On a discrete version of the ambiguity function

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Def Some definitions.

For $d=2s+1\in\{1,3,5,...\}$ and $\mathcal{H}\equiv\mathbb{C}^d\equiv\{\ \psi:\{-s,-s+1,...,s-1,s\}\to\mathbb{C}\ \}\equiv\{\ \psi:\mathbb{Z}_d\to\mathbb{C}\ \}$

Def1
$$\langle\varphi|\psi\rangle=\sum_{n=-s}^{s}\overline{\varphi(n)}\,\psi(n)\equiv\sum_{n\in\mathbb{Z}_d}\overline{\varphi(n)}\,\psi(n).$$

Def2 The *canonical orthonormal basis* is formed by $\delta_{-s},\delta_{-s+1},...,\delta_s:\{-s,-s+1,...,s-1,s\}\to\mathbb{C}$,
$$\delta_m(n)=\delta_{nm}=\begin{cases}1 & \text{for}\quad n=m,\\ 0 & \text{for}\quad n\neq m.\end{cases}$$
By using the notation $|m\rangle=|\delta_m\rangle$, we have
$$\boxed{\langle j|m\rangle=\delta_{jm}\quad,\quad\sum_{m=-s}^{s}|m\rangle\langle m|=\mathbb{I}}.$$
Alternatively, we can regard it as the set $\{\,\delta_m:\mathbb{Z}_d\to\mathbb{C}\ |\ m\in\mathbb{Z}_d\,\}$, where
$$\delta_m(n)=\delta_{nm}^{[d]}=\begin{cases}1 & \text{for}\quad n=m \bmod(d),\\ 0 & \text{for}\quad n\neq m \bmod(d).\end{cases}$$

Def3 *Finite Fourier transform* $F:\mathcal{H}\to\mathcal{H}$ is
$$\begin{array}{l}\psi:\mathbb{Z}_d\longrightarrow\mathbb{C}\\ \qquad\downarrow\\ F[\psi]:\mathbb{Z}_d\longrightarrow\mathbb{C}\end{array}\qquad \boxed{F[\psi](k)=\frac{1}{\sqrt{d}}\sum_{n\in\mathbb{Z}_d}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}kn}\,\psi(n)}$$

Def4 *Position* operator $Q:\mathcal{H}\to\mathcal{H}$, $\boxed{Q\psi(n)=\mathring{n}\psi(n)}$

Def5 *Momentum* operator $P:\mathcal{H}\to\mathcal{H}$, $\boxed{P=F^\dagger QF}$

where $\mathring{n}$ is the representative modulo d of n satisfying $-s\leq\mathring{n}\leq s$.

Def6 *Translation* operators $(n,k\in\mathbb{Z}_d)$ $\mathcal{H}\to\mathcal{H}$,
$$\boxed{\begin{array}{l}\psi\mapsto\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}\psi\\ \psi\mapsto\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}\psi\end{array}}$$

Def7 *Displacement* operators $(-s\leq n,k\leq s)$
$$\boxed{D(n,k)=\mathrm{e}^{-\frac{\pi\mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kQ}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}nP}}$$

Def8 *Parity operators* $(n,k\in\mathbb{Z}_d)$
$$\boxed{\Pi(n,k)=D(n,k)\,\Pi\,D(n,k)^\dagger}$$
where $\Pi\psi(n)=\psi(-n)$.

Def9 *Weyl transform* of a linear operator $A:\mathcal{H}\to\mathcal{H}$ is
$$A^{\mathrm{w}}:\mathbb{Z}_d\times\mathbb{Z}_d\longrightarrow\mathbb{C},\qquad \boxed{A^{\mathrm{w}}(n,k)\stackrel{\text{def}}{=}\langle\Pi(n,k),A\rangle=\mathrm{tr}(\Pi(n,k)A)}$$

Def10 *Wigner function* of a linear operator $A:\mathcal{H}\to\mathcal{H}$ is
$$\mathcal{W}_A:\mathbb{Z}_d\times\mathbb{Z}_d\longrightarrow\mathbb{C},\qquad \boxed{\mathcal{W}_A(n,k)\stackrel{\text{def}}{=}\frac{1}{d}\langle\Pi(n,k),A\rangle=\frac{1}{d}\,\mathrm{tr}(\Pi(n,k)A)}$$

Def11 **Ambiguity function** (characteristic function) of a linear operator $A:\mathcal{H}\to\mathcal{H}$ is
$$\boxed{\chi_A(n,k)\stackrel{\text{def}}{=}\frac{1}{d}\langle D(n,k),A\rangle=\frac{1}{d}\,\mathrm{tr}(D(n,k)^\dagger A)}$$

R Some remarks:

R1
$$\langle m|D(n,k)|j\rangle=\mathrm{e}^{\frac{\pi\mathrm{i}}{d}kn}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kj}\,\delta_m(j+n)$$
$$\langle m|\Pi(n,k)|j\rangle=\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k(m-j)}\,\delta_{2n}(m+j)$$

Proof of **R1** A proof is presented in the note *On a discrete version of the Weyl-Wigner description* available at https://unibuc.ro/user/nicolae.cotfas/.

R2 The definition of $D(n,k)$ is not modulo d invariant, but the definition of $\Pi(n,k)$ is modulo d invariant.

Proof of **R2**
$$D(n,k)=(-1)^k\,D(n+d,k)$$
$$D(n,k)=(-1)^n\,D(n,k+d).$$

R3 $\left\{\frac{1}{\sqrt{d}}D(n,k)\right\}_{-s\leq n,k\leq s}$ and $\left\{\frac{1}{\sqrt{d}}\Pi(n,k)\right\}_{n,k\in\mathbb{Z}_d}$ are orthonormal bases in the complex Hilbert space of all the linear operators $A:\mathcal{H}\to\mathcal{H}$.

Proof of **R3** A proof is presented in the note *On a discrete version of the Weyl-Wigner description* available at https://unibuc.ro/user/nicolae.cotfas/.

R4 $A:\mathcal{H}\to\mathcal{H}$ is linear operator $\Rightarrow$
$$\boxed{\begin{array}{l}A=\frac{1}{d}\sum\limits_{n,k=-s}^{s}A^{\mathrm{w}}(n,k)\,\Pi(n,k)\\ A=\sum\limits_{n,k=-s}^{s}\mathcal{W}_A(n,k)\,\Pi(n,k)\\ A=\sum\limits_{n,k=-s}^{s}\chi_A(n,k)\,D(n,k)\end{array}}$$

R5
$$\boxed{\langle\Pi(n,k),D(n',k')\rangle=\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}kn'}\sum_{m=-s}^{s}\mathrm{e}^{\frac{\pi\mathrm{i}}{d}k'(n'+2m)}\delta_{2n}(n'+2m)}$$

Proof of **R5**
$$\begin{aligned}\langle\Pi(n,k),D(n',k')\rangle&=\mathrm{tr}(\Pi(n,k)\,D(n',k'))\\ &=\sum_{m,j=-s}^{s}\langle m|\Pi(n,k)|j\rangle\langle j|D(n',k')|m\rangle\\ &=\sum_{m,j=-s}^{s}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k(m-j)}\,\delta_{2n}(m+j)\mathrm{e}^{\frac{\pi\mathrm{i}}{d}k'n'}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k'm}\,\delta_j(m+n')\\ &=\sum_{m=-s}^{s}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}kn'}\,\mathrm{e}^{\frac{\pi\mathrm{i}}{d}k'n'}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k'm}\sum_{j=-s}^{s}\delta_{2n-m}(j)\,\delta_{m+n'}(j)\\ &=\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}kn'}\,\mathrm{e}^{\frac{\pi\mathrm{i}}{d}k'n'}\sum_{m=-s}^{s}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k'm}\,\delta_{2n-m}(m+n')\\ &=\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}kn'}\sum_{m=-s}^{s}\mathrm{e}^{\frac{\pi\mathrm{i}}{d}k'(n'+2m)}\delta_{2n}(n'+2m)\end{aligned}$$

R6 The formula from **R5** can not be written as
$$\langle\Pi(n,k),D(n',k')\rangle=\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(k'n-n'k)}$$

Proof of **R6** $\delta_{2n}(n'+2m)=1\ \Leftrightarrow\ n'+2m-2n\in\mathbb{Z}d$, but $n'+2m-2n=\ell d\ \Rightarrow\ \mathrm{e}^{\frac{\pi\mathrm{i}}{d}k'(n'+2m)}=\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k'n}\,(-1)^{k'\ell}$.

R7 The relations
$$D(n',k')=\frac{1}{d}\sum_{n,k=-s}^{s}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(k'n-n'k)}\,\Pi(n,k),$$
$$\chi_A(n',k')=\frac{1}{d}\sum_{n,k=-s}^{s}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}(kn'-nk')}\,\mathcal{W}_A(n,k)$$
can not be true for any $n,k\in\mathbb{Z}$.

Proof of **R7** The right hand side is modulo d invariant, but the left hand side is not modulo d invariant.

R8
$$\boxed{\begin{array}{l}D(n',k')=\frac{1}{d}\sum\limits_{n,k=-s}^{s}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}kn'}\sum\limits_{m=-s}^{s}\mathrm{e}^{\frac{\pi\mathrm{i}}{d}k'(n'+2m)}\delta_{2n}(n'+2m)\,\Pi(n,k)\\ \chi_A(n',k')=\frac{1}{d}\sum\limits_{n,k=-s}^{s}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kn'}\sum\limits_{m=-s}^{s}\mathrm{e}^{-\frac{\pi\mathrm{i}}{d}k'(-n'+2m)}\delta_{2n}(-n'+2m)\,\mathcal{W}_A(n,k)\end{array}}$$

Proof of **R8**
$$D(n',k')=\frac{1}{d}\sum_{n,k=-s}^{s}\langle\Pi(n,k),D(n',k')\rangle\,\Pi(n,k).$$
$$\chi_A(n',k')=\frac{1}{d}\,\mathrm{tr}(D(n',k')^\dagger A)=\frac{1}{d}\,\mathrm{tr}(D(-n',-k')A)$$
$$=\frac{1}{d}\sum_{n,k=-s}^{s}\langle\Pi(n,k),D(-n',-k')\rangle\,\frac{1}{d}\,\mathrm{tr}(\Pi(n,k)A).$$

Numerical checks by using Mathematica

Check of R3

```
s=1; d=2 s+1
Displacement[n_, k_]:= Table[Exp[Pi I n k/d] Exp[2 Pi I k j/d] DiscreteDelta[ Mod[j + n - m, d]], {m, -s, s}, {j, -s, s}]
Do[Print[N[(1/d) Tr[ConjugateTranspose[Displacement[n, k]].Displacement[nn, kk]]]],{n,-s,s},{k,-s,s},{nn,-s,s},{kk,-s,s}]
```

```
s=1; d=2 s+1
Parity[n_, k_] := Table[Exp[2 Pi I k (m - j)/d] DiscreteDelta[ Mod[2 n - m - j, d]], {m, -s, s}, {j, -s, s}]
Do[Print[(1/d) N[Tr[Parity[n, k]. Parity[nn, kk]]]], {n, -s,   s}, {k, -s, s}, {nn, -s, s}, {kk, -s, s}]
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Check of R5

```
s=1; d=2 s+1
Displacement[n_, k_]:= Table[Exp[Pi I n k/d] Exp[2 Pi I k j/d] DiscreteDelta[Mod[j+n-m,d]],{m,-s,s},{j,-s,s}]
Parity[n_, k_] := Table[Exp[2 Pi I k (m - j)/d] DiscreteDelta[Mod[2 n-m-j,d]],{m,-s,s},{j,-s,s}]
Do[Print[N[Tr[Parity[n, k]. Displacement[nn, kk]]- Exp[-2 Pi I k nn/d] Sum[Exp[Pi I kk (nn+2 m)/d]
                          DiscreteDelta[Mod[2 n-2 m-nn,d]],{m,-s,s}]]]],{n,-s,s},{k,-s,s},{nn,-s,s},{kk,-s,s}]
```

Check of R6

```
s=1; d=2 s+1
Displacement[n_, k_]:=Table[Exp[Pi I n k/d] Exp[2 Pi I k j/d] DiscreteDelta[Mod[j+n-m,d]],{m,-s,s},{j,-s,s}]
Parity[n_, k_] := Table[Exp[2 Pi I k (m - j)/d] DiscreteDelta[Mod[2 n - m - j, d]], {m, -s, s}, {j, -s, s}]
Do[Print[N[Tr[Parity[n, k].Displacement[nn,kk]]- Exp[2 Pi I (kk n-nn k)/d]]],{n,-s,s},{k,-s,s},{nn,-s,s},{kk,-s,s}]
```

Check of R8

```
s=1; d=2 s+1
Displacement[n_, k_]:= Table[Exp[Pi I n k/d] Exp[2 Pi I k j/d] DiscreteDelta[Mod[j+n-m,d]],{m,-s,s},{j,-s,s}]
Parity[n_, k_] := Table[Exp[2 Pi I k (m - j)/d] DiscreteDelta[Mod[2 n-m-j,d]],{m,-s,s},{j,-s,s}]
Do[Print[N[ Displacement[nn, kk]]- (1/d)Sum[ Exp[-2 Pi I k nn/d] Sum[Exp[Pi I kk (nn+2 m)/d]
             DiscreteDelta[Mod[2 n-2 m-nn,d]],{m,-s,s}] Parity[n,k],{n,-s,s},{k,-s,s}]]],{nn,-s,s},{kk,-s,s}]
```