An overview on **QUASIPROBABILITY REPRESENTATIONS OF FINITE QUANTUM SYSTEMS**

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DISCRETE WIGNER FUNCTION

($d=2j+1$ is a positive odd integer)

■ By regarding $\mathbb{C}^d$ as a Hilbert space of functions,

$$\mathbb{C}^d\equiv\{\psi:\{-j,-j+1,...,j-1,j\}\to\mathbb{C}\}\equiv\{\psi:\mathbb{Z}_d\to\mathbb{C}\},$$

and using *Fourier transform, position and momentum operators*

$$\mathbf{F}[\psi](k)\overset{\text{def}}{=}\frac{1}{\sqrt{d}}\sum_{n\in\mathbb{Z}_d}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}kn}\,\psi(n),\qquad (\mathbf{Q}\psi)(n)\overset{\text{def}}{=}\tilde{n}\,\psi(n),\quad \mathbf{P}\overset{\text{def}}{=}\mathbf{F}^\dagger\mathbf{Q}\mathbf{F},$$

where $-j\leq\tilde{n}\leq j$ is the representative mod(d) of n, we define the *translation operators*

$$\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k\mathbf{Q}},\ \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}n\mathbf{P}}\quad\text{acting as}\quad \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k\mathbf{Q}}\psi(n)=\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kn}\psi(n),\quad \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}n\mathbf{P}}\psi(m)=\psi(m-n).$$

■ The *displacement operators*

$$\mathbf{D}(n,k)\overset{\text{def}}{=}\mathrm{e}^{-\frac{\pi\mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k\mathbf{Q}}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}n\mathbf{P}}=\mathrm{e}^{\frac{\pi\mathrm{i}}{d}nk}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}n\mathbf{P}}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k\mathbf{Q}}$$

and the *parity operators*

$$\Pi(n,k)\overset{\text{def}}{=}\mathbf{D}(n,k)\,\Pi\,\mathbf{D}(n,k)^\dagger,\qquad \Pi\psi(n)\overset{\text{def}}{=}\psi(-n).$$

act as

$$\mathbf{D}(n,k)\psi(m)=\mathrm{e}^{-\frac{\pi\mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}km}\,\psi(m-n),$$
$$\Pi(n,k)\psi(m)=\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}k(n-m)}\,\psi(2n-m).$$

■ **Definition**

Discrete Wigner function of ρ	is	$\mathbf{W}_\rho:\mathbb{Z}_d\times\mathbb{Z}_d\longrightarrow\mathbb{R}$, $\boxed{\mathbf{W}_\rho(n,k)\overset{\text{def}}{=}\frac{1}{d}\,\mathrm{tr}(\rho\,\Pi(n,k))}$

■ In the space of Hermitian operators $\{\Pi(n,k\}_{n,k\in\mathbb{Z}_d}$ is orthogonal basis $\mathrm{tr}(\,\Pi(n,k)\ \Pi(m,\ell)\,)=d\ \delta_{nm}\,\delta_{k\ell}$ $\Rightarrow$ $\boxed{\rho=\sum_{n,k\in\mathbb{Z}_d}\mathbf{W}_\rho(n,k)\,\Pi(n,k)}$

■ In terms of the canonical basis $\{|m\rangle\equiv\delta_m\}_{m\in\mathbb{Z}_d}$ $\quad\boxed{\mathbf{W}_\varrho(n,k)=\frac{1}{d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\,\langle n+m|\varrho|n-m\rangle}$

Proof.
$$\begin{aligned}\mathbf{W}_\rho(n,k)&=\frac{1}{d}\sum_{\ell,s\in\mathbb{Z}_d}\langle\ell|\varrho|s\rangle\langle s|\Pi(n,k)|\ell\rangle\\&=\frac{1}{d}\sum_{\ell,s\in\mathbb{Z}_d}\langle\ell|\varrho|s\rangle\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}k(n-s)}\delta_\ell(2n-s)\\&=\frac{1}{d}\sum_{\ell,m\in\mathbb{Z}_d}\langle\ell|\varrho|s\rangle\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\delta_\ell(n+m)\\&=\frac{1}{d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\,\langle n+m|\varrho|n-m\rangle.\end{aligned}$$

■ For any state ϱ, we have $\boxed{\sum_{n,k\in\mathbb{Z}_d}\mathbf{W}_\rho(n,k)=1}$

Proof. $\sum_{k\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}=d\,\delta_{2m,0}=d\,\delta_{m,0}\Rightarrow\sum_{n,k\in\mathbb{Z}_d}\mathbf{W}_\rho(n,k)=\mathrm{tr}\,\varrho.$

■ Pure state $\varrho=|\psi\rangle\langle\psi|$ $\Rightarrow$ $\boxed{\mathbf{W}_\psi(n,k)=\frac{1}{d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\,\overline{\psi(n-m)}\,\psi(n+m)}$

Proof.
$$\begin{aligned}\mathbf{W}_\psi(n,k)&=\frac{1}{d}\,\mathrm{tr}(\,|\psi\rangle\langle\psi|\ \Pi(n,k)\,)=\frac{1}{d}\,\langle\psi|\Pi(n,k)|\psi\rangle\\&=\frac{1}{d}\sum_{\ell\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}k(n-\ell)}\,\overline{\psi(\ell)}\,\psi(2n-\ell)\\&=\frac{1}{d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\,\overline{\psi(n-m)}\,\psi(n+m).\end{aligned}$$

■ Pure state $\varrho=|\psi\rangle\langle\psi|$ $\Rightarrow$ $\boxed{\sum_{k\in\mathbb{Z}_d}\mathbf{W}_\psi(n,k)=|\psi(n)|^2}$

Proof. For odd d, $\sum_{k\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}=d\,\delta_{2m,0}=d\,\delta_{m,0}$.

■ Pure state $\varrho=|\psi\rangle\langle\psi|$ $\Rightarrow$ $\boxed{\sum_{n\in\mathbb{Z}_d}\mathbf{W}_\psi(n,k)=|\mathbf{F}[\psi](k)|^2}$

Proof. $\mathbb{Z}_d\times\mathbb{Z}_d\to\mathbb{Z}_d\times\mathbb{Z}_d:(n,m)\mapsto(n-m,n+m)$ is bijective for odd d. Therefore

$$\begin{aligned}\sum_{n\in\mathbb{Z}_d}\mathbf{W}_\psi(n,k)&=\frac{1}{d}\sum_{n,m\in\mathbb{Z}_d}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{d}km}\,\overline{\psi(n-m)}\,\psi(n+m)\\&=\frac{1}{d}\sum_{\alpha,\beta\in\mathbb{Z}_d}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k(\alpha-\beta)}\,\overline{\psi(\alpha)}\,\psi(\beta)=|\mathbf{F}[\psi](k)|^2.\end{aligned}$$

QUASIPROBABILITY REPRESENTATIONS

■ Discrete Wigner function is completely determined by the operators $\Pi(n,k)$ $\Rightarrow$ We can regard Wigner function as the set of operators $\{\frac{1}{d}\Pi(n,k)\}$.

■ **Definition**

Discrete Wigner function	is	the orthogonal basis $\{\frac{1}{d}\Pi(n,k)\}$ satisfying $\sum_{n,k\in\mathbb{Z}_d}\frac{1}{d}\Pi(n,k)=\mathbf{I}$.

The associated quasiprobability is $\varrho\mapsto$ $\boxed{\mathbf{W}_\rho:\mathbb{Z}_d\times\mathbb{Z}_d\longrightarrow\mathbb{R},\ \mathbf{W}_\rho(n,k)=\frac{1}{d}\,\mathrm{tr}(\rho\,\Pi(n,k))}$

■ **Definition** [J H DeBrota, B C Stacey, arXiv:1912.07554]

Minimal Wigner function	is	an orthogonal basis $\{\mathfrak{F}_m\}_{m\in\mathcal{M}}$ in $\mathcal{L}(\mathbb{C}^d)$ satisfying $\sum_{m\in\mathcal{M}}\mathfrak{F}_m=\mathbf{I}$.

The associated quasiprobability is $\varrho\mapsto$ $\boxed{\mathbf{W}_\rho:\mathcal{M}\longrightarrow\mathbb{R},\ \mathbf{W}_\rho(m)=\mathrm{tr}(\rho\,\mathfrak{F}_m)}$

■ **Definition** [J H DeBrota, B C Stacey, arXiv:1912.07554]

Unbiased Wigner function	is	an orthogonal basis $\{\mathfrak{F}_m\}_{m\in\mathcal{M}}$ in $\mathcal{L}(\mathbb{C}^d)$ satisfying $\mathrm{tr}\,\mathfrak{F}_m=\frac{1}{d}$ and $\sum_{m\in\mathcal{M}}\mathfrak{F}_m=\mathbf{I}$.

The associated quasiprobability is $\varrho\mapsto$ $\boxed{\mathbf{W}_\rho:\mathcal{M}\longrightarrow\mathbb{R},\ \mathbf{W}_\rho(m)=\mathrm{tr}(\rho\,\mathfrak{F}_m)}$

PROBABILITY REPRESENTATIONS

■ **Definition** [J H DeBrota, B C Stacey, arXiv:1912.07554]

MIC (Minimal Informationally Complete POVM)	is	a basis $\{\mathfrak{E}_m\}_{m\in\mathcal{M}}$ in $\mathcal{L}(\mathbb{C}^d)$ satisfying $\mathfrak{E}_m\geq 0$, $\sum_{m\in\mathcal{M}}\mathfrak{E}_m=\mathbf{I}$

The associated probability is $\varrho\mapsto$ $\boxed{\mathcal{M}\longrightarrow\mathbb{R},\ m\mapsto\mathrm{tr}(\rho\,\mathfrak{E}_m)}$

■ **Definition** [J H DeBrota, B C Stacey, arXiv:1912.07554]

Unbiased MIC	is	a basis $\{\mathfrak{E}_m\}_{m\in\mathcal{M}}$ in $\mathcal{L}(\mathbb{C}^d)$ satisfying $\mathfrak{E}_m\geq 0$, $\mathrm{tr}\,\mathfrak{E}_m=\frac{1}{d}$, $\sum_{m\in\mathcal{M}}\mathfrak{E}_m=\mathbf{I}$.

The associated probability is $\varrho\mapsto$ $\boxed{\mathcal{M}\longrightarrow\mathbb{R},\ m\mapsto\mathrm{tr}(\rho\,\mathfrak{E}_m)}$

REMARKS

■ A MIC can not be an orthogonal basis.

■ The relation **quasiprobabilities-probabilities**, that is, the relation **Wigner function-MIC** is analyzed in [J H DeBrota, B C Stacey, arXiv:1912.07554].

■ More general definitions (frames instead of bases), and the negativity of a quantum state with respect to various Wigner type functions are analyzed in [H Zhu, arXiv:1604.06974].