DIFFERENTIAL EQUATIONS - Solved Exercises 4

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EXERCISES

■ **Exercise 1**

Find the general solution of the equations:

$\boxed{1}$ $y''-4y'+3y=0.$

$\boxed{2}$ $y''-4y'+4y=0.$

$\boxed{3}$ $y''-4y'+5y=0.$

$\boxed{4}$ $y''''+y''-6y=0.$

■ **Exercise 2**

Solve the equations

$\boxed{5}$ $y'=-y.$

$\boxed{6}$ $y'=-y+x\mathrm{e}^{-x}+1.$

$\boxed{7}$ $x^2y''+xy'-y=0.$

■ **Exercise 3**

Solve the equations

$\boxed{8}$ $(x+2y)dx+2xdy=0.$

$\boxed{9}$ $(1+2\frac{y}{x})dx+2dy=0.$

■ **Exercise 4**

Solve the sytem of differential equations

$\boxed{10}$ $\begin{cases} y_1'=3y_1-y_2 \\ y_2'=y_1+y_2 \end{cases}$

■ **Exercise 5**

$\boxed{11}$ Find the eigenfunctions of the differential operator

$$A=-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}+1$$

corresponding to the eigenvalues $\lambda\in\mathbb{R}$, that is, solve the differential equation

$$-\mathrm{i}\frac{\mathrm{d}\psi}{\mathrm{d}x}+\psi=\lambda\psi,$$

by looking for complex functions as solutions.

■ **Exercise 6**

$\boxed{12}$ Find the linear second-order differential quation with variable coefficients satisfied by the function

$$\psi(x)=\mathrm{e}^{-\frac{1}{2}x^2}.$$

SOME DEFINITIONS, THEOREMS and REMARKS

$\boxed{T1}$ **Theorem** (*Linear equations with constant coefficients).*

The space of all the real solutions of

$$a_0y^{(n)}+a_1y^{(n-1)}+\cdots+a_{n-1}y'+a_ny=0 \qquad (1)$$

where $a_0,\dots,a_n\in\mathbb{R}$, is a real vector space of dimension n.

$\boxed{D1}$ **Definition**. The polynomial

$$P(\lambda)=a_0\lambda^n+a_1\lambda^{n-1}+\cdots+a_{n-1}\lambda+a_n$$

is called the *characteristic polynomial* of the equation (1).

$\boxed{T2}$ **Theorem** (Particular solutions).

$$\boxed{y(x)=\mathrm{e}^{\lambda x} \text{ is a solution of (1)}} \Leftrightarrow \boxed{P(\lambda)=0}$$

$\boxed{D2}$ **Definition** (*Complex exponential*).

$$\boxed{\mathrm{e}^{(\alpha+\beta\mathrm{i})x}=\mathrm{e}^{\alpha x}\cos\beta x+\mathrm{i}\,\mathrm{e}^{\alpha x}\sin\beta x}$$

$\boxed{T3}$ **Theorem**. General solution of $a_0y''+a_1y'+a_2y=0.$

$P(\lambda)=a_0\lambda^2+a_1\lambda+a_2$ has the roots $\lambda_{1,2}=\frac{-a_1\pm\sqrt{a_1^2-4a_0\,a_2}}{2a_0}.$

General solution:

- $\lambda_1\neq\lambda_2\in\mathbb{R} \quad\Rightarrow\quad y(x)=C_1\,\mathrm{e}^{\lambda_1x}+C_2\,\mathrm{e}^{\lambda_2x}.$
- $\lambda_1=\lambda_2=\lambda \quad\Rightarrow\quad y(x)=C_1\,\mathrm{e}^{\lambda x}+C_2\,x\,\mathrm{e}^{\lambda x}.$
- $\lambda_{1,2}=\alpha\pm\beta\mathrm{i} \quad\Rightarrow\quad y(x)=C_1\,\mathrm{e}^{\alpha x}\,\cos\beta x+C_2\,\mathrm{e}^{\alpha x}\,\sin\beta x.$

$\boxed{T4}$ **Theorem** (Euler's equation).

$$\boxed{a_0x^ny^{(n)}+\ldots+a_{n-1}xy'+a_ny=0} \xrightarrow[x=\mathrm{e}^t]{\text{change}} \text{linear equation with constant coefficients}$$

$\boxed{T5}$ **Theorem** (*Primitives* of a continuous function)

$f:(a,b)\to\mathbb{R}$ continuous $\Rightarrow$ Primitives of f are: $F:(a,b)\to\mathbb{R}$, $\boxed{F(x)=\int_{x_0}^{x}f(t)\,dt+C}$ $x_0\in(a,b)$ is fixed

We have $F'(x)=f(x)$, that is

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{x_0}^{x}f(t)\,dt\right)=f(x)}$$

for any $x\in(a,b)$.

$\boxed{T6}$ **Theorem** (*Separable equations*).

The solution $y(x)$ of $\boxed{y'=f(x)\,g(y)}$ is defined by $\boxed{\int_{y_0}^{y}\frac{1}{g(u)}du=\int_{x_0}^{x}f(t)\,dt+C}$

x_0, y_0 constants, $g(y_0)\neq0$.

$\boxed{T7}$ **Linear equation**.

$\boxed{y'=f(x)y}$ has the general solution $\boxed{y(x)=C\,\mathrm{e}^{\int_{x_0}^{x}f(t)dt}}$

$\boxed{T8}$ **Method of the variation of parameter**.

$\boxed{y'=f(x)y+g(x)}$ admits a particular solution of the form $y_p(x)=C(x)\,\mathrm{e}^{\int_{x_0}^{x}f(t)dt}$.

$\boxed{T9}$ **Linear non-homogeneous equation**.

$$\boxed{\begin{array}{c}\text{general solution of}\\ y'=f(x)y+g(x)\end{array}}=\boxed{\begin{array}{c}\text{general solution of}\\ y'=f(x)y\end{array}}+\boxed{\begin{array}{c}\text{a particular solution of}\\ y'=f(x)y+g(x)\end{array}}$$

$\boxed{T10}$ **Theorem** (*Exact equations*).

$\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$ in a simply connected domain D $\Rightarrow$

The function $F:D\to\mathbb{R}$,

$$F(x,y)=\int_\gamma Pdx+Qdy,$$

where $\gamma:[a,b]\to D$ is an arbitrary path connecting a fixed point (x_0,y_0) with (x,y), defines a function satisfying the relation

$$P(x,y)dx+Q(x,y)dy=dF$$

$\boxed{R3}$ **Remark**.

In D, the equation $P(x,y)dx+Q(x,y)dy=0$ can be written as $dF=0$, and its solution is described implicitly by $F(x,y)=C$.

$\boxed{R4}$ **Remark**. By denoting $Y(x)=\begin{pmatrix}y_1(x)\\y_2(x)\end{pmatrix}$, $A=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}$,

$$\boxed{\begin{cases}y_1'=a_{11}y_1+a_{12}y_2\\y_2'=a_{21}y_1+a_{22}y_2\end{cases}} \text{ can be written as } \boxed{Y'=AY}$$

$\boxed{T11}$ **Theorem**.

The space of all the real solutions of $Y'=AY$, where $a_{ij}\in\mathbb{R}$, is a real vector space of dimension 2.

$\boxed{D4}$ **Definition**. The polynomial

$$P(\lambda)=\begin{vmatrix}a_{11}-\lambda & a_{12}\\ a_{21} & a_{22}-\lambda\end{vmatrix}=\lambda^2-(a_{11}+a_{22})\lambda+a_{11}a_{22}-a_{12}a_{21}$$

is called the *characteristic polynomial* of $Y'=AY$.

$\boxed{T12}$ **Theorem** (Particular non-null solutions)

$$Y(x)=\begin{pmatrix}p\\q\end{pmatrix}\mathrm{e}^{\lambda x}\neq\begin{pmatrix}0\\0\end{pmatrix} \text{ satisfies } Y'=AY \;\Leftrightarrow\; \begin{cases}P(\lambda)=0 \quad\text{and}\\ A\begin{pmatrix}p\\q\end{pmatrix}=\lambda\begin{pmatrix}p\\q\end{pmatrix}.\end{cases}$$

$\boxed{T13}$ **Theorem.** If λ_1 and λ_2 are the solutions of $P(\lambda)=0$, then:

- $\left.\begin{array}{l}\lambda_1,\ \lambda_2\in\mathbb{R}\\ A\begin{pmatrix}p_j\\q_j\end{pmatrix}=\lambda_j\begin{pmatrix}p_j\\q_j\end{pmatrix}\\ \text{linearly independent}\end{array}\right\}\Rightarrow\; Y(x)=C_1\begin{pmatrix}p_1\\q_1\end{pmatrix}\mathrm{e}^{\lambda_1x}+C_2\begin{pmatrix}p_2\\q_2\end{pmatrix}\mathrm{e}^{\lambda_2x}.$
- $\lambda_1=\lambda_2$ is a double root, then we look for solutions of the form $\begin{pmatrix}y_1(x)\\y_2(x)\end{pmatrix}=\begin{pmatrix}C_1+C_2x\\C_3+C_4x\end{pmatrix}\mathrm{e}^{\lambda_1x}$, and eliminate two constants C_k by substituting into the given system.

SOLUTIONS

Exercise 1.

We use T2 and T3.

$\boxed{1}$ $\lambda^2+4\lambda+3=0 \quad\Rightarrow\quad \lambda_1=1,\ \lambda_2=3.$

$$T3 \quad\Rightarrow\quad y(x)=C_1\,e^{x}+C_2\,e^{3x}.$$

$\boxed{2}$ $\lambda^2+4\lambda+4=0 \quad\Rightarrow\quad \lambda_1=\lambda_2=2.$

$$T3 \quad\Rightarrow\quad y(x)=C_1\,e^{2x}+C_2\,x\,e^{2x}.$$

$\boxed{3}$ $\lambda^2+4\lambda+5=0 \quad\Rightarrow\quad \lambda_{1,2}=2\pm i.$

$$T3 \quad\Rightarrow\quad y(x)=C_1\,e^{2x}\cos x+C_2\,e^{2x}\sin x.$$

$\boxed{4}$ $(\lambda^2-2)(\lambda^2+3)=0 \quad\Rightarrow\quad \lambda_{1,2}=\pm\sqrt{2},\ \ \lambda_{3,4}=\pm i\sqrt{3}.$

$$T3 \quad\Rightarrow\quad y(x)=C_1\,e^{\sqrt{2}x}+C_2\,e^{-\sqrt{2}x} +C_3\,\cos\sqrt{3}x+C_4\,\sin\sqrt{3}x.$$

Exercise 2

$\boxed{5}$ We use T7.

The equation can be written successively as follows:

$$y'=-y,\qquad \frac{y'}{y}=-1,\qquad (\ln y)'=-1,$$
$$\ln y=-x+\ln C,\quad y=Ce^{-x}.$$

$\boxed{6}$ We use T8 and T9. Looking for a particular solution of the form $y_p(x)=C(x)\,e^{-x}$ we get successively:

$$C'\,e^{-x}-Ce^{-x}=-\ C\ e^{-x}+xe^{-x}+1,\qquad C'(x)=x+e^{x},$$

$C(x)=\int(x+e^{-x})dx=\frac{x^2}{2}+e^{x}$, and consequently $y_p(x)=\frac{x^2}{2}\,e^{-x}+1$. T9 $\Rightarrow$ the general solution of $y'=-y+xe^{-x}+1$ is $y(x)=Ce^{-x}+\frac{x^2}{2}\,e^{-x}+1.$

$\boxed{7}$ We use T4, T2 and T3.

We use the change of variables:

$$\begin{matrix} x\mapsto t, \\ y\mapsto z, \end{matrix}\quad \text{satisfying}\quad \begin{matrix} x=e^{t}, & y(x)=z(\ln x), \\ t=\ln x, & z(t)=y(e^{t}). \end{matrix}$$

Since $y'(x)=z'(\ln x)(\ln x)'=\frac{1}{x}\,z'(\ln x),$

$$y''(x)=-\frac{1}{x^2}\,z'(\ln x)+\frac{1}{x^2}\,z''(\ln x),$$

in the new variables, the equation becomes

$$e^{2t}\left(-\frac{1}{e^{2t}}\,z'+\frac{1}{e^{2t}}\,z''\right)+e^{t}\frac{1}{e^{t}}\,z'-z=0,$$

that is $\quad z''-z=0.$

Because $\lambda^2-1=0 \quad\Rightarrow\quad \lambda_{1,2}=\pm1,$

and consequently, $\quad T3 \quad\Rightarrow\quad z(t)=C_1\,e^{t}+C_2\,e^{-t},$

we get $\quad y(x)=z(\ln x)=C_1\,e^{\ln x}+C_2\,e^{-\ln x}.$

$$=C_1\,x+C_2\,\frac{1}{x}.$$

Remark.

The given equation can be written as

$$\left(x^2\frac{d^2}{dx^2}+x\frac{d}{dx}-1\right)y=0.$$

and by using the operatorial relation $\frac{d}{dx}=\frac{dt}{dx}\frac{d}{dt}=e^{-t}\frac{d}{dt}$, we get the equation for $z(t)$,

$$\left(e^{2t}e^{-t}\frac{d}{dt}e^{-t}\frac{d}{dt}+e^{t}e^{-t}\frac{d}{dt}-1\right)z=0.$$

namely, $\quad z''-z=0.$

Exercise 3

$\boxed{8}$ The equation is exact in $D=\mathbb{R}^2$,

$$\frac{\partial(x+2y)}{\partial y}=2=\frac{\partial\,2x}{\partial x}.$$

By using T10, R3 and the path

$$\gamma:[0,1]\to\mathbb{R}^2,\quad \gamma(t)=(xt,yt)$$

connecting $(0,0)$ with (x,y), we get

$$F(x,y)=\int_{\gamma}(x+2y)dx+2xdy=\int_0^1[(xt+2yt)x+2xt\,y]\,dt$$
$$=(x^2+4xy)\int_0^1 t\,dt=(x^2+4xy)\frac{t^2}{2}\Big|_{t=0}^{t=1}=\frac{x^2}{2}+2xy.$$

The given equation can be written as (see R3)

$$d\left(\frac{x^2}{2}+2xy\right)=0,$$

and its solution is described by

$$\frac{x^2}{2}+2xy=C,$$

namely $\quad y(x)=\frac{C}{2x}-\frac{x}{4}\quad$ for $\ x\neq0.$

$\boxed{9}$ This equation can be reduced to the exact equation $\boxed{8}$ by multiplying it with the integrating factor $\mu(x)=x$. The two equations have the same solutions, namely

$$\frac{x^2}{2}+2xy=C.$$

Exercise 4

$\boxed{10}$ The system can be written as $Y'=AY$, where.

$$Y(x)=\begin{pmatrix} y_1(x)\\ y_2(x)\end{pmatrix},\ A=\begin{pmatrix} 3 & -1\\ 1 & 1\end{pmatrix},$$

In this case,

$$\begin{vmatrix} 3-\lambda & -1\\ 1 & 1-\lambda\end{vmatrix}=0 \ \Rightarrow\ \lambda_1=\lambda_2=2.$$

We look for a solution of the form

$$\begin{pmatrix} y_1(x)\\ y_2(x)\end{pmatrix}=\begin{pmatrix} C_1+C_2x\\ C_3+C_4x\end{pmatrix}e^{2x},$$

that is

$$y_1(x)=C_1\,e^{2x}+C_2\,xe^{2x},$$
$$y_2(x)=C_3\,e^{2x}+C_4\,xe^{2x}.$$

By substituting into system, we get

$$2C_1\,e^{2x}+C_2\,e^{2x}+2C_2\,xe^{2x}=3C_1\,e^{2x}+3C_2\,xe^{2x}-C_3\,e^{2x}-C_4\,xe^{2x},$$
$$2C_3\,e^{2x}+C_4\,e^{2x}+2C_4\,xe^{2x}=C_1\,e^{2x}+C_2\,xe^{2x}+C_3\,e^{2x}+C_4\,xe^{2x},$$

that is, the relations

$$2C_1+C_2+2C_2\,x=3C_1+3C_2\,x-C_3-C_4\,x,$$
$$2C_3+C_4+2C_4\,x=C_1+C_2\,x+C_3+C_4\,x,$$

leading to

$$2C_1+C_2=3C_1-C_3,$$
$$2C_2=3C_2-C_4,$$
$$2C_3+C_4=C_1+C_3,$$
$$2C_4=C_2+C_4,$$

and finaly to

$C_3=C_1-C_2,\qquad C_4=C_2,\qquad$ and

$$y_1(x)=C_1\,e^{2x}+C_2\,xe^{2x},$$
$$y_2(x)=C_1\,e^{2x}-C_2(1-x)e^{2x}.$$

Exercise 5.

$\boxed{11}$ The equation can be written successively as follows:

$$-i\,\psi'=(\lambda-1)\,\psi,$$
$$\psi'=i\,(\lambda-1)\,\psi,$$
$$\frac{\psi'}{\psi}=i\,(\lambda-1)$$
$$\psi(x)=C\,e^{i\,(\lambda-1)x}.$$
$$\psi(x)=C\,\cos(\lambda-1)x+iC\,\sin(\lambda-1)x.$$

Exercise 6.

$\boxed{12}$ Because

$$\psi(x)=e^{-\frac{1}{2}x^2},$$
$$\psi'(x)=-xe^{-\frac{1}{2}x^2}$$
$$\psi''(x)=-e^{-\frac{1}{2}x^2}+x^2\,e^{-\frac{1}{2}x^2},$$

the function ψ satisfies the equation

$$-\psi''+x^2\,\psi=\psi.$$

Remark.

The last equation written in the form

$$-\frac{1}{2}\psi''+\frac{1}{2}x^2\,\psi=\frac{1}{2}\psi.$$

shows that ψ is an eigenfunction of the Hamiltonian

$$\hat{H}=-\frac{1}{2}\frac{d^2}{dx^2}+\frac{1}{2}x^2$$

of the quantum harmonic oscillator

corresponding to the eigenvalue $E_0=\frac{1}{2}$

Normalized as

$$\psi(x)=\frac{1}{\sqrt[4]{\pi}}\,e^{-\frac{1}{2}x^2},$$

it describes the ground state of the harmonic oscillator.