}^{0+}_{2\sigma^{-1}}(k_1,k_2)-\mathbf{g}^{++}_{2\sigma^{-1}}(k_1,k_2)\right]\\&+\frac{1}{2d\sqrt{\det\sigma}}\ \mathbf{g}^{0+}_{2\sigma}(n_1,n_2)\left[\mathbf{g}_{2\sigma^{-1}}(k_1,k_2)+\mathbf{g}^{+0}_{2\sigma^{-1}}(k_1,k_2)-\mathbf{g}^{0+}_{2\sigma^{-1}}(k_1,k_2)-\mathbf{g}^{++}_{2\sigma^{-1}}(k_1,k_2)\right]\\&+\frac{1}{2d\sqrt{\det\sigma}}\ \mathbf{g}^{++}_{2\sigma}(n_1,n_2)\left[\mathbf{g}_{2\sigma^{-1}}(k_1,k_2)-\mathbf{g}^{+0}_{2\sigma^{-1}}(k_1,k_2)-\mathbf{g}^{0+}_{2\sigma^{-1}}(k_1,k_2)+\mathbf{g}^{++}_{2\sigma^{-1}}(k_1,k_2)\right].\end{aligned}}$$

CONTINUOUS-DISCRETE CORRESPONDENCE

(C_κ and C_σ are constants) [arXiv:1912.01998]

$$\boxed{\mathbf{g}_\kappa(n)=\sum_{\alpha=-\infty}^{\infty}g_\kappa\Big((n+\alpha d)\sqrt{\tfrac{2\pi}{d}}\Big)}\quad\boxed{\mathbf{F}[\mathbf{g}_\kappa](k)=\sum_{\beta=-\infty}^{\infty}\mathcal{F}[g_\kappa]\Big((k+\beta d)\sqrt{\tfrac{2\pi}{d}}\Big)}$$

$$\boxed{\mathbf{W}_{\mathbf{g}_\kappa}(n,k)=C_\kappa\sum_{\alpha,\beta=-\infty}^{\infty}(-1)^{\alpha\beta}\mathcal{W}_{g_\kappa}\Big((n+\alpha\tfrac{d}{2})\sqrt{\tfrac{2\pi}{d}},(k+\beta\tfrac{d}{2})\sqrt{\tfrac{2\pi}{d}}\Big)}$$

$$\boxed{\mathbf{g}_\sigma(n_1,n_2)=\sum_{\alpha_1,\alpha_2=-\infty}^{\infty}g_\sigma\Big((n_1+\alpha_1d)\sqrt{\tfrac{2\pi}{d}},(n_2+\alpha_2d)\sqrt{\tfrac{2\pi}{d}}\Big)}$$

$$\boxed{\mathbf{F}[\mathbf{g}_\sigma](k_1,k_2)=\sum_{\beta_1,\beta_2=-\infty}^{\infty}\mathcal{F}[g_\sigma]\ \Big((k_1+\beta_1d)\sqrt{\tfrac{2\pi}{d}},(k_2+\beta_2d)\sqrt{\tfrac{2\pi}{d}}\Big)}$$

$$\boxed{\mathbf{W}_{\mathbf{g}_\sigma}(n_1,n_2,k_1,k_2)=C_\sigma\sum_{\alpha_1,\alpha_2=-\infty}^{\infty}\ \sum_{\beta_1,\beta_2=-\infty}^{\infty}(-1)^{\alpha_1\beta_1+\alpha_2\beta_2}\mathcal{W}_{g_\sigma}\Big((n_1+\alpha_1\tfrac{d}{2})\sqrt{\tfrac{2\pi}{d}},(n_2+\alpha_2\tfrac{d}{2})\sqrt{\tfrac{2\pi}{d}},(k_1+\beta_1\tfrac{d}{2})\sqrt{\tfrac{2\pi}{d}},(k_2+\beta_2\tfrac{d}{2})\sqrt{\tfrac{2\pi}{d}}\Big)}$$