An overview on $\boxed{\textbf{FOURIER ANALYSIS}}$

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TRIGONOMETRIC **FOURIER SERIES** with period 2π

- To a periodic function $\varphi:\mathbb{R}\to\mathbb{C}$ with period 2π such that $\varphi(t+2\pi)=\varphi(t),\ \forall t$
$$\boxed{\begin{array}{l} a_n \overset{\text{def}}{=} \frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(t)\,\cos nt\,dt \\ b_n \overset{\text{def}}{=} \frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(t)\,\sin nt\,dt \end{array}}$$
exist, we associate the *trigonometric Fourier series* $\boxed{\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt)}$

- Depending on φ, Fourier series can $\begin{cases}\text{-converge to } \varphi\\ \text{-converge to a different function}\\ \text{-be divergent.}\end{cases}$
- If certain conditions are satisfied, a partial sum $\boxed{S_k(t)=\frac{1}{2}a_0+\sum_{n=1}^{k}(a_n\cos nt+b_n\sin nt)}$ approximates very well φ for relatively small k.

Thus, φ can be well described by the coefficients a_0, a_1, b_1, ..., a_k, b_k.

- Due to periodicity, we can integrate on any interval of length 2π
$$\boxed{a_n=\frac{1}{\pi}\int_{\lambda}^{\lambda+2\pi}\varphi(t)\,\cos nt\,dt,\qquad b_n=\frac{1}{\pi}\int_{\lambda}^{\lambda+2\pi}\varphi(t)\,\sin nt\,dt.}$$

TRIGONOMETRIC **FOURIER SERIES** with period T

$$\boxed{\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos\tfrac{2\pi}{T}nt+b_n\sin\tfrac{2\pi}{T}nt\right)}\ \text{with}\ \boxed{\begin{array}{l} a_n\overset{\text{def}}{=}\frac{2}{T}\int_{\lambda}^{\lambda+T}\varphi(t)\,\cos\frac{2\pi}{T}nt\,dt\\ b_n\overset{\text{def}}{=}\frac{2}{T}\int_{\lambda}^{\lambda+T}\varphi(t)\,\sin\frac{2\pi}{T}nt\,dt\end{array}}$$

FOURIER SERIES with period 2π

- *Euler's formula* $\boxed{e^{it}\overset{\text{def}}{=}\cos t+i\sin t}\ \Leftrightarrow\ \boxed{\begin{array}{l}\cos t=\frac{e^{it}+e^{-it}}{2}\\ \sin t=\frac{e^{it}-e^{-it}}{2i}\end{array}}$
- *Fourier series* $\boxed{S(t)=\sum_{n=-\infty}^{\infty}c_n\,e^{int}}$ with $\boxed{c_n\overset{\text{def}}{=}\frac{1}{2\pi}\int_{\lambda}^{\lambda+2\pi}\varphi(t)\,e^{-int}\,dt}$
- If certain conditions are satisfied, a partial sum $\boxed{S_k(t)=\sum_{n=-k}^{k}c_n\,e^{int}}$ approximates very well φ for relatively small $|k|$.

Thus, φ can be well described by the coefficients c_{-k}, c_{-k+1}, ..., c_k.

FOURIER SERIES with period T (λ=arbitrary constant)

- *Fourier series* $\boxed{S(t)=\sum_{n=-\infty}^{\infty}c_n\,e^{\frac{2\pi i}{T}nt}}$ with $\boxed{c_n\overset{\text{def}}{=}\frac{1}{T}\int_{\lambda}^{\lambda+T}\varphi(t)\,e^{-\frac{2\pi i}{T}nt}\,dt.}$

DISCRETE (FINITE) **FOURIER TRANSFORM**

- We regard $\mathbb{C}^N$ as a space of periodic functions by using
$$\left\{\varphi:\mathbb{Z}\to\mathbb{C}\ \middle|\ \begin{array}{c}\varphi(n+N)=\varphi(n)\\ \text{for all } n\in\mathbb{Z}\end{array}\right\}\to\mathbb{C}^N:\ \boxed{\varphi\mapsto(\varphi(0),...,\varphi(N-1))}$$
- **Definition**.

Fourier transform of $\varphi:\mathbb{Z}\to\mathbb{C}$ is $F[\varphi]:\mathbb{Z}\to\mathbb{C}$, $\boxed{F[\varphi](k)\overset{\text{def}}{=}\sum_{n=0}^{N-1}e^{-\frac{2\pi i}{N}kn}\varphi(n)}$

Inverse Fourier transform of $\varphi:\mathbb{Z}\to\mathbb{C}$ is $F^{-1}[\varphi]:\mathbb{Z}\to\mathbb{C}$, $\boxed{F^{-1}[\varphi](k)\overset{\text{def}}{=}\frac{1}{N}\sum_{n=0}^{N-1}e^{\frac{2\pi i}{N}kn}\varphi(n)}$

- Due to periodicity, in these definitions, we can replace $\sum_{n=0}^{N-1}$ with $\sum_{n=m}^{m+N-1}$ for any $m\in\mathbb{Z}$.
- We have $\varphi=F[F^{-1}[\varphi]]$, $\varphi=F^{-1}[F[\varphi]]$, that is
$$\varphi(n)=\sum_{k=0}^{N-1}e^{-\frac{2\pi i}{N}nk}F^{-1}[\varphi](k),\qquad \boxed{\varphi(n)=\frac{1}{N}\sum_{k=0}^{N-1}e^{\frac{2\pi i}{N}nk}F[\varphi](k)}$$
- If certain conditions are satisfied, $\boxed{\varphi_m(n)=\frac{1}{N}\sum_{k=-m}^{m}e^{\frac{2\pi i}{N}nk}F[\varphi](k)}$ approximates very well φ for relatively small $|m|$.

Thus, φ can be obtained with a good approximation from the values of $F[\varphi]$ taken in relatively small neighbourhood of 0.

2D DISCRETE **FOURIER TRANSFORM**

- $\boxed{F[\varphi](k,\ell)\overset{\text{def}}{=}\sum_{n=0}^{N_x-1}\sum_{m=0}^{N_y-1}e^{-\frac{2\pi i}{N_x}kn}e^{-\frac{2\pi i}{N_y}\ell m}\varphi(n,m)}$
- $\boxed{F^{-1}[\psi](k,\ell)\overset{\text{def}}{=}\frac{1}{N_xN_y}\sum_{n=0}^{N_x-1}\sum_{m=0}^{N_y-1}e^{\frac{2\pi i}{N_x}kn}e^{\frac{2\pi i}{N_y}\ell m}\varphi(n,m)}$
- If certain conditions are satisfied,
$$\boxed{\varphi_{n_x,n_y}(n,m)=\frac{1}{N_xN_y}\sum_{k=-n_x}^{n_x}\sum_{\ell=-n_y}^{n_y}e^{\frac{2\pi i}{N_x}kn}e^{\frac{2\pi i}{N_y}\ell m}F[\varphi](k,\ell)}$$
approximates very well φ for relatively small $|n_x|$ and $|n_y|$.

Thus, φ can be obtained with a good approximation from the values of $F[\varphi]$ taken in relatively small neighbourhood of $(0,0)$.

FOURIER TRANSFORM OF FUNCTIONS

- **Definition** ($\mathcal{F}^{\pm1}[\varphi]$ exists only if the integral is convergent).

Fourier transform of $\varphi:\mathbb{R}\to\mathbb{C}$ is $\mathcal{F}[\varphi]:\mathbb{R}\to\mathbb{C}$, $\boxed{\mathcal{F}[\varphi](\xi)\overset{\text{def}}{=}\int_{-\infty}^{\infty}e^{i\xi x}\varphi(x)\,dx}$

Inverse Fourier transform of $\varphi:\mathbb{R}\to\mathbb{C}$ is $\mathcal{F}^{-1}[\varphi]:\mathbb{R}\to\mathbb{C}$, $\boxed{\mathcal{F}^{-1}[\varphi](\xi)\overset{\text{def}}{=}\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\xi x}\varphi(x)\,dx}$

- **Theorem**. $\boxed{(\mathcal{F}[\varphi])'=\mathcal{F}[ix\varphi]}$ and $\boxed{\mathcal{F}[\varphi'](\xi)=-i\xi\,\mathcal{F}[\varphi]}$
- **Examples** ($a\in(0,\infty)$ is a parameter).
  - $\boxed{\mathcal{F}\left[e^{-ax^2}\right](\xi)=\sqrt{\frac{\pi}{a}}\ e^{-\frac{\xi^2}{4a}}}$ $\quad\boxed{\mathcal{F}[e^{-a|x|}](\xi)=\frac{2a}{a^2+\xi^2}}$
  - $\chi_{[-a,a]}(x)=\begin{cases}1 & \text{for } x\in[-a,a]\\ 0 & \text{for } x\notin[-a,a]\end{cases}\Rightarrow\boxed{\mathcal{F}[\chi_{[-a,a]}](\xi)=\frac{2}{\xi}\sin a\xi}$

FOURIER TRANSFORM OF DISTRIBUTIONS

- Vector space of rapidly decreasing functions
$$\mathcal{S}(\mathbb{R})=\left\{\varphi:\mathbb{R}\longrightarrow\mathbb{C}\ \middle|\ \begin{array}{l}\varphi\in C^\infty(\mathbb{R})\text{ satisfying the relation}\\ \lim_{x\to\pm\infty}x^k\varphi^{(m)}(x)=0,\quad\forall k,m\in\mathbb{N}\end{array}\right\}$$
- $\lim_{n\to\infty}\varphi_n=0\ \overset{\text{def}}{\Leftrightarrow}\ \lim_{n\to\infty}\sup_{x\in\mathbb{R}}\left|x^k\varphi_n^{(m)}(x)\right|=0,\quad\forall k,m\in\mathbb{N}.$
- Vector space of *tempered distributions*
$$\mathcal{S}'(\mathbb{R})=\left\{\begin{array}{r}f:\mathcal{S}(\mathbb{R})\longrightarrow\mathbb{C}:\\ \varphi\mapsto\langle f,\varphi\rangle\end{array}\ \middle|\ \begin{array}{l}\langle f,\alpha\varphi+\beta\psi\rangle=\alpha\langle f,\varphi\rangle+\beta\langle f,\psi\rangle,\\ \lim_{n\to\infty}\varphi_n=0\ \Rightarrow\ \lim_{n\to\infty}\langle f,\varphi_n\rangle=0\end{array}\right\}$$
- When it exists, the distribution corresponding to a usual function $f:\mathbb{R}\longrightarrow\mathbb{C}$ is defined by the relation $\boxed{\langle f,\varphi\rangle\overset{\text{def}}{=}\int_{-\infty}^{\infty}f(x)\,\varphi(x)\,dx}$
- *Dirac distribution* ($a\in\mathbb{R}$ is a parameter) $\boxed{\langle\delta_a,\varphi\rangle\overset{\text{def}}{=}\varphi(a)}$
- **Theorem**. $\varphi\in\mathcal{S}(\mathbb{R})\ \Rightarrow\ \mathcal{F}^{\pm1}[\varphi]\in\mathcal{S}(\mathbb{R})$.
- **Definition.**
  - *Multiplication of a distribution f* by a function $\eta(x)$ (when it exists) $\boxed{\langle\eta f,\varphi\rangle\overset{\text{def}}{=}\langle f,\eta\varphi\rangle}$
  - *Derivative of a distribution f* $\boxed{\langle f',\varphi\rangle\overset{\text{def}}{=}-\langle f,\varphi'\rangle}$
  - *Fourier transform of a distribution f* $\boxed{\langle\mathcal{F}[f],\varphi\rangle\overset{\text{def}}{=}\langle f,\mathcal{F}[\varphi]\rangle}$
  - *Inverse Fourier transform of a distribution f* $\boxed{\langle\mathcal{F}^{-1}[f],\varphi\rangle\overset{\text{def}}{=}\langle f,\mathcal{F}^{-1}[\varphi]\rangle}$
- **Theorem**. $\boxed{(\mathcal{F}[f])'=\mathcal{F}[ix\,f]}$ and $\boxed{\mathcal{F}[f']=-i\xi\,\mathcal{F}[f]}$.
- **Examples** ($a\in(0,\infty)$ is a parameter).

$\boxed{\mathcal{F}[\delta_a]=e^{iax}}$ $\quad\boxed{\mathcal{F}[1]=2\pi\,\delta}$ $\quad\boxed{\mathcal{F}[x^k]=2\pi(-i)^k\,\delta^{(k)}}$

$\boxed{\mathcal{F}[\delta]=1}$ $\quad\boxed{\mathcal{F}[\delta^{(k)}]=(-i\xi)^k}$ $\quad\boxed{\mathcal{F}[\cos ax]=\pi(\delta_a+\delta_{-a})}$