COMPLEX ANALYSIS - Solved Problems 4

Nicolae Cotfas, version 02 June 2026 (for future updates see https://unibuc.ro/user/nicolae.cotfas/)

PROBLEMS

Problem 1 Compute:

1a $(1-3\mathrm{i})^2+(\overline{1-2\mathrm{i}})+\frac{1-\mathrm{i}}{1+\mathrm{i}}+\mathrm{i}^7+|5-12\mathrm{i}|$.

1b $\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}+\mathrm{e}^{1+\pi\mathrm{i}}+\cos(2\mathrm{i})$.

1c $(z^4+\frac{1}{z^2}-\sin z^2+z\mathrm{e}^{2\mathrm{i}z})'$.

1d $\log(2+2\sqrt{2}\,\mathrm{i})$.

Problem 2 Expand the function

$$f:\mathbb{C}\backslash\{0,1\}\to\mathbb{C},\quad f(z)=\frac{1}{z-z^2}$$

2a in a power series around $z_0=0$ (series of powers of z),

2b in a power series around $z_0=\mathrm{i}$ (series of powers of $(z-\mathrm{i})$).

Problem 3

3 By using the definiton of the complex integral, compute

$$I=\int_\gamma \frac{2}{z-1}dz,$$

where γ is the circle $\{\, z \mid |z-1|=\frac{1}{2}\,\}$.

Problem 3

Compute the residues of the functions

4a $f:\mathbb{C}\backslash\{1,\mathrm{i},-\mathrm{i}\}\to\mathbb{C},\quad f(z)=\frac{1}{(z-1)^2(z^2+1)}$

4b $g:\mathbb{C}\backslash\{0\}\to\mathbb{C},\quad g(z)=\frac{1}{z\sin z}$

4c $h:\mathbb{C}\backslash\{0\}\to\mathbb{C},\quad h(z)=z^3\,\mathrm{e}^{\frac{1}{z}}\,\cos\frac{1}{z}$

at the corresponding singular points.

Problem 5

Compute the integrals

5a $I_a=\int_0^{2\pi}\frac{1}{(2+\cos t)^2}dt.$

5b $I_b=\int_{-\infty}^{\infty}\frac{1}{1+x+x^2}dx.$

SOME DEFINITIONS AND THEOREMS

D1 **Definition**

$(x_1+y_1\mathrm{i})(x_2+y_2\mathrm{i})=(x_1x_2-y_1y_2)+(x_1y_2+x_2y_1)\mathrm{i}$

$\overline{x+y\mathrm{i}}=x-y\mathrm{i}$,

$|x+y\mathrm{i}|=\sqrt{x^2+y^2}$,

$|z_1-z_2|$ = distance between z_1 and z_2,

$|z|$ = distance between z and 0,

$\mathrm{e}^{\mathrm{i}t}\overset{\mathrm{def}}{=}\cos t+\mathrm{i}\sin t$ (Euler's formula),

$\mathrm{e}^{x+y\mathrm{i}}=\mathrm{e}^x\cos y+\mathrm{i}\,\mathrm{e}^x\sin y$,

$z=|z|\,\mathrm{e}^{\mathrm{i}\arg z}$, where $-\pi<\arg z\leq\pi$,

$\cos z=\frac{1}{2}(\mathrm{e}^{\mathrm{i}z}+\mathrm{e}^{-\mathrm{i}z})$,

$\sin z=\frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}z}-\mathrm{e}^{-\mathrm{i}z})$,

$\log z=\ln|z|+\mathrm{i}\arg z$, $\quad\log_k z=\ln|z|+\mathrm{i}(\arg z+2k\pi)$.

D2 **Definition** Let $D\subseteq\mathbb{C}$ be an open set, and $z_0\in D$.

A function $f:D\longrightarrow\mathbb{C}$ is **complex-differentiable** ($\mathbb{C}$-differentiable) at z_0 $\overset{\mathrm{def}}{\Longleftrightarrow}$ there exists and is finite $f'(z_0)\overset{\mathrm{def}}{=}\lim\limits_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$.

D3 **Definition**

$f:D\to\mathbb{C}$ defined on an open set D is called $\mathbb{C}$-*differentiable* if (or *holomorphic function*) f is $\mathbb{C}$-differentiable at any point $z_0\in D$.

T1 **Theorem**

$(f\pm g)'=f'\pm g'$ $\qquad(z^n)'=nz^{n-1}$

$(fg)'=f'g+fg'$ $\qquad(\mathrm{e}^z)'=\mathrm{e}^z$

$\left(\frac{f}{g}\right)'=\frac{f'g-fg'}{g^2}$ $\qquad(\sin z)'=\cos z$

$(f(\varphi(z)))'=f'(\varphi(z))\,\varphi'(z)$ $\qquad(\cos z)'=-\sin z$.

T2 **Theorem**

$\frac{1}{1-z}=1+z+z^2+z^3+\ldots$ for $|z|<1$

$\mathrm{e}^z=1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\ldots$ for any z

D4 **Definition** Let $D\subseteq\mathbb{C}$ be an open set, $f:D\longrightarrow\mathbb{C}$ be a continuous function, $\gamma:[a,b]\longrightarrow D$ be a path of class C^1.

The complex line integral of f along γ is $\int_\gamma f(z)\,dz=\int_a^b f(\gamma(t))\,\gamma'(t)dt$

T3 **Theorem**

$g:D\to\mathbb{C}$ is a primitive of f (that is $g'=f$) $\Rightarrow$ $\int_\gamma f(z)\,dz=g(\gamma(b))-g(\gamma(a))$

If $f:D\to\mathbb{C}$ admits a primitive and γ is closed, then $\int_\gamma f(z)\,dz=0$.

D5 **Definition** Let D be an open set and $f:D\to\mathbb{C}$ holomorphic.

$z_0\in\mathbb{C}\backslash D$ is an *isolated singular point* if there exists $r>0$ such that $\{\,z\mid 0<|z-z_0|<r\,\}\subset D$.

$z_0\in D$ is a *zero of multiplicity k* if $f(z_0)=f'(z_0)=\ldots=f^{(k-1)}(z_0)=0$ and $f^{(k)}(z_0)\neq0$.

$z_0\in\mathbb{C}\backslash D$ is a *pole of order k of f* if z_0 is a zero of multiplicity k of $\frac{1}{f}$.

T4 **Theorem** Let D be an open set and $f:D\longrightarrow\mathbb{C}$ holomorphic.

z_0 is a an isolated singular point and $\{\,z\mid 0<|z-z_0|<r\,\}\subset D$ $\Rightarrow$ there exists a unique Laurent series such that $f(z)=\sum\limits_{n=-\infty}^{\infty}a_n\,(z-z_0)^n$, for z satisfying $0<|z-z_0|<r$.

The number $\mathbf{Res}_{z_0}f\overset{\mathrm{def}}{=}a_{-1}$ is the **residue** of f in z_0.

T5 **Theorem**

If z_0 pole of order k, then around z_0 the function f admits an expansion of the form

$$f(z)=\frac{a_{-k}}{(z-z_0)^k}+\ldots+\frac{a_{-1}}{(z-z_0)}+a_0+a_1(z-z_0)+\ldots$$

and

$$\mathbf{Res}_{z_0}f=\frac{1}{(k-1)!}\lim_{z\to z_0}\left((z-z_0)^k\,f(z)\right)^{(k-1)}$$

D6 **Definition** (*Index* of a point z_0 with respect to a path).
For a closed path γ not passing through z_0,

$n(z_0,\gamma)\overset{\mathrm{def}}{=}\frac{1}{2\pi\mathrm{i}}\int_\gamma\frac{1}{z-z_0}dz$ shows how many turns around z_0, γ makes.

D7 **Definition**. Let D be an open set .
A closed path $\gamma:[a,b]\to D$ is called *homotopic to zero in D* if γ can be continuosly deformed inside D up to a constant path (a path having as image just a point).

T6 **Theorem of Residues**. For an open set $D\subseteq\mathbb{C}$:

$f:D\longrightarrow\mathbb{C}$ holomorphic function, S set of isolated singular points, γ path homotopic to zero in $D\cup S$ $\Rightarrow$ $\int_\gamma f(z)dz=2\pi\mathrm{i}\sum\limits_{z\in S}n(z,\gamma)\mathbf{Res}_z f$

T7 **Lemma 1.** For $\gamma_r:[\alpha,\beta|\to\mathbb{C},\ \gamma_r(t)=r\,\mathrm{e}^{\mathrm{i}t}$

$\lim\limits_{z\to\infty}z\,f(z)=0\Rightarrow\lim\limits_{r\to\infty}\int_{\gamma_r}f(z)\,dz=0$

SOLUTIONS

Problem 1

1a By using D1, we get

$$(1-3\mathrm{i})^2=1-6\mathrm{i}-9=-8-6\mathrm{i},$$
$$\overline{(1-2\mathrm{i})}=1+2\mathrm{i},$$
$$\frac{1-\mathrm{i}}{1+\mathrm{i}}=\frac{(1-\mathrm{i})^2}{(1-\mathrm{i})(1+\mathrm{i})}=\frac{1-2\mathrm{i}-1}{2}=-\mathrm{i},$$
$$\mathrm{i}^7=\mathrm{i}^4\,\mathrm{i}^2\,\mathrm{i}=1\,(-1)\,\mathrm{i}=-\mathrm{i},$$
$$|5-12\mathrm{i}|=\sqrt{5^2+12^2}=13,$$

whence

$$(1-3\mathrm{i})^2+\overline{(1-2\mathrm{i})}+\frac{1-\mathrm{i}}{1+\mathrm{i}}+\mathrm{i}^7+|5-12\mathrm{i}|=6-6\mathrm{i}.$$

1b By using D1, we get

$$\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}=\cos\frac{3\pi}{4}+\mathrm{i}\sin\frac{3\pi}{4}=-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\mathrm{i},$$
$$\mathrm{e}^{1+\pi\mathrm{i}}=\mathrm{e}\,\mathrm{e}^{\pi\mathrm{i}}=\mathrm{e}\,(\cos\pi+\mathrm{i}\sin\pi)=-\mathrm{e},$$
$$\cos(2\mathrm{i})=\frac{\mathrm{e}^{2\mathrm{i}^2}+\mathrm{e}^{-2\mathrm{i}^2}}{2}=\frac{\mathrm{e}^{-2}+\mathrm{e}^2}{2},$$

whence

$$\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}+\mathrm{e}^{1+\pi\mathrm{i}}+\cos(2\mathrm{i})=-\mathrm{e}+\frac{\mathrm{e}^{-2}+\mathrm{e}^2}{2}-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\mathrm{i}.$$

1c By using D1 and T1, we obtain

$$\left(z^4+\frac{1}{z^2}-\sin z^2+z\mathrm{e}^{2\mathrm{i}z}\right)'=4z^3-\frac{2}{z^3}-2z\cos z^2+\mathrm{e}^{2\mathrm{i}z}+2\mathrm{i}z\mathrm{e}^{2\mathrm{i}z}.$$

1d By using D1, we get $\log(2+2\sqrt{2}\,\mathrm{i})=$

$$=\ln|2+2\sqrt{2}\,\mathrm{i}|+\mathrm{i}\arg(2+2\sqrt{3}\,\mathrm{i})=\ln(2\sqrt{3})+\mathrm{i}\arctan\sqrt{2}.$$

Problem 2

2a By using T2, around $z_0=0$, we get

$$f(z)=\frac{1}{z-z^2}=\frac{1}{z}\frac{1}{1-z}$$
$$=\frac{1}{z}(1+z+z^2+z^3+\dots)=\frac{1}{z}+1+z+z^2+z^3+\dots\quad\text{for } |z|<1.$$

2b By using T2, around $z_1=\mathrm{i}$, we get

$$f(z)=\frac{1}{z-z^2}=\frac{1}{z}+\frac{1}{1-z}\quad\text{and}$$
$$\frac{1}{z}=\frac{1}{\mathrm{i}+z-\mathrm{i}}=\frac{1}{\mathrm{i}}\frac{1}{1-(-\frac{z-\mathrm{i}}{\mathrm{i}})}=-\mathrm{i}\,\frac{1}{1-\mathrm{i}(z-\mathrm{i})}=-\mathrm{i}[1+\mathrm{i}(z-\mathrm{i})+\mathrm{i}^2(z-\mathrm{i})^2+\dots]$$
$$=-\mathrm{i}+(z-\mathrm{i})+\mathrm{i}(z-\mathrm{i})^2+\dots\quad\text{for } |z-\mathrm{i}|<1,$$
$$\frac{1}{1-z}=\frac{1}{1-\mathrm{i}-(z-\mathrm{i})}=\frac{1}{1-\mathrm{i}}\,\frac{1}{1-\frac{z-\mathrm{i}}{1-\mathrm{i}}}=\frac{1}{1-\mathrm{i}}\,\frac{1}{1-\frac{(1+\mathrm{i})(z-\mathrm{i})}{2}}=\frac{1+\mathrm{i}}{2}\,\frac{1}{1-\frac{1+\mathrm{i}}{2}(z-\mathrm{i})}$$
$$=\frac{1+\mathrm{i}}{2}\left[1+\frac{1+\mathrm{i}}{2}(z-\mathrm{i})+\frac{(1+\mathrm{i})^2}{2^2}(z-\mathrm{i})^2+\dots\right]$$
$$=\frac{1+\mathrm{i}}{2}+\frac{(1+\mathrm{i})^2}{2^2}(z-\mathrm{i})+\frac{(1+\mathrm{i})^3}{2^3}(z-\mathrm{i})^2+\dots$$

for $\left|\frac{1+\mathrm{i}}{2}(z-\mathrm{i})\right|<1$, that is $|z-\mathrm{i}|<\sqrt{2}$.

Problem 3

3 By using the definition D4 and the parametrization

$$\gamma:[0,2\pi]\to\mathbb{C}\backslash\{1\},\quad \gamma(t)=1+\tfrac{1}{2}\mathrm{e}^{\mathrm{i}t}$$

we get

$$\int_\gamma\frac{2}{z-1}dz=\int_0^{2\pi}\frac{2}{1+\frac{1}{2}\mathrm{e}^{\mathrm{i}t}-1}\,(1+\tfrac{1}{2}\mathrm{e}^{\mathrm{i}t})'\,dt$$
$$=\int_0^{2\pi}\frac{2}{\frac{1}{2}\mathrm{e}^{\mathrm{i}t}}\,\tfrac{1}{2}\mathrm{i}\,\mathrm{e}^{\mathrm{i}t}\,dt=2\mathrm{i}\int_0^{2\pi}dt=4\pi\mathrm{i}.$$

Problem 4

4a We use D1, D5, T5. The singular points are 1, i and $-$i.

The point $z_0=1$ is a pole of order 2, and consequently

$$\mathbf{Res}_1 f=\frac{1}{1!}\lim_{z\to1}\left((z-1)^2f(z)\right)'=\lim_{z\to1}\left((z-1)^2\frac{1}{(z-1)^2(z^2+1)}\right)'$$
$$=\lim_{z\to1}\left(\frac{1}{z^2+1}\right)'=\lim_{z\to1}\frac{-2z}{(z^2+1)^2}=\frac{-2}{2^2}=\frac{-1}{2}.$$

The points $z_{1,2}=\pm\mathrm{i}$ are poles of order 1, and consequently

$$\mathbf{Res}_{\pm\mathrm{i}}f=\frac{1}{0!}\lim_{z\to\pm\mathrm{i}}(z\mp\mathrm{i})f(z)$$
$$=\lim_{z\to\pm\mathrm{i}}(z\mp\mathrm{i})\frac{1}{(z-1)^2(z-\mathrm{i})(z+\mathrm{i})}=\lim_{z\to\pm\mathrm{i}}\frac{1}{(z-1)^2(z\pm\mathrm{i})}$$
$$=\frac{1}{(\pm\mathrm{i}-1)^2(\pm\mathrm{i}\pm\mathrm{i})}=\frac{1}{4}.$$

4b We use D1, D5, T5. Since $\frac{1}{g(z)}=z\sin z$, and

$$\left(\frac{1}{g(z)}\right)'=\sin z+z\cos z,\quad \left(\frac{1}{g(z)}\right)''=2\cos z-z\sin z,$$
$$\frac{1}{g(0)}=\left(\frac{1}{g(z)}\right)'\Big|_{z=0}=0,\quad \left(\frac{1}{g(z)}\right)''\Big|_{z=0}=2\neq0,$$

the point 0 is a pole of order 2, and consequently

$$\mathbf{Res}_0 g=\frac{1}{1!}\lim_{z\to0}(z^2g(z))'=\lim_{z\to0}\left(z^2\frac{1}{z\sin z}\right)'=\lim_{z\to0}\left(\frac{z}{\sin z}\right)'$$
$$=\lim_{z\to0}\frac{\sin z-z\cos z}{\sin^2 z}=\lim_{z\to0}\frac{z-\frac{1}{3!}z^3+\frac{1}{5!}z^5-\ldots-z(1-\frac{1}{2!}z^2+\frac{1}{4!}z^4-\ldots)}{(z-\frac{1}{3!}z^3+\frac{1}{5!}z^5-\ldots)^2}=0.$$

4c We use T2 and T4. The only singular point 0 is not a pole. So, we have to use the Laurent series of h around 0,

$$h(z)=z^3\left(1+\frac{1}{1!}\frac{1}{z}+\frac{1}{2!}\frac{1}{z^2}+\frac{1}{3!}\frac{1}{z^3}+\dots\right)\left(1-\frac{1}{2!}\frac{1}{z^2}+\frac{1}{4!}\frac{1}{z^4}-\frac{1}{6!}\frac{1}{z^6}+\dots\right)$$
$$=\left(z^3+\frac{1}{1!}z^2+\frac{1}{2!}z+\frac{1}{3!}+\frac{1}{4!}\frac{1}{z}+\dots\right)\left(1-\frac{1}{2!}\frac{1}{z^2}+\frac{1}{4!}\frac{1}{z^4}-\frac{1}{6!}\frac{1}{z^6}+\dots\right)$$
$$=\dots+\left(\frac{1}{24}-\frac{1}{4}+\frac{1}{24}\right)\frac{1}{z}=\dots\quad \text{Consequently } \mathbf{Res}_0 h=-\frac{1}{6}.$$

Problem 5

5a We use D1, D4, T5, T6. By using D1, the expression

$$I=\int_0^{2\pi}\frac{1}{(2+\cos t)^2}dt$$

can be written as

$$I=\int_0^{2\pi}\frac{1}{(2+\frac{1}{2}(\mathrm{e}^{\mathrm{i}t}+\mathrm{e}^{-\mathrm{i}t}))^2}dt$$

or

$$I=\int_0^{2\pi}\frac{4}{(4+\mathrm{e}^{\mathrm{i}t}+\mathrm{e}^{-\mathrm{i}t})^2}\frac{1}{\mathrm{i}\mathrm{e}^{\mathrm{i}t}}(\mathrm{e}^{\mathrm{i}t})'dt\,.$$

This formula, written as

$$I=-4\mathrm{i}\int_0^{2\pi}\frac{1}{(4+\mathrm{e}^{\mathrm{i}t}+\mathrm{e}^{-\mathrm{i}t})^2}\frac{1}{\mathrm{e}^{\mathrm{i}t}}(\mathrm{e}^{\mathrm{i}t})'dt\,.$$

is the complex integral $\displaystyle I=-4\mathrm{i}\int_\gamma\frac{1}{(4+z+\frac{1}{z})^2}\frac{1}{z}\,dz$

that is

$$I=-4\mathrm{i}\int_\gamma\frac{z}{(z^2+4z+1)^2}\,dz$$

along the circular path $\gamma:[0,2\pi]\to\mathbb{C}$, $\gamma(t)=\mathrm{e}^{\mathrm{i}t}$.
It can be computed by using the theorem T6. Since
$z^2+4z+1=(z+2)^2-(\sqrt{3})^2=(z-z_1)(z-z_2)$,
the singular points are $z_1=-2+\sqrt{3}$ and $z_2=-2-\sqrt{3}$,
but only z_1 is inside the domain with the frontier γ.
The point z_1 is a pole of order 2, and by using T5,

$$\mathbf{Res}_{z_1}\frac{z}{(z^2+4z+1)^2}=\lim_{z\to z_1}\left((z-z_1)^2\frac{z}{(z-z_1)^2(z-z_2)^2}\right)'=\lim_{z\to z_1}\left(\frac{z}{(z-z_2)^2}\right)'$$
$$=\lim_{z\to z_1}\frac{(z-z_2)^2-2z(z-z_2)}{(z-z_2)^4}=\frac{z_1-z_2-2z_1}{(z_1-z_2)^3}=\frac{2\sqrt{3}-2(-2+\sqrt{3})}{(2\sqrt{3})^3}=\frac{1}{6\sqrt{3}}.$$

Therefore,

$$I=-4\mathrm{i}\,2\pi\mathrm{i}\,\mathbf{Res}_{z_1}\frac{z}{(z^2+4z+1)^2}=\frac{8\pi}{6\sqrt{3}}=\frac{4\pi}{3\sqrt{3}}.$$

5b In view of T6, for $r>1$, by integrating the function

$$f:D=\mathbb{C}\backslash\{z_1,z_2\}\to\mathbb{C},\quad f(z)=\frac{1}{1+z+z^2}=\frac{1}{(z-z_1)(z-z_2)}$$

where $z_1=-\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}$ and $z_2=-\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}$
along the closed path obtained by composing the paths

$$\gamma_r:[0,\pi]\to D,\quad \gamma_r(t)=r\,\mathrm{e}^{\mathrm{i}t}\quad\text{and}$$
$$\gamma:[-r,r]\to D,\quad \gamma(t)=t,$$

we get the relation

$$\int_{\gamma_r}\frac{1}{1+z+z^2}\,dz+\int_{-r}^{r}\frac{1}{1+x+x^2}\,dx=2\pi\mathrm{i}\,\mathbf{Res}_{z_1}\frac{1}{1+z+z^2}\qquad(*).$$

The point z_1 is a pole of order 1, and by using T5,

$$\mathbf{Res}_{z_1}\frac{1}{1+z+z^2}=\lim_{z\to z_1}(z-z_1)\frac{1}{(z-z_1)(z-z_2)}$$
$$=\lim_{z\to z_1}\frac{1}{z-z_2}=\frac{1}{z_1-z_2}=\frac{1}{\mathrm{i}\sqrt{3}}=-\frac{\mathrm{i}}{\sqrt{3}}.$$

By using the relations

$$|z+z'|\le|z|+|z'|\quad\text{and}\quad\big||z|-|z'|\big|\le|z-z'|$$

satisfied for any $z,z'\in\mathbb{C}$, we get

$$0\le\left|z\frac{1}{1+z+z^2}\right|=\frac{|z|}{|1+z+z^2|}=\frac{|z|}{|z^2-(-z-1)|}\le\frac{|z|}{\big||z^2|-|-z-1|\big|}$$
$$=\frac{|z|}{\big||z|^2-|z+1|\big|}\le\frac{|z|}{\big||z|^2-|z|-1\big|}\le\frac{1}{\big||z|-1-\frac{1}{|z|}\big|}\xrightarrow{|z|\to\infty}0.$$

By using T7, we get $\displaystyle\lim_{r\to\infty}\int_{\gamma_r}\frac{1}{1+z+z^2}\,dz=0$, and consequently, for $r\to\infty$, the relation (*) becomes

$$0+\int_{-\infty}^{\infty}\frac{1}{1+x+x^2}\,dx=2\pi\mathrm{i}\,\mathbf{Res}_{z_1}\frac{1}{1+z+z^2}=\frac{2\pi}{\sqrt{3}}.$$