Explicit expression of Wigner function of a Gaussian function of discrete variable

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■ An explicit description of the discrete Wigner function of a Gaussian function of one discrete variable

The Gaussian function of one discrete variable

$$\boxed{\begin{aligned} &g_\kappa:\{-s,-s+1,...,s-1,s\}\to\mathbb{R},\\ &g_\kappa(n)=\sum_{\alpha=-\infty}^{\infty}\mathrm{e}^{-\frac{\kappa\pi}{d}(n+\alpha d)^2}\\ &\quad=\sum_{\alpha=-\infty}^{\infty}\mathrm{G}_\kappa\left((n+\alpha d)\sqrt{\tfrac{2\pi}{d}}\right),\end{aligned}}\quad \textit{where}\quad \begin{aligned} &s\in\{1,2,3,...\}\\ &d=2s+1\\ &\kappa\in(0,\infty)\\ &\mathrm{G}_\kappa:\mathbb{R}\to\mathbb{R},\\ &\mathrm{G}_\kappa(q)=\mathrm{e}^{-\frac{\kappa}{2}q^2}\end{aligned}$$

has the discrete Fourier transform

$$\boxed{\begin{aligned} &F[g_\kappa]:\{-s,-s+1,...,s-1,s\}\to\mathbb{R},\\ &F[g_\kappa](k)=\tfrac{1}{\sqrt{d}}\sum_{n=-s}^{s}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}kn}\,g_\kappa(n)=\tfrac{1}{\sqrt{\kappa}}\sum_{\alpha=-\infty}^{\infty}\mathrm{e}^{-\frac{\pi}{\kappa d}(k+\alpha d)^2}\\ &\quad=\sum_{\alpha=-\infty}^{\infty}F[\mathrm{G}_\kappa]\left((k+\alpha d)\sqrt{\tfrac{2\pi}{d}}\right)=\tfrac{1}{\sqrt{\kappa}}\,g_{\kappa^{-1}}(k),\end{aligned}}\quad \textit{where}\quad \begin{aligned} &F[\mathrm{G}_\kappa]:\mathbb{R}\to\mathbb{R},\\ &F[\mathrm{G}_\kappa](p)=\tfrac{1}{\sqrt{\kappa}}\mathrm{e}^{-\frac{1}{2\kappa}p^2},\\ &F[\mathrm{G}_\kappa]=\tfrac{1}{\sqrt{\kappa}}\mathrm{G}_{\kappa^{-1}},\end{aligned}$$

and the discrete Wigner function

$$\mathcal{W}_{g_\kappa}(n,k)=C_d\sum_{\alpha,\beta=-\infty}^{\infty}(-1)^{\alpha\beta}\,\mathcal{W}_{\mathrm{G}_\kappa}\left((n+\alpha\tfrac{d}{2})\sqrt{\tfrac{2\pi}{d}},(k+\beta\tfrac{d}{2})\sqrt{\tfrac{2\pi}{d}}\right),$$

where C_d is a normalizing factor, and

$$\mathcal{W}_{\mathrm{G}_\kappa}(q,p)=\mathrm{e}^{-(q\ \ p)\begin{pmatrix}\kappa^{-1}&0\\0&\kappa\end{pmatrix}^{-1}\begin{pmatrix}q\\p\end{pmatrix}}$$

is the Wigner function of G_κ, up to a multiplicative constant.

Proof. The formulas obtained in

[N. Cotfas and D. Dragoman, Properties of finite Gaussians and the discrete-continuous transition *J. Phys. A: Math. and Theor.* **45** (2012) 425305 (https://arxiv.org/pdf/1205.6302.pdf)]

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■ An explicit description of the discrete Wigner function of a Gaussian function of two discrete variables

The Gaussian function of two discrete variables

$$\boxed{\begin{aligned} &g_\sigma:\{-s,-s+1,...,s-1,s\}\times\{-s,-s+1,...,s-1,s\}\longrightarrow\mathbb{R},\\ &g_\sigma(n_1,n_2)=\sum_{\alpha_1,\alpha_2=-\infty}^{\infty}\mathrm{e}^{-\frac{\pi}{d}(n_1+\alpha_1 d\ \ n_2+\alpha_2 d)\begin{pmatrix}a&b\\b&c\end{pmatrix}\begin{pmatrix}n_1+\alpha_1 d\\n_2+\alpha_2 d\end{pmatrix}}\\ &\quad=\sum_{\alpha_1,\alpha_2=-\infty}^{\infty}\mathrm{G}_\sigma((n_1+\alpha_1 d)\sqrt{\tfrac{2\pi}{d}},(n_2+\alpha_2 d)\sqrt{\tfrac{2\pi}{d}})\end{aligned}}\quad \textit{where}\quad \begin{aligned} &\sigma=\begin{pmatrix}a&b\\b&c\end{pmatrix}>0\\ &\mathrm{G}_\sigma:\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R},\\ &\mathrm{G}_\sigma(q_1,q_2)=\mathrm{e}^{-\frac{1}{2}(q_1\ \ q_2)\begin{pmatrix}a&b\\b&c\end{pmatrix}\begin{pmatrix}q_1\\q_2\end{pmatrix}}\end{aligned}$$

has the discrete Fourier transform

$$\boxed{\begin{aligned} &F[g_\sigma]:\{-s,-s+1,...,s-1,s\}\times\{-s,-s+1,...,s-1,s\}\longrightarrow\mathbb{R},\\ &F[g_\sigma](k_1,k_2)=\tfrac{1}{d}\sum_{n_1=-s}^{s}\sum_{n_2=-s}^{s}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}(k_1n_1+k_2n_2)}\,g_\sigma(n_1,n_2).\\ &\quad=\tfrac{1}{\sqrt{\det\sigma}}\sum_{\beta_1,\beta_2=-\infty}^{\infty}\mathrm{G}_{\sigma^{-1}}\left((k_1+\beta_1 d)\sqrt{\tfrac{2\pi}{d}},(k_2+\beta_2 d)\sqrt{\tfrac{2\pi}{d}}\right)\\ &\quad=\sum_{\beta_1,\beta_2=-\infty}^{\infty}F[\mathrm{G}_\sigma]\left((k_1+\beta_1 d)\sqrt{\tfrac{2\pi}{d}},(k_2+\beta_2 d)\sqrt{\tfrac{2\pi}{d}}\right)=\tfrac{1}{\sqrt{\det\sigma}}\,g_{\sigma^{-1}}(k_1,k_2),\end{aligned}}\quad \textit{where}\quad \begin{aligned} &F[\mathrm{G}_\sigma]:\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R},\\ &F[\mathrm{G}_\sigma]=\tfrac{1}{\sqrt{\det\sigma}}\,\mathrm{G}_{\sigma^{-1}}\end{aligned}$$

and the discrete Wigner function

$$\mathcal{W}_{g_\sigma}(n_1,n_2,k_1,k_2)=C_d\sum_{\alpha_1,\alpha_2=-\infty}^{\infty}\ \sum_{\beta_1,\beta_2=-\infty}^{\infty}(-1)^{\alpha_1\beta_1+\alpha_2\beta_2}$$

$$\times\mathcal{W}_{\mathrm{G}_\sigma}\left((n_1+\alpha_1\tfrac{d}{2})\sqrt{\tfrac{2\pi}{d}},(n_2+\alpha_2\tfrac{d}{2})\sqrt{\tfrac{2\pi}{d}},(k_1+\beta_1\tfrac{d}{2})\sqrt{\tfrac{2\pi}{d}},(k_2+\beta_2\tfrac{d}{2})\sqrt{\tfrac{2\pi}{d}}\right),$$

where C_d is a normalizing factor, and

$$\mathcal{W}_{\mathrm{G}_\sigma}(q_1,q_2,p_1,p_2)=\mathrm{e}^{-\left\{(q_1\ \ q_2\ \ p_1\ \ p_2)\begin{pmatrix}\sigma^{-1}&\begin{matrix}0&0\\0&0\end{matrix}\\\begin{matrix}0&0\\0&0\end{matrix}&\sigma\end{pmatrix}^{-1}\begin{pmatrix}q_1\\q_2\\p_1\\p_2\end{pmatrix}\right\}}$$

is the Wigner function of G_σ, up to a multiplicative constant.

Proof. The formulas obtained in

[N. Cotfas , On the Gaussian functions of two discrete variables https://arxiv.org/pdf/1912.01998.pdf]

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