An overview on POISSON SUMMATION FORMULA and ZAK TRANSFORM

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FOURIER TRANSFORM OF FUNCTIONS

Definition ($\mathcal{F}^{\pm1}[\varphi]$ exists only if the integral is convergent).

Fourier transform of $\varphi:\mathbb{R}\to\mathbb{C}$ is $\mathcal{F}[\varphi]:\mathbb{R}\to\mathbb{C}$, $\boxed{\mathcal{F}[\varphi](\xi)\overset{\text{def}}{=}\int_{-\infty}^{\infty} e^{i\xi x}\varphi(x)\,dx}$

Inverse Fourier transform of $\varphi:\mathbb{R}\to\mathbb{C}$ is $\mathcal{F}^{-1}[\varphi]:\mathbb{R}\to\mathbb{C}$, $\boxed{\mathcal{F}^{-1}[\varphi](\xi)\overset{\text{def}}{=}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x}\varphi(x)\,dx}$

Theorem. $\boxed{(\mathcal{F}[\varphi])'=\mathcal{F}[ix\varphi]}$ and $\boxed{\mathcal{F}[\varphi'](\xi)=-i\xi\,\mathcal{F}[\varphi]}$

Example $\boxed{\mathcal{F}\left[e^{-ax^2}\right](\xi)=\sqrt{\frac{\pi}{a}}\,e^{-\frac{\xi^2}{4a}}}$ $a\in(0,\infty)$ is a parameter.

FOURIER TRANSFORM OF DISTRIBUTIONS

Vector space of rapidly decreasing functions

$$\mathcal{S}(\mathbb{R})=\left\{\varphi:\mathbb{R}\longrightarrow\mathbb{C}\;\middle|\;\begin{array}{l}\varphi\in C^\infty(\mathbb{R}) \text{ satisfying the relation}\\ \lim_{x\to\pm\infty} x^k\,\varphi^{(m)}(x)=0,\quad \forall k,m\in\mathbb{N}\end{array}\right\}$$

$$\lim_{n\to\infty}\varphi_n=0 \quad\overset{\text{def}}{\iff}\quad \lim_{n\to\infty}\sup_{x\in\mathbb{R}}\left|x^k\,\varphi_n^{(m)}(x)\right|=0,\quad \forall k,m\in\mathbb{N}.$$

Vector space of *tempered distributions*

$$\mathcal{S}'(\mathbb{R})=\left\{\begin{array}{l} f:\mathcal{S}(\mathbb{R})\longrightarrow\mathbb{C}:\\ \qquad\varphi\mapsto\langle f,\varphi\rangle\end{array}\;\middle|\;\begin{array}{l}\langle f,\alpha\varphi+\beta\psi\rangle=\alpha\langle f,\varphi\rangle+\beta\langle f,\psi\rangle,\\ \lim_{n\to\infty}\varphi_n=0\;\Rightarrow\;\lim_{n\to\infty}\langle f,\varphi_n\rangle=0\end{array}\right\}$$

When it exists, the distribution corresponding to a usual function $f:\mathbb{R}\longrightarrow\mathbb{C}$ is defined by the relation $\boxed{\langle f,\varphi\rangle\overset{\text{def}}{=}\int_{-\infty}^{\infty} f(x)\,\varphi(x)\,dx}$

Dirac distribution ($a\in\mathbb{R}$ is a parameter) $\boxed{\langle\delta_a,\varphi\rangle\overset{\text{def}}{=}\varphi(a)}$

Theorem. $\varphi\in\mathcal{S}(\mathbb{R})\;\Rightarrow\;\mathcal{F}^{\pm1}[\varphi]\in\mathcal{S}(\mathbb{R})$.

Definition.

Multiplication and Derivative $\boxed{\begin{array}{l}\langle\eta\, f,\varphi\rangle\overset{\text{def}}{=}\langle f,\eta\,\varphi\rangle\\ \langle f',\varphi\rangle\overset{\text{def}}{=}-\langle f,\varphi'\rangle\end{array}}$

Fourier transform $\boxed{\begin{array}{l}\langle\mathcal{F}[f],\varphi\rangle\overset{\text{def}}{=}\langle f,\mathcal{F}[\varphi]\rangle\\ \langle\mathcal{F}^{-1}[f],\varphi\rangle\overset{\text{def}}{=}\langle f,\mathcal{F}^{-1}[\varphi]\rangle\end{array}}$

Theorem. $\boxed{(\mathcal{F}[f])'=\mathcal{F}[ix\,f]}$ and $\boxed{\mathcal{F}[f']=-i\xi\,\mathcal{F}[f]}$.

Example $\boxed{\mathcal{F}[\delta_a]=e^{iax}}$ $a\in(0,\infty)$ is a parameter.

POISSON SUMMATION FORMULA

Theorem. *Let $\kappa\in(0,\infty)$. If $\psi\in\mathcal{S}(\mathbb{R})$ then*

$$\boxed{\sum_{\alpha=-\infty}^{\infty}\psi(q+\alpha\kappa)=\frac{1}{\kappa}\sum_{\beta=-\infty}^{\infty}\mathcal{F}[\psi]\left(\beta\tfrac{2\pi}{\kappa}\right)e^{-\frac{2\pi i}{\kappa}\beta q}}$$

Proof. Fourier series $\sum_{\alpha=-\infty}^{\infty}\psi(q+\alpha\kappa)=\sum_{\beta=-\infty}^{\infty}c_\beta\, e^{-\frac{2\pi i}{\kappa}\beta q}$ has

$$c_\beta=\frac{1}{\kappa}\int_0^\kappa\sum_{\alpha=-\infty}^{\infty}\psi(q+\alpha\kappa)e^{\frac{2\pi i}{\kappa}\beta q}dq=\frac{1}{\kappa}\sum_{\alpha=-\infty}^{\infty}\int_0^\kappa\psi(q+\alpha\kappa)e^{\frac{2\pi i}{\kappa}\beta q}dq$$

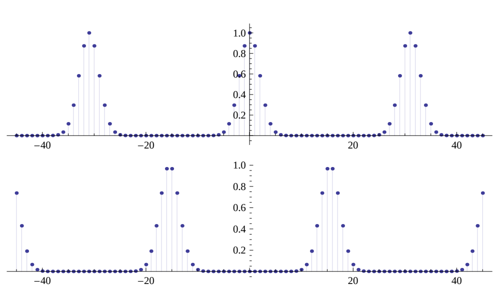
$$=\frac{1}{\kappa}\sum_{\alpha=-\infty}^{\infty}\int_{\alpha\kappa}^{(\alpha+1)\kappa}\psi(x)e^{\frac{2\pi i}{\kappa}\beta x}dx=\frac{1}{\kappa}\int_{-\infty}^{\infty}\psi(x)e^{\frac{2\pi i}{\kappa}\beta x}dx=\frac{1}{\kappa}\mathcal{F}[\psi]\left(\beta\tfrac{2\pi}{\kappa}\right).$$

By choosing $q=0$, $\psi(\alpha\kappa)=\langle\delta_{\alpha\kappa},\psi\rangle$, etc. we get

$$\boxed{\sum_{\alpha=-\infty}^{\infty}\psi(\alpha\kappa)=\frac{1}{\kappa}\sum_{\beta=-\infty}^{\infty}\mathcal{F}[\psi]\left(\beta\tfrac{2\pi}{\kappa}\right)}\Rightarrow\boxed{\sum_{\alpha=-\infty}^{\infty}\delta_{\alpha\kappa}=\frac{1}{\kappa}\sum_{\beta=-\infty}^{\infty}\mathcal{F}\left[\delta_{\beta\frac{2\pi}{\kappa}}\right]}$$

$\Swarrow$ $\Downarrow$

$$\boxed{\sum_{\alpha=-\infty}^{\infty}\delta_{\alpha\kappa}=\frac{1}{\kappa}\sum_{\beta=-\infty}^{\infty}e^{i\beta\frac{2\pi}{\kappa}}}\qquad\boxed{\mathcal{F}\left[\sum_{\alpha=-\infty}^{\infty}\delta_{\alpha\kappa}\right]=\frac{2\pi}{\kappa}\sum_{\beta=-\infty}^{\infty}\delta_{\beta\frac{2\pi}{\kappa}}}$$



ZAK TRANSFORM

Definition (using the notation $\mathbb{T}_\kappa=[-\frac{\kappa}{2},\frac{\kappa}{2})\times[-\frac{\pi}{\kappa},\frac{\pi}{\kappa})$)

$L^2(\mathbb{R})\xrightarrow{\mathcal{Z}}L^2(\mathbb{T}_\kappa)$, $\psi\mapsto\mathcal{Z}[\psi]$ $\boxed{\mathcal{Z}[\psi](q,p)\overset{\text{def}}{=}\sqrt{\frac{\kappa}{2\pi}}\sum_{\alpha=-\infty}^{\infty}e^{-i\alpha\kappa p}\,\psi(q+\alpha\kappa)}$

is an isomorphism, and $\boxed{\psi(q)=\sqrt{\frac{\kappa}{2\pi}}\int_{-\frac{\pi}{\kappa}}^{\frac{\pi}{\kappa}}\mathcal{Z}[\psi](q,p)\,dp}$

Convergence (for almost any q) of the Fourier series from def. follows from $\int_{-\frac{\kappa}{2}}^{\frac{\kappa}{2}}\sum_{\alpha=-\infty}^{\infty}|\psi(q+\alpha\kappa)|^2dq=\int_{-\infty}^{\infty}|\psi(q)|^2dq<\infty$.

$\left\{\sqrt{\frac{\kappa}{2\pi}}e^{-i\alpha\kappa p}\right\}_{\alpha\in\mathbb{Z}}$ is an orthonormal basis in $L^2[-\frac{\pi}{\kappa},\frac{\pi}{\kappa})$ $\Rightarrow$ $\int_{-\frac{\pi}{\kappa}}^{\frac{\pi}{\kappa}}|\mathcal{Z}[\psi](q,p)|^2\,dp=\sum_{\alpha=-\infty}^{\infty}|\psi(q+\alpha\kappa)|^2$

$L^2(\mathbb{R})\xrightarrow{\mathcal{Z}}L^2(\mathbb{T}_\kappa)$ is a unitary transform: $||\mathcal{Z}[\psi]||^2=\int_{-\frac{\kappa}{2}}^{\frac{\kappa}{2}}\sum_{\alpha=-\infty}^{\infty}|\psi(q+\alpha\kappa)|^2dq=||\psi||^2$.

By extending $\mathcal{Z}$ to distributions, we get

$\boxed{\mathcal{Z}[\delta_{q_0}](q,p)=\sqrt{\frac{\kappa}{2\pi}}\sum_{\alpha=-\infty}^{\infty}e^{-i\alpha\kappa p}\,\delta_{q_0-\alpha\kappa}}$ Dirac's notation $|q\rangle\overset{\text{def}}{\equiv}\delta_q$

The **Zak basis states** $\boxed{|q,p\rangle_z\overset{\text{def}}{=}\sqrt{\frac{\kappa}{2\pi}}\sum_{\alpha=-\infty}^{\infty}e^{-ip\alpha\kappa}\,|q-\alpha\kappa\rangle}$ $\boxed{\begin{array}{c}{}_z\langle q,p|a,b\rangle_z=\delta(q-a)\,\delta(p-b)\\ \mathbb{I}=\iint_{\mathbb{T}_\kappa}dqdp\,|q,p\rangle_{z\,z}\langle q,p|.\end{array}}$

are common eigenvectors of the commuting operators $e^{\frac{2\pi i}{\kappa}Q}e^{-\frac{i}{\hbar}\kappa P}=e^{-\frac{i}{\hbar}\kappa P}e^{\frac{2\pi i}{\kappa}Q}$ where $Q\psi(q)=q\psi(q)$, $P=-i\hbar\frac{d}{dq}$

$\boxed{\begin{array}{l}e^{\frac{2\pi i}{\kappa}Q}|q,p\rangle_z=e^{\frac{2\pi i}{\kappa}q}\,|q,p\rangle_z,\\ e^{-\frac{i}{\hbar}\kappa P}|q,p\rangle_z=e^{-i\kappa p}\,|q,p\rangle_z\end{array}}$

GAUSSIAN FUNCTIONS OF DISCRETE VARIABLE

($d=2j+1$ is an odd positive integer)

Definition For $\kappa\in(0,\infty)$, we define $\boxed{\mathbf{g}_\kappa(n)\overset{\text{def}}{=}\sum_{\alpha=-\infty}^{\infty}e^{-\frac{\kappa\pi}{d}(n+\alpha d)^2}}$ and $\mathbf{g}^+_\kappa(n)\overset{\text{def}}{=}\sum_{\alpha=-\infty}^{\infty}e^{-\frac{\kappa\pi}{d}(n+(\alpha+\frac{1}{2})d)^2}$ (see the left hand side of the fig.)

In terms of the Zak transform: $\mathbf{g}_\kappa(n)=\sqrt{\frac{2\pi}{d}}\,\mathcal{Z}\left[e^{-\frac{\kappa\pi}{d}q^2}\right](n,0)$.

Discrete Fourier transform. By using the definition $\mathbf{F}[\psi](k)\overset{\text{def}}{=}\frac{1}{\sqrt{d}}\sum_{n=-j}^{j}e^{-\frac{2\pi i}{d}kn}\,\psi(n)$ we get $\boxed{\mathbf{F}[\mathbf{g}_\kappa]=\frac{1}{\sqrt{\kappa}}\,\mathbf{g}_{\kappa^{-1}}}$

More general, for any $\psi\in C^0(\mathbb{R})\cap L^1(\mathbb{R})$ with $\mathcal{F}[\psi]\in L^1(\mathbb{R})$,

$$\boxed{\mathbf{F}\left[\sum_{\alpha=-\infty}^{\infty}\psi\left((n+\alpha d)\sqrt{\tfrac{2\pi}{d}}\right)\right](k)=\sum_{\beta=-\infty}^{\infty}\mathcal{F}[\psi]\left((k+\beta d)\sqrt{\tfrac{2\pi}{d}}\right)}$$

Discrete Wigner function. By using the definition

$$\mathbf{W}_\psi(n,k)\overset{\text{def}}{=}\frac{1}{d}\sum_{m=-j}^{j}e^{-\frac{4\pi i}{d}km}\,\overline{\psi(n-m)}\;\psi(n+m)$$

we get [arXiv:1205.6302] (see the figure)

$$\boxed{\begin{array}{l}\mathbf{W}_{\mathbf{g}_\kappa}(n,k)=\frac{1}{\sqrt{2\kappa d}}\,\mathbf{g}_{2\kappa}(n)\,\left[\mathbf{g}_{2\kappa^{-1}}(k)+\mathbf{g}^+_{2\kappa^{-1}}(k)\right]\\ \qquad+\frac{1}{\sqrt{2\kappa d}}\,\mathbf{g}^+_{2\kappa}(n)\,\left[\mathbf{g}_{2\kappa^{-1}}(k)-\mathbf{g}^+_{2\kappa^{-1}}(k)\right]\end{array}}$$