Chapter 10

Elements of Complex Analysis

10.1 Complex numbers

10.1.1 The set of complex numbers

$$\mathbb{C} = \mathbb{R} + \mathbb{R}\mathrm{i} = \{ z = x + y\mathrm{i} \mid x, y \in \mathbb{R} \}$$

considered together with the addition

$$(x + y\mathrm{i}) + (x' + y'\mathrm{i}) = (x + x') + (y + y')\mathrm{i}$$

and multiplication by a real number

$$\alpha(x + y\mathrm{i}) = \alpha x + \alpha y\mathrm{i}$$

is a real vector space of dimension 2. The usual form of a complex number $z = x+y\mathrm{i}$ represents its expansion in the basis $\{1, \mathrm{i}\}$. The linear isomorphism

$$\mathbb{R}^2 \longrightarrow \mathbb{C} : (x, y) \mapsto x + y\mathrm{i}$$

allows us to identify the two real vector spaces and leads to a natural geometric interpretation of the complex numbers (the complex plane).

10.1.2 The relation $\mathrm{i}^2 = -1$ allows us to define an additional operation in $\mathbb{C}$

$$(x + y\mathrm{i})(x' + y'\mathrm{i}) = (xx' - yy') + (xy' + yx')\mathrm{i}.$$

called the *multiplication* of complex numbers. The set $\mathbb{C}$ considered together with the addition and multiplication of complex numbers is a commutative field. Particularly, any non-null complex number admits an inverse

$$(x+y\mathrm{i})^{-1} = \frac{1}{x+y\mathrm{i}} = \frac{x-y\mathrm{i}}{x^2+y^2} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}\mathrm{i}.$$

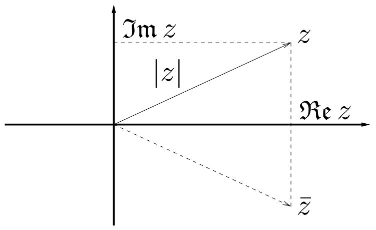

Figure 10.1: Conjugatul unui număr complex

10.1.3 Definition. Let $z = x + y\mathrm{i}$ be a complex number.

The number $\mathfrak{Re}\, z = x$ is called the *real part* of z.

The number $\mathfrak{Im}\, z = y$ is called the *imaginary part* of z.

The number $\bar{z} = x - y\mathrm{i}$ is called the *complex conjugate* of z.

The number $|z| = \sqrt{x^2+y^2}$ is called the *modulus* of z.

10.1.4 MATHEMATICA `Re[x+y I]`, `Im[x+y I]`, `Abs[x+y I]`, `Conjugate[x+y I]`

`In[1]:=I`	$\mapsto$	Out[1]= $\mathbb{i}$	`In[5]:=Re[3+4 I]`	$\mapsto$	Out[5]=3
`In[2]:=Sqrt[-4]`	$\mapsto$	Out[2]=2 $\mathbb{i}$	`In[6]:=Im[3+4 I]`	$\mapsto$	Out[6]=4
`In[3]:=(3+2 I)^2`	$\mapsto$	Out[3]=5+12 $\mathbb{i}$	`In[7]:=Abs[3+4 I]`	$\mapsto$	Out[7]=5
`In[4]:=(3+2 I)/(5-I)`	$\mapsto$	Out[4]= $\frac{1}{2}+\frac{\mathbb{i}}{2}$	`In[8]:=Conjugate[3+4 I]`	$\mapsto$	Out[8]=3−4 $\mathbb{i}$.

10.1.5 Proposition. *The relations*

$$\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2 \qquad \overline{z_1\, z_2} = \bar{z}_1\, \bar{z}_2 \qquad \overline{(z^n)} = (\bar{z})^n$$

$$|\bar{z}| = |z| \qquad |z|^2 = z\,\bar{z} \qquad \overline{(\bar{z})} = z$$

$$\mathfrak{Re}\, z = \tfrac{z+\bar{z}}{2} \qquad \mathfrak{Im}\, z = \tfrac{z-\bar{z}}{2\,\mathrm{i}} \qquad z = \mathfrak{Re}\, z + \mathrm{i}\, \mathfrak{Im}\, z.$$

are satisfied for any complex numbers z_1, z_2 şi z.

Proof. These relations are direct consequences of the previous definition.

10.1.6 For any $\varphi,\ \psi \in \mathbb{R}$ we have

$$(\cos\varphi + \mathrm{i}\sin\varphi)(\cos\psi + \mathrm{i}\sin\psi) = (\cos\varphi\cos\psi - \sin\varphi\sin\psi)$$
$$+\mathrm{i}(\cos\varphi\sin\psi + \sin\varphi\cos\psi) = \cos(\varphi+\psi) + \mathrm{i}\sin(\varphi+\psi).$$

By using Euler's notation

$$\mathrm{e}^{\mathrm{i}t} = \cos t + \mathrm{i}\sin t$$

the previous relation becomes

$$\mathrm{e}^{\mathrm{i}\varphi}\,\mathrm{e}^{\mathrm{i}\psi} = \mathrm{e}^{\mathrm{i}(\varphi+\psi)}.$$

10.1.7 For any non-null number $z = x + y\mathrm{i}$ there exists $\arg z \in (-\pi, \pi]$ such that

$$z = |z|(\cos(\arg z) + \mathrm{i}\sin(\arg z)) = |z|\,\mathrm{e}^{\mathrm{i}\arg z}.$$

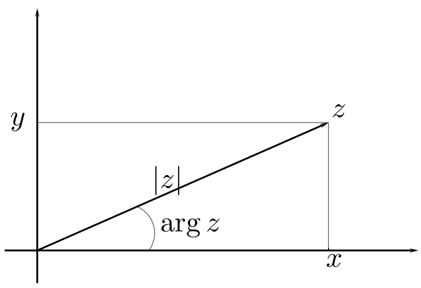

Figure 10.2: Modulus and the argument of a complex number.

The number $\arg z$, called the *principal value* of the argument of $z = x + y\mathrm{i}$, is

$$\arg z = \begin{cases} \operatorname{arctg}\frac{y}{x} & \text{for} \quad x > 0 \\ \pi + \operatorname{arctg}\frac{y}{x} & \text{for} \quad x < 0, \quad y > 0 \\ -\pi + \operatorname{arctg}\frac{y}{x} & \text{for} \quad x < 0, \quad y < 0 \\ \frac{\pi}{2} & \text{for} \quad x = 0, \quad y > 0 \\ \frac{\pi}{2} & \text{for} \quad x = 0, \quad y < 0. \end{cases}$$

10.1.8 MATHEMATICA `Arg[x+y I]`, `N[Arg[x+y I]]`

`In[1]:=Arg[-1]` $\mapsto$ Out[1]=π `In[3]:=Arg[2+3 I]` $\mapsto$ Out[3]=ArcTan$\left[\frac{3}{2}\right]$

`In[2]:=Arg[I^15]` $\mapsto$ Out[2]=$-\frac{\pi}{2}$ `In[4]:=N[Arg[2+3 I],9]` $\mapsto$ Out[4]=0.982793723

10.1.9 The function

$$\arg : \mathbb{C}^* \longrightarrow (-\pi, \pi]$$

is discontinuous on the negative half line

$$(-\infty,0)=\{\ z\ |\ \ \mathfrak{Re}\,z<0,\ \mathfrak{Im}\,z=0\ \}$$

because for $x\in(-\infty,0)$ we have

$$\lim_{y\nearrow 0}\arg(x+y\mathrm{i})=-\pi \qquad \text{and} \qquad \lim_{y\searrow 0}\arg(x+y\mathrm{i})=\pi.$$

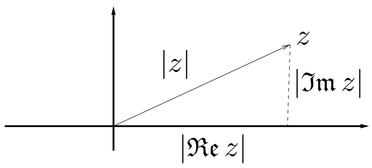

Figure 10.3: Relation among $|z|$, $|\mathfrak{Re}\,z|$ and $|\mathfrak{Im}\,z|$.

10.1.10 Proposition. *For any complex number $z=x+y\mathrm{i}$ we have*

$$\left.\begin{array}{r}|x|\\ \\ |y|\end{array}\right\}\leq|x+y\mathrm{i}|\leq|x|+|y|$$

that is,

$$\left.\begin{array}{r}|\mathfrak{Re}\,z|\\ \\ |\mathfrak{Im}\,z|\end{array}\right\}\leq|z|\leq|\mathfrak{Re}\,z|+|\mathfrak{Im}\,z|.$$

Proof. We have

$$|x+y\mathrm{i}|=\sqrt{x^2+y^2}\geq\sqrt{x^2}=|x| \qquad |x+y\mathrm{i}|=\sqrt{x^2+y^2}\geq\sqrt{y^2}=|y|$$

and the inequality

$$\sqrt{x^2+y^2}\leq|x|+|y|$$

is equivalent with the obvious relation

$$x^2+y^2\leq(|x|+|y|)^2.$$

10.1.11 Proposition. *The mapping $|\ |:\mathbb{C}\longrightarrow\mathbb{R}$,*

$$|z|=|x+y\mathrm{i}|=\sqrt{x^2+y^2}$$

is a norm in the real vector space $\mathbb{C}$, and $d:\mathbb{C}\times\mathbb{C}\longrightarrow\mathbb{R}$

$$d(z_1,z_2)=|z_1-z_2|=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

is the corresponding distance.

Demonstraţie. For any complex number $z = x + y\mathrm{i}$ we have

$$|z| = \sqrt{x^2 + y^2} \geq 0$$

and

$$|z| = 0 \quad \Longleftrightarrow \quad z = 0.$$

If α is a real number then

$$|\alpha z| = |(\alpha x) + (\alpha y)\mathrm{i}| = \sqrt{(\alpha x)^2 + (\alpha y)^2} = \sqrt{\alpha^2(x^2 + y^2)} = |\alpha|\,|z|.$$

For any $z_1 = x_1 + y_1\mathrm{i}$ and $z_2 = x_2 + y_2\mathrm{i}$ we have

$$\begin{aligned} |z_1 + z_2|^2 = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) &= |z_1|^2 + |z_2|^2 + z_1\,\bar{z}_2 + \bar{z}_1\,z_2 \\ &= |z_1|^2 + |z_2|^2 + 2\mathfrak{Re}\,(z_1\,\bar{z}_2) \leq |z_1|^2 + |z_2|^2 + 2|\mathfrak{Re}\,(z_1\,\bar{z}_2)| \\ &\leq |z_1|^2 + |z_2|^2 + 2|z_1\,\bar{z}_2| = (|z_1| + |z_2|)^2 \end{aligned}$$

whence

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

10.1.12 If we consider $\mathbb{R}^2$ endowed with the usual norm

$$||\ || : \mathbb{R}^2 \longrightarrow \mathbb{R}, \qquad ||(x, y)|| = \sqrt{x^2 + y^2}$$

then

$$||(x, y)|| = \sqrt{x^2 + y^2} = |x + y\mathrm{i}.|$$

This means that the linear mapping

$$\mathbb{R}^2 \longrightarrow \mathbb{C} : \ (x, y) \mapsto x + y\mathrm{i}$$

is an isomorphism which allows us to identify the normed vector spaces $(\mathbb{R}^2, ||\ ||)$ and $(\mathbb{C}, |\ |)$. If we take into consideration only the structure of vector space, the spaces $(\mathbb{R}^2, ||\ ||)$ and $(\mathbb{C}, |\ |)$ differ only in the used notations. The distance

$$d(z_1, z_2) = |z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

between two numbers $z_1 = x_1 + y_1\mathrm{i}$ and $z_2 = x_2 + y_2\mathrm{i}$ in the complex space correspond to the distance between the corresponding points in the Euclidean plane (Figure 10.4)

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

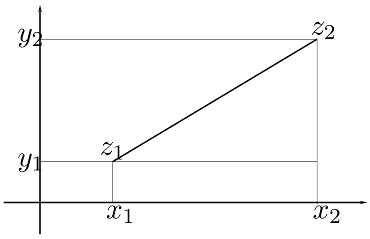

Figure 10.4: Distance between two points.

10.1.13 In the complex plane:

$$|z_1 - z_2| = \text{distance between } z_1 \text{ and } z_2.$$

$$|z| = |z - 0| = \text{distance between } z \text{ and the origin.}$$

Let $a \in \mathbb{C}$ be a fixed point and $r > 0$. The set

$$B_r(a) = \{ \ z \ | \ \ |z-a| < r \ \}$$

is called the (open) *disk* of center a and radius r (Figure 10.5).

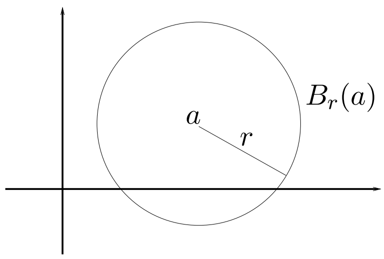

Figure 10.5: The disk of center a and radius r.

10.1.14 Definition. We say that $M \subset \mathbb{C}$ is a *bounded set* if there exist $a \in \mathbb{C}$ and $r > 0$ such that $M \subseteq B_r(a)$.

10.1.15 Exercise. M is a bounded set if and only if there exists $r > 0$ such that $|z| \leq r$, for any $z \in M$.

10.1.16 Definition. A set $D \subseteq \mathbb{C}$ is called an *open set* if for any $a \in D$ there exists $r > 0$ such that $B_r(a) \subset D$.

A set $F \subseteq \mathbb{C}$ is called a *closed set* if $\mathbb{C} - F$ is an open set.

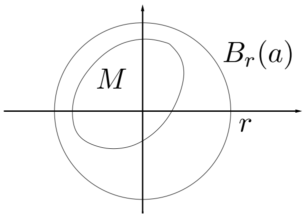

Figure 10.6: A bounded set.

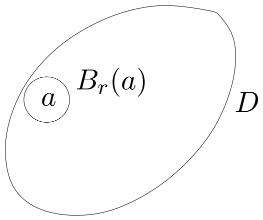

Figure 10.7: An open set.

10.1.17 Example.

a) The open disk $B_1(0)$ is an open set.

b) The half-plane $\{ \ z \mid \ \mathfrak{Im}\ z > 0 \ \}$ is an open set.

c) Any finite set $F \subseteq \mathbb{C}$ is a closed set.

d) The half-plane $\{ \ z \mid \ \mathfrak{Re}\ z \geq 0 \ \}$ is a closed set.

10.1.18 Definition. A set $K \subseteq \mathbb{C}$ is called a *compact set* if it is a closed and bounded set.

10.1.19 Exercise. Prouve that the relations

$$a) \qquad |z_1 \, z_2| = |z_1| \, |z_2|$$

$$b) \qquad |\, |z_1| - |z_2| \,| \leq |z_1 - z_2|$$

$$c) \qquad |z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\,|z_1|^2 + 2\,|z_2|^2$$

hold for any complex numbers z_1 and z_2.

Solution. a) We have

$$(x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2 = (x_1^2 + y_1^2)(x_2^2 + y_2^2).$$

b) From

$$|z_1| = |z_1 - z_2 + z_2| \leq |z_1 - z_2| + |z_2|, \qquad |z_2| = |z_2 - z_1 + z_1| \leq |z_2 - z_1| + |z_1|$$

we get the relation

$$-|z_1 - z_2| \leq |z_1| - |z_2| \leq |z_1 - z_2|$$

equivalent to

$$|\,|z_1| - |z_2|\,| \leq |z_1 - z_2|.$$

c) By direct computation we get

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) + (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) = 2\,|z_1|^2 + 2\,|z_2|^2.$$

10.2 Sequences of complex numbers

10.2.1 Definition. We say that the sequence $(z_n)_{n\geq 0}$ *converges* to a and write

$$\lim_{n\to\infty} z_n = a$$

if

$$\lim_{n\to\infty} |z_n - a| = 0.$$

10.2.2 From the relation

$$\left.\begin{array}{l} |x_n - \alpha| \\ \\ |y_n - \beta| \end{array}\right\} \leq |(x_n + y_n\mathrm{i}) - (\alpha + \beta\mathrm{i})| \leq |x_n - \alpha| + |y_n - \beta|$$

we get

$$\lim_{n\to\infty} (x_n + y_n\mathrm{i}) = \alpha + \beta\mathrm{i} \quad \Longleftrightarrow \quad \left\{\begin{array}{l} \lim_{n\to\infty} x_n = \alpha \\ \\ \lim_{n\to\infty} y_n = \beta. \end{array}\right.$$

that is, the sequence of complex numbers $(z_n)_{n\geq 0}$ converges if and only if the sequences of real numbers $(\mathfrak{Re}\, z_n)_{n\geq 0}$ and $(\mathfrak{Im}\, z_n)_{n\geq 0}$ are convergent, and

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \mathfrak{Re}\, z_n + \mathrm{i} \lim_{n\to\infty} \mathfrak{Im}\, z_n.$$

10.2.3 Example.

$$\lim_{n\to\infty} \left(\frac{n}{n+1} + \mathrm{i} \left(1 + \frac{1}{n}\right)^n \right) = \lim_{n\to\infty} \frac{n}{n+1} + \mathrm{i} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = 1 + \mathrm{e}\,\mathrm{i}.$$

10.2.4 MATHEMATICA: `Lim[z[n],n->Infinity]`

`In[1]:=Lim[n/(n+1),n->Infinity]`	$\mapsto$	Out[1]=1
`In[2]:=Lim[(1+1/n)^n,n->Infinity]`	$\mapsto$	Out[2]=**e**
`In[3]:=Lim[n/(n+1)+I (1+1/n)^n,n->Infinity]`	$\mapsto$	Out[3]=1+ⅈ**e**

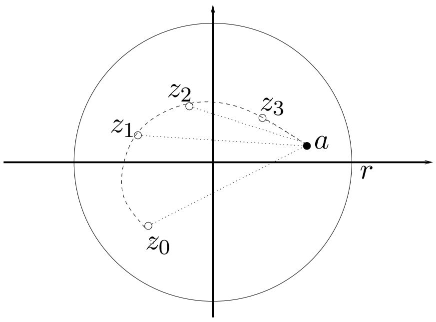

Figure 10.8: A bounded convergent sequence.

10.2.5 Definition. A sequence $(z_n)_{n\geq 0}$ is called a *bounded sequence* if there exists $r>0$ such that

$$|z_n| \leq r, \qquad \text{for any} \quad n \geq 0.$$

10.2.6 From the relation

$$\left.\begin{array}{l} |x_n| \\ \\ |y_n| \end{array}\right\} \leq |x_n + y_n \mathrm{i}| \leq |x_n| + |y_n|$$

it follows that the sequence of complex numbers $(z_n)_{n\geq 0}$ is bounded if and only if the sequences of real numbers $(\mathfrak{Re}\, z_n)_{n\geq 0}$ and $(\mathfrak{Im}\, z_n)_{n\geq 0}$ are bounded.

10.2.7 Exercise. Prouve that

$$\begin{array}{llll} a) & |z|<1 & \Longrightarrow & \lim\limits_{n\to\infty} z^n = 0 \\ b) & |z|<1 & \Longrightarrow & \sum\limits_{n=0}^{\infty} z^n = \frac{1}{1-z} \\ c) & |z|<1 & \Longrightarrow & \frac{1}{1-z} = 1+z+z^2+z^3+\cdots \end{array}$$

Solution. We have

$$\lim_{n\to\infty} |z^n - 0| = \lim_{n\to\infty} |z|^n = 0$$

$$\sum_{n=0}^{\infty} z^n = \lim_{k\to\infty} \sum_{n=0}^{k} z^n = \lim_{k\to\infty} \tfrac{1-z^{k+1}}{1-z} = \tfrac{1}{1-z}$$

$$\tfrac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1+z+z^2+z^3+\cdots$$

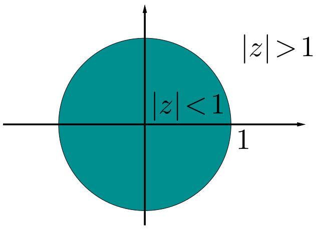

Figure 10.9: The set of all the points z with $|z|<1$.

10.2.8 Definition. We say that the sequence $(z_n)_{n\geq 0}$ *tends to infinity*

$$\lim_{n\to\infty} z_n = \infty$$

if

$$\lim_{n\to\infty} |z_n| = \infty.$$

10.2.9 If $|z|>1$ then $\lim_{n\to\infty} z^n = \infty$.

10.3 Complex functions of a complex variable

10.3.1 *Complex function* means a function with complex values.

10.3.2 Definition. We say that the function of a real variable

$$f:(a,b)\longrightarrow \mathbb{R}$$

is *differentiable* at the point $x_0\in(a,b)$ if there exists and is finite the limit

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$$

called the *derivative* of the function f at the point x_0.

10.3.3 Derivatives of some real functions of a real variable

Function		Derivative	Domain	Conditions
$f:\mathbb{R}\longrightarrow\mathbb{R}$	$f(x)=c$	$f'(x)=0$	$\mathbb{R}$	
$f:\mathbb{R}\longrightarrow\mathbb{R}$	$f(x)=x^n$	$f'(x)=nx^{n-1}$	$\mathbb{R}$	$n\in\mathbb{N}^*$
$f:(0,\infty)\longrightarrow\mathbb{R}$	$f(x)=x^\alpha$	$f'(x)=\alpha x^{\alpha-1}$	$(0,\infty)$	$\alpha\in\mathbb{R}$
$f:\mathbb{R}^*\longrightarrow\mathbb{R}$	$f(x)=\frac{1}{x}$	$f'(x)=-\frac{1}{x^2}$	$\mathbb{R}^*$	
$f:[0,\infty)\longrightarrow\mathbb{R}$	$f(x)=\sqrt{x}$	$f'(x)=\frac{1}{2\sqrt{x}}$	$(0,\infty)$	
$f:[0,\infty)\longrightarrow\mathbb{R}$	$f(x)=\sqrt[n]{x}$	$f'(x)=\frac{1}{n\sqrt[n]{x^{n-1}}}$	$(0,\infty)$	$n\in 2\mathbb{N}^*$
$f:\mathbb{R}\longrightarrow\mathbb{R}$	$f(x)=\sqrt[n]{x}$	$f'(x)=\frac{1}{n\sqrt[n]{x^{n-1}}}$	$\mathbb{R}^*$	$n\in 2\mathbb{N}+1$
$f:(0,\infty)\longrightarrow\mathbb{R}$	$f(x)=\ln x$	$f'(x)=\frac{1}{x}$	$(0,\infty)$	
$f:\mathbb{R}\longrightarrow\mathbb{R}$	$f(x)=a^x$	$f'(x)=a^x\ln a$	$\mathbb{R}$	$0<a\neq 1$
$f:\mathbb{R}\longrightarrow\mathbb{R}$	$f(x)=\mathrm{e}^x$	$f'(x)=\mathrm{e}^x$	$\mathbb{R}$	
$f:\mathbb{R}\longrightarrow\mathbb{R}$	$f(x)=\sin x$	$f'(x)=\cos x$	$\mathbb{R}$	
$f:\mathbb{R}\longrightarrow\mathbb{R}$	$f(x)=\cos x$	$f'(x)=-\sin x$	$\mathbb{R}$	
$f:\mathbb{R}-(\frac{\pi}{2}+\mathbb{Z}\pi)\rightarrow\mathbb{R}$	$f(x)=\tan x$	$f'(x)=\frac{1}{\cos^2 x}$	$\mathbb{R}-(\frac{\pi}{2}+\mathbb{Z}\pi)$	
$f:\mathbb{R}-\mathbb{Z}\pi\longrightarrow\mathbb{R}$	$f(x)=\cot x$	$f'(x)=-\frac{1}{\sin^2 x}$	$\mathbb{R}-\mathbb{Z}\pi$	
$f:[-1,1]\longrightarrow\mathbb{R}$	$f(x)=\arcsin x$	$f'(x)=\frac{1}{\sqrt{1-x^2}}$	$(-1,1)$	
$f:[-1,1]\longrightarrow\mathbb{R}$	$f(x)=\arccos x$	$f'(x)=-\frac{1}{\sqrt{1-x^2}}$	$(-1,1)$	
$f:\mathbb{R}\longrightarrow\mathbb{R}$	$f(x)=\arctan x$	$f'(x)=\frac{1}{1+x^2}$	$\mathbb{R}$	
$f:\mathbb{R}\longrightarrow\mathbb{R}$	$f(x)=\mathrm{arccot} x$	$f'(x)=-\frac{1}{1+x^2}$	$\mathbb{R}$	
$f:\mathbb{R}\longrightarrow\mathbb{R}$	$f(x)=\mathrm{sh}\, x$	$f'(x)=\mathrm{ch}\, x$	$\mathbb{R}$	
$f:\mathbb{R}\longrightarrow\mathbb{R}$	$f(x)=\mathrm{ch}\, x$	$f'(x)=\mathrm{sh}\, x$	$\mathbb{R}$	

10.3.4 Previous definition can not be directly extended to functions of two variables

$$f : D\subseteq\mathbb{R}^2\longrightarrow\mathbb{R}$$

because the relation

$$f'(x_0,y_0)=\lim_{(x,y)\to(x_0,y_0)}\frac{f(x,y)-f(x_0,y_0)}{(x,y)-(x_0,y_0)}$$

is meaningless (divison by the vector $(x,y)-(x_0,y_0)$ is not defined). But, the possibility to divide by a non-null complex number allows us to define the differentiability of a function of complex variable by following the analogy with the real case.

10.3.5 Definition. Let $D\subseteq\mathbb{C}$ be an open set.We say that the complex function

$$f : D\longrightarrow\mathbb{C}$$

is *complex-differentiable* (or $\mathbb{C}$-*differentiable*) at the point $z_0 \in D$ if there exists and is finite the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

called the *derivative* of f at z_0. Instead of $f'(z_0)$ we sometimes write $\frac{df}{dz}(z_0)$.

10.3.6 Example. The function

$$f : \mathbb{C} \longrightarrow \mathbb{C}, \qquad f(z) = z^3$$

is $\mathbb{C}$-differentiable at any point $z_0 \in \mathbb{C}$

$$f'(z_0) = \lim_{z \to z_0} \frac{z^3 - z_0^3}{z - z_0} = \lim_{z \to z_0} (z^2 + z_0 z + z_0^2) = 3z_0^2$$

and $f'(z) = 3z^2$, that is, we have

$$(z^3)' = 3z^2.$$

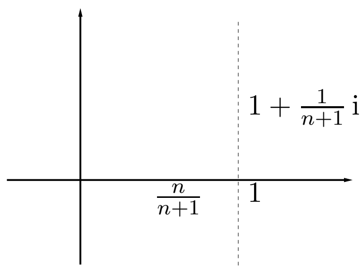

Figure 10.10: The function $f(z) = \bar{z}$ is not $\mathbb{C}$-differentiable at $z_0 = 1$.

10.3.7 The function

$$f : \mathbb{C} \longrightarrow \mathbb{C}, \qquad f(z) = \bar{z}$$

is not $\mathbb{C}$-differentiable at the point $z_0 = 1$ because the limit

$$\lim_{z \to 1} \frac{\bar{z} - 1}{z - 1}$$

does not exist. By choosing the sequence $z_n = \frac{n}{n+1}$ with $\lim_{n\to\infty} z_n = 1$ we get

$$\lim_{n \to \infty} \frac{\bar{z}_n - 1}{z_n - 1} = 1$$

but, by choosing the sequence $z_n = 1 + \frac{1}{n+1}\mathrm{i}$ also with $\lim_{n\to\infty} z_n = 1$ we get

$$\lim_{n \to \infty} \frac{\bar{z}_n - 1}{z_n - 1} = -1.$$

10.3.8 Based on the identification of $\mathbb{C}$ with $\mathbb{R}^2$

$$\mathbb{C} \longrightarrow \mathbb{R}^2 : \ x + y\mathrm{i} \mapsto (x, y)$$

we can describe any complex function of a complex variable

$$f : D \longrightarrow \mathbb{C}$$

by using two real functions of two real variables

$$f(x + yi) = u(x, y) + v(x, y)\,\mathrm{i}$$

where

$$u = \mathfrak{Re}\ f : D \longrightarrow \mathbb{R} \quad \text{is the } \textit{real part} \text{ of } f$$

$$v = \mathfrak{Im}\ f : D \longrightarrow \mathbb{R} \quad \text{is the } \textit{imaginary part} \text{ of } f.$$

10.3.9 Examples. a) In the case of the function

$$f : \mathbb{C} \longrightarrow \mathbb{C}, \qquad f(z) = \bar{z}$$

we have

$$f(x + y\mathrm{i}) = x - y\mathrm{i}$$

that is,

$$u(x, y) = x, \qquad v(x, y) = -y.$$

b) In the case of the function

$$f : \mathbb{C} \longrightarrow \mathbb{C}, \qquad f(z) = z^2$$

we have

$$f(x + y\mathrm{i}) = (x + y\mathrm{i})^2 = (x^2 - y^2) + 2xy\mathrm{i}$$

and hence

$$u(x, y) = x^2 - y^2, \qquad v(x, y) = 2xy.$$

10.3.10 In view of the definition, the function

$$f : D \longrightarrow \mathbb{C}, \qquad f(x + yi) = u(x, y) + v(x, y)\,\mathrm{i}$$

is $\mathbb{C}$-differentiable at $z_0 = x_0 + y_0\mathrm{i}$ if and only if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists and is finite. In order to have

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}=\alpha+\beta\mathrm{i}$$

it is necessary to have

$$\lim_{t\to 0}\frac{f(z_0+t)-f(z_0)}{t}=\alpha+\beta\mathrm{i},\qquad \lim_{t\to 0}\frac{f(z_0+t\mathrm{i})-f(z_0)}{t\mathrm{i}}=\alpha+\beta\mathrm{i}$$

that is, the relations

$$\lim_{t\to 0}\frac{u(x_0+t,y_0)-u(x_0,y_0)}{t}+\lim_{t\to 0}\frac{v(x_0+t,y_0)-v(x_0,y_0)}{t}\mathrm{i}=\alpha+\beta\mathrm{i}$$
$$\lim_{t\to 0}\frac{u(x_0,y_0+t)-u(x_0,y_0)}{t\mathrm{i}}+\lim_{t\to 0}\frac{v(x_0,y_0+t)-v(x_0,y_0)}{t\mathrm{i}}\mathrm{i}=\alpha+\beta\mathrm{i}$$

equivalent to

$$\frac{\partial u}{\partial x}(x_0,y_0)=\alpha=\frac{\partial v}{\partial y}(x_0,y_0),\qquad \frac{\partial v}{\partial x}(x_0,y_0)=\beta=-\frac{\partial u}{\partial y}(x_0,y_0).$$

Particularly, if f is $\mathbb{C}$-derivable at $z_0=x_0+y_0\mathrm{i}$ then

$$f'(x_0+y_0\mathrm{i})=\frac{\partial u}{\partial x}(x_0,y_0)+\frac{\partial v}{\partial x}(x_0,y_0)\,\mathrm{i}.$$

10.3.11 Theorem (Cauchy-Riemann) *The function*

$$f:D\longrightarrow\mathbb{C},\qquad f(x+yi)=u(x,y)+v(x,y)\,\mathrm{i}$$

defined on the open set $D\subseteq\mathbb{C}$ is $\mathbb{C}$-differentiable at the point $z_0=x_0+y_0\mathrm{i}\in D$ if and only if the real functions

$$u:D\longrightarrow\mathbb{R},\qquad v:D\longrightarrow\mathbb{R}$$

are $\mathbb{R}$-differentiable at (x_0,y_0) and the Cauchy-Riemann relations

$$\frac{\partial u}{\partial x}(x_0,y_0)=\frac{\partial v}{\partial y}(x_0,y_0),\qquad \frac{\partial u}{\partial y}(x_0,y_0)=-\frac{\partial v}{\partial x}(x_0,y_0)$$

are satisfied. If all these conditions are satisfied then

$$f'(x_0+y_0\mathrm{i})=\frac{\partial u}{\partial x}(x_0,y_0)+\frac{\partial v}{\partial x}(x_0,y_0)\,\mathrm{i}.$$

A proof can be found in [**?**].

10.3.12 Definition. Let $D\subseteq\mathbb{C}$ be an open set. A function

$$f:D\longrightarrow\mathbb{C}$$

is called $\mathbb{C}$-*differentiable* (or *holomorphic*) if it is $\mathbb{C}$-differentiable at any point of D.

10.3.13 Exercise. Prove that the function
$$f : \mathbb{C} \longrightarrow \mathbb{C}, \qquad f(z) = z^2$$
is holomorphic and find its derivative $f'(z)$.

Solution. We use the Cauchy-Riemann theorem. We have
$$f(x + y\mathrm{i}) = (x + y\mathrm{i})^2 = (x^2 - y^2) + 2xy\mathrm{i}$$
and hence
$$u(x, y) = x^2 - y^2, \qquad v(x, y) = 2xy.$$
The functions u and v are $\mathbb{R}$-differentiable at any point and
$$\frac{\partial u}{\partial x}(x, y) = 2x = \frac{\partial v}{\partial y}(x, y), \qquad \frac{\partial u}{\partial y}(x, y) = -2y = -\frac{\partial v}{\partial x}(x, y).$$
The derivative of f is
$$f'(x + y\mathrm{i}) = \frac{\partial u}{\partial x}(x, y) + \frac{\partial v}{\partial x}(x, y)\,\mathrm{i} = 2x + 2y\mathrm{i}$$
that is, $f'(z) = 2z$.

10.3.14 Exercise. Prove that the function
$$f : \mathbb{C} \longrightarrow \mathbb{C}, \qquad f(z) = \bar{z}$$
is not $\mathbb{C}$-differentiable.

Solution. We use the Cauchy-Riemann theorem. We have
$$f(x + y\mathrm{i}) = x - y\mathrm{i}$$
that is,
$$u(x, y) = x, \qquad v(x, y) = -y.$$
In this case the Cauchy-Riemann relations are verified at no point because
$$\frac{\partial u}{\partial x}(x, y) = 1, \qquad \frac{\partial v}{\partial y}(x, y) = -1.$$

10.3.15 Definition. The function
$$f : \mathbb{C} \longrightarrow \mathbb{C}, \qquad f(z) = \mathrm{e}^z$$
where
$$\mathrm{e}^{x+y\mathrm{i}} = \mathrm{e}^x\, \mathrm{e}^{y\mathrm{i}} = \mathrm{e}^x\, (\cos y + \mathrm{i}\, \sin y) = \mathrm{e}^x\, \cos y + \mathrm{i}\, \mathrm{e}^x\, \sin y$$
is called the (complex) *exponential function.*

10.3.16 MATHEMATICA: `Exp[x+y\, I]`, `N[Exp[x+y\, I]]`

```
In[1]:=Exp[x+y I]                  ↦  Out[1]=e^(x+ⅈy)
In[2]:=ComplexExpand[Exp[x+y I]]   ↦  Out[2]=e^x Cos[y]+ⅈ e^x Sin[y]
In[3]:=Exp[2+3 I]                  ↦  Out[3]=e^(2+3 ⅈ)
In[4]:=N[Exp[2+3 I]]               ↦  Out[4]=−7.31511+1.04274 ⅈ
In[5]:=N[Exp[2+3 I],15]            ↦  Out[5]=−7.31511009490110+1.04274365623590 ⅈ
```

10.3.17 The exponential function is a periodic function with the period $2\pi\mathrm{i}$

$$\mathrm{e}^{z+2\pi\mathrm{i}}=\mathrm{e}^{z}$$

and

$$\mathrm{e}^{z_1+z_2}=\mathrm{e}^{z_1}\,\mathrm{e}^{z_2}$$

for any $z_1,\ z_2\in\mathbb{C}$.

10.3.18 Exercise. Prove that the exponential function

$$f:\mathbb{C}\longrightarrow\mathbb{C},\qquad f(z)=\mathrm{e}^{z}$$

is holomorphic and

$$(\mathrm{e}^{z})'=\mathrm{e}^{z}.$$

Solution. We use the Cauchy-Riemann theorem. From the relation

$$f(x+y\mathrm{i})=\mathrm{e}^{x}\,\cos y+\mathrm{i}\,\mathrm{e}^{x}\,\sin y$$

we get $u(x,y)=\mathrm{e}^{x}\,\cos y$ and $v(x,y)=\mathrm{e}^{x}\,\sin y$. The real functions u and v are $\mathbb{R}$-differentiable at any point and

$$\frac{\partial u}{\partial x}(x,y)=\mathrm{e}^{x}\,\cos y=\frac{\partial v}{\partial y}(x,y),\qquad \frac{\partial u}{\partial y}(x,y)=-\mathrm{e}^{x}\,\sin y=-\frac{\partial v}{\partial x}(x,y).$$

The derivative of f is

$$f'(z)=f'(x+y\mathrm{i})=\frac{\partial u}{\partial x}(x,y)+\frac{\partial v}{\partial x}(x,y)\,\mathrm{i}=\mathrm{e}^{x}\,\cos y+\mathrm{i}\,\mathrm{e}^{x}\,\sin y=\mathrm{e}^{z}.$$

10.3.19 Exercise. Find the holomorphic function

$$f:\mathbb{C}\longrightarrow\mathbb{C}$$

satisfying the conditions

$$\mathfrak{Im}\,f(x,y)=2xy+y,\qquad f(\mathrm{i})=\mathrm{i}.$$

Solution. By looking for a function f of the form

$$f(x+y\mathrm{i}) = u(x,y) + (2xy+y)\mathrm{i}$$

from the Cauchy-Riemann theorem we deduce the relations

$$\frac{\partial u}{\partial x}(x,y) = 2x+1, \qquad \frac{\partial u}{\partial y}(x,y) = -2y$$

whence $u(x,y) = x^2 - y^2 + x + c$, where c is a constant. By imposing the additional conditions $f(\mathrm{i}) = \mathrm{i}$ we get

$$f(x+y\mathrm{i}) = x^2 - y^2 + x + 1 + (2xy+y)\mathrm{i} = (x+y\mathrm{i})^2 + (x+y\mathrm{i}) + 1$$

that is, $f(z) = z^2 + z + 1$.

10.3.20 a) If the functions $f, g : D \longrightarrow \mathbb{C}$ are holomorphic then

$$(\alpha f \pm \beta g)' = \alpha\, f' \pm \beta\, g' \qquad\qquad (fg)' = f'g + fg'$$

for any $\alpha, \beta \in \mathbb{C}$. If, in addition, $g(z) \neq 0$ for any $z \in D$, then

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.$$

b) If the functions $D \xrightarrow{f} \mathbb{C} \xrightarrow{g} \mathbb{C}$ are holomorphic then

$$\frac{d}{dz}(g(f(z)) = g'(f(z))\, f'(z).$$

10.3.21 MATHEMATICA `D[f[z],z]`

`In[1]:=D[a f[z]+b g[z],z]`	$\mapsto$	$\mathrm{Out[1]{=}a\,f'[z]{+}b\,g'[z]}$
`In[2]:=D[f[z] g[z],z]`	$\mapsto$	$\mathrm{Out[2]{=}f'[z]\,g[z]{+}f[z]\,g'[z]}$
`In[3]:=D[f[z]/g[z],z]`	$\mapsto$	$\mathrm{Out[3]{=}}\frac{\mathrm{f'[z]}}{\mathrm{g[z]}} - \frac{\mathrm{f[z]\,g'[z]}}{\mathrm{g[z]^2}}$
`In[4]:=D[g[f[z]],z]`	$\mapsto$	$\mathrm{Out[4]{=}}g'[f[z]]\,f'[z]$

10.3.22 Exercise. The complex functions

$$\cos : \mathbb{C} \longrightarrow \mathbb{C}, \qquad \cos z = \frac{\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}}{2}$$

$$\sin : \mathbb{C} \longrightarrow \mathbb{C}, \qquad \sin z = \frac{\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}}{2\mathrm{i}}$$

$$\mathrm{ch} : \mathbb{C} \longrightarrow \mathbb{C}, \qquad \mathrm{ch}\, z = \frac{\mathrm{e}^{z} + \mathrm{e}^{-z}}{2}$$

$$\mathrm{sh} : \mathbb{C} \longrightarrow \mathbb{C}, \qquad \mathrm{sh}\, z = \frac{\mathrm{e}^{z} - \mathrm{e}^{-z}}{2}$$

are holomorphic and

$$(\cos z)' = -\sin z \qquad (\sin z)' = \cos z$$
$$(\mathrm{ch}\, z)' = \mathrm{sh}\, z \qquad (\mathrm{sh}\, z)' = \mathrm{ch}\, z.$$

Hint. Direct computation.

10.3.23 MATHEMATICA `D[f[z],z]`

`In[1]:=D[z^n,z]`	$\mapsto$	Out[1]=n z^{-1+n}	`In[4]:=D[Exp[z],z]`	$\mapsto$	Out[4]=e^{z}
`In[2]:=D[Cos[z],z]`	$\mapsto$	Out[2]=−Sin[z]	`In[5]:=D[Sin[z],z]`	$\mapsto$	Out[5]=Cos[z]
`In[3]:=D[Cosh[z],z]`	$\mapsto$	Out[3]=Sinh[z]	`In[6]:=D[Sinh[z],z]`	$\mapsto$	Out[6]=Cosh[z]

10.3.24 MATHEMATICA Figura 10.11 s-a obţinut utilizând

```
In[1]:=Plot[{Exp[x], x, Log[x]}, {x, -3, 3}, PlotStyle -> {Red, Dashed, Thick},
                                                     AspectRatio -> Automatic]
```

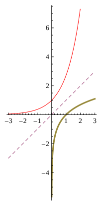

Figure 10.11: The natural logarithm ln is the inverse of the exponential function e^x.

10.3.25 The real exponential function

$$\mathbb{R} \longrightarrow (0,\infty): \ x \mapsto \mathrm{e}^x$$

is bijective. Its inverse is the natural logarithm function

$$(0,\infty) \longrightarrow \mathbb{R}: \ x \mapsto \ln x.$$

We have

$$x = \mathrm{e}^{\ln x}$$

for any $x \in (0,\infty)$. In the complex case, we can obtain a rather similar relation

$$z = |z|\,\mathrm{e}^{\mathrm{i}\arg z} = \mathrm{e}^{\ln|z|}\,\mathrm{e}^{\mathrm{i}\arg z} = \mathrm{e}^{\ln|z|+\mathrm{i}(\arg z + 2k\pi)}$$

satisfied for any $k \in \mathbb{Z}$.

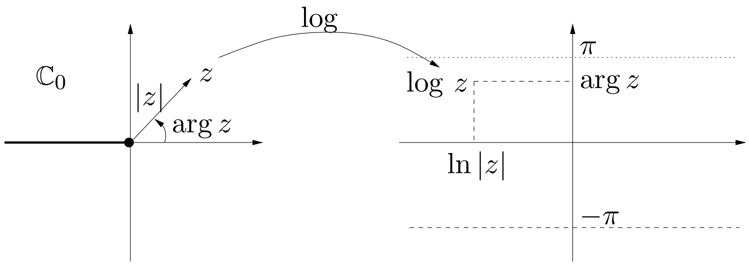

Figure 10.12: The principal branch $\log z = \ln|z| + \mathrm{i}\arg z$.

10.3.26 Definition. Let us consider the set

$$\mathbb{C}_0 = \mathbb{C} - \{\ z \mid \ \mathfrak{Im}\ z = 0,\ \mathfrak{Re}\ z \le 0\ \}$$

obtained by removing from $\mathbb{C}$ the "cut" $\{\ z \mid \mathfrak{Im}\ z = 0,\ \mathfrak{Re}\ z \le 0\}$ connecting 0 with ∞. The continuous functions

$$\log_k : \mathbb{C}_0 \longrightarrow \mathbb{C}, \qquad \log_k z = \ln|z| + \mathrm{i}(\arg z + 2k\pi)$$

depending on the parameter $k \in \mathbb{Z}$ are called the *branches of the complex logarithmic function.* The *principal branch* $\log_0$ is usually denoted by log, that is,

$$\log : \mathbb{C}_0 \longrightarrow \mathbb{C}, \qquad \log z = \ln|z| + \mathrm{i}\arg z.$$

10.3.27 MATHEMATICA `ComplexExpand[Log[x+I y]]`

`In[1]:=ComplexExpand[Log[x+I y]]`	$\mapsto$	Out[1]=$\mathbb{i}$ Arg[x+$\mathbb{i}$y]+$\frac{1}{2}$ Log[x^2+y^2]
`In[2]:=ComplexExpand[Log[1+I]]`	$\mapsto$	Out[2]=$\frac{\mathbb{i}\,\pi}{4}+\frac{\mathrm{Log}[2]}{2}$
`In[3]:=N[ComplexExpand[Log[1+I]]]`	$\mapsto$	Out[3]=0.346574+0.785398 $\mathbb{i}$
`In[4]:=N[ComplexExpand[Log[1+I]],10]`	$\mapsto$	Out[4]=0.3465735903+0.7853981634 $\mathbb{i}$

10.3.28 At the level of the cut $\{\ z \mid \mathfrak{Im}\ z = 0,\ \mathfrak{Re}\ z \le 0\}$ we have

$$\lim_{t \nearrow 0} \log(-2 + t\mathrm{i}) = \ln 2 - \mathrm{i}\pi \qquad \text{and} \qquad \lim_{t \searrow 0} \log(-2 + t\mathrm{i}) = \ln 2 + \mathrm{i}\pi.$$

Therefore, log can not be extended by continuity at the points of the cut.

10.3.29 MATHEMATICA `Limit[Log[-2 + t I], t -> 0, Direction -> 1]`

`In[1]:=Limit[Log[-2 + t I], t -> 0, Direction -> 1]`	$\mapsto$	Out[1]=$-\mathbb{i}\,\pi$+Log[2]
`In[2]:=Limit[Log[-2 + t I], t -> 0, Direction -> -1]`	$\mapsto$	Out[2]=$\mathbb{i}\,\pi$+Log[2]

Exercise. Prove that

$$(\log_k z)' = \frac{1}{z}.$$

Solution. By denoting $f(z)=\log_k z=u(x,y)+\mathrm{i}\,v(x,y)$ we get the relations

$$\frac{\partial u}{\partial x}(x,y) = \frac{x}{x^2+y^2} = \frac{\partial v}{\partial y}(x,y), \qquad \frac{\partial u}{\partial y}(x,y) = \frac{y}{x^2+y^2} = -\frac{\partial v}{\partial x}(x,y)$$

whence

$$f'(x+y\mathrm{i}) = \frac{\partial u}{\partial x}(x,y) + \mathrm{i}\,\frac{\partial v}{\partial x}(x,y) = \frac{x-y\mathrm{i}}{x^2+y^2} = \frac{1}{x+y\mathrm{i}}.$$

10.3.30 The branches of the *power function* z^α with complex exponent α are

$$\mathbb{C}_0 \longrightarrow \mathbb{C}: \ z \mapsto z^\alpha = \mathrm{e}^{\alpha \log_k z}.$$

In the case $\alpha=\frac{1}{n}$ with $n\in\mathbb{N}^*$ there exist only n distinct branches

$$\mathbb{C}_0 \longrightarrow \mathbb{C}: \ z \mapsto z^{\frac{1}{n}} = \mathrm{e}^{\frac{1}{n}\log_k z} = \sqrt[n]{|z|}\ \mathrm{e}^{\frac{\mathrm{i}}{n}(\arg z+2k\pi)}$$

for example, those corresponding to $k\in\{0,1,...,n-1\}$.

10.3.31 MATHEMATICA `ComplexExpand[Sqrt[x+I y]]`

`In[1]:=ComplexExpand[Sqrt[x+I y]]`

$\mapsto$ Out[1]=$(\mathrm{x}^2+\mathrm{y}^2)^{1/4}\mathrm{Cos}\left[\frac{1}{2}\mathrm{Arg}[\mathrm{x}+\mathrm{i}\mathrm{y}]\right]+\mathrm{i}(\mathrm{x}^2+\mathrm{y}^2)^{1/4}\mathrm{Sin}\left[\frac{1}{2}\mathrm{Arg}[\mathrm{x}+\mathrm{i}\mathrm{y}]\right]$

`In[2]:=ComplexExpand[Sqrt[1+I]]` $\mapsto$ Out[2]=$2^{1/2}\mathrm{Cos}\left[\frac{\pi}{8}\right]+\mathrm{i}\,2^{1/2}\mathrm{Sin}\left[\frac{\pi}{8}\right]$

`In[3]:=N[ComplexExpand[Sqrt[1+I]],10]` $\mapsto$ Out[3]=1.098684113+0.4550898606 i

`In[4]:=Limit[Sqrt[-1 + I x], x -> 0, Direction -> 1]` $\mapsto$ Out[4]=$-\mathrm{i}$

`In[5]:=Limit[Sqrt[-1 + I x], x -> 0, Direction -> -1]` $\mapsto$ Out[5]=i

10.3.32 MATHEMATICA `ComplexExpand[(x + I y)^(1/n)]`

`In[1]:=ComplexExpand[(x + I y)^(1/3)]`

$\mapsto$ Out[1]=$(\mathrm{x}^2+\mathrm{y}^2)^{\frac{1}{2\mathrm{n}}}\mathrm{Cos}\left[\frac{\mathrm{Arg}[\mathrm{x}+\mathrm{i}\mathrm{y}]}{\mathrm{n}}\right]+\mathrm{i}(\mathrm{x}^2+\mathrm{y}^2)^{\frac{1}{2\mathrm{n}}}\mathrm{Sin}\left[\frac{\mathrm{Arg}[\mathrm{x}+\mathrm{i}\mathrm{y}]}{\mathrm{n}}\right]$

10.3.33 Exercise. Describe the branch of the function

$$f(z) = \sqrt[3]{\frac{z}{\mathrm{i}-z}} \qquad \text{with} \quad f(1) = \frac{1}{\sqrt[6]{2}}\,\mathrm{e}^{\mathrm{i}\frac{5\pi}{12}}.$$

Solution. We have

$$\frac{z}{\mathrm{i}-z} \in \mathbb{C}_0$$

if and only if z belongs to the domain

$$D = \mathbb{C} - [0,\mathrm{i}] = \mathbb{C} - \{\ z \mid \mathfrak{Re}\,z = 0, \quad 0 \le \mathfrak{Im}\,z \le 1\ \}$$

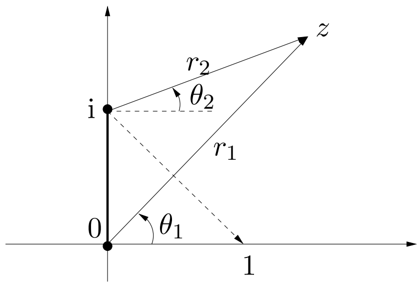

Figure 10.13: The relation $z = r_1\, e^{i\theta_1} = i + r_2\, e^{i\theta_2}$.

obtained by removing from $\mathbb{C}$ the "cut" $[0, i]$. By denoting

$$z = r_1\, e^{i\theta_1} = i + r_2\, e^{i\theta_2}$$

we get $i - z = -r_2\, e^{i\theta_2} = r_2\, e^{i(\theta_2+\pi)}$ and

$$f(z) = \sqrt[3]{\frac{r_1}{r_2}}\; e^{i\frac{\theta_1-\theta_2-\pi+2k\pi}{3}}.$$

From $1 = e^{i0} = i + \sqrt{2}\, e^{-i\frac{\pi}{4}}$ it follows $k = 1$ and hence

$$f(z) = \sqrt[3]{\frac{r_1}{r_2}}\; e^{i\frac{\theta_1-\theta_2+\pi}{3}}.$$

10.4 Complex line integral

10.4.1 Proposition. *Let $D \subseteq \mathbb{C}$. The complex mapping*

$$\gamma : [a, b] \longrightarrow D, \qquad \gamma(t) = \varphi(t) + \psi(t)\, i$$

is continuous if and only if the real mappings

$$\varphi = \mathfrak{Re}\, \gamma : [a, b] \longrightarrow \mathbb{R}, \qquad \psi = \mathfrak{Im}\, \gamma : [a, b] \longrightarrow \mathbb{R}$$

are continuous.

Proof. The statement follows from the relation

$$\left.\begin{array}{l} |\varphi(t) - \varphi(t_0)| \\ \\ |\psi(t) - \psi(t_0)| \end{array}\right\} \leq |\gamma(t) - \gamma(t_0)| \leq |\varphi(t) - \varphi(t_0)| + |\psi(t) - \psi(t_0)|.$$

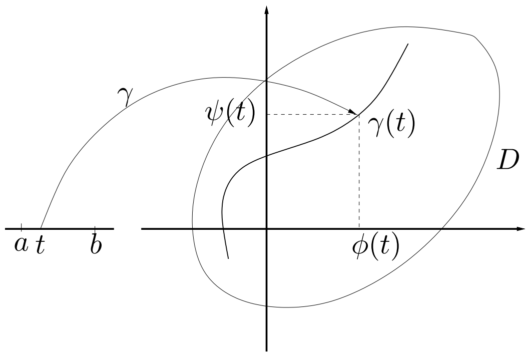

Figure 10.14: Path of class C^1 in D.

10.4.2 Definition. A mapping

$$\gamma : (a,b) \longrightarrow D$$

is *differentiable* at the point $t_0 \in (a,b)$ if there exists and is finite the limit

$$\gamma'(t_0) = \lim_{t\to t_0} \frac{\gamma(t)-\gamma(t_0)}{t-t_0}.$$

The mapping γ is called *differentiable* if it is differentiable at any point.

10.4.3 In the case of a mapping

$$\gamma : [a,b] \longrightarrow D$$

the derivateives $\gamma'(a)$ and $\gamma'(b)$ are defined as

$$\gamma'(a) = \lim_{t\searrow a} \frac{\gamma(t)-\gamma(a)}{t-a}, \qquad \gamma'(b) = \lim_{t\nearrow b} \frac{\gamma(t)-\gamma(b)}{t-b}.$$

10.4.4 Proposition. *The mapping*

$$\gamma : [a,b] \longrightarrow D, \qquad \gamma(t) = \varphi(t) + \psi(t)\,\mathrm{i}$$

is diferrentiable if and only if the real mappings

$$\varphi = \mathfrak{Re}\,\gamma : [a,b] \longrightarrow \mathbb{R}, \qquad \psi = \mathfrak{Im}\,\gamma : [a,b] \longrightarrow \mathbb{R}$$

are differentiable, and

$$\gamma'(t) = \varphi'(t) + \psi'(t)\,\mathrm{i}.$$

Proof. We have

$$\gamma'(t_0) = \lim_{t\to t_0} \frac{\gamma(t)-\gamma(t_0)}{t-t_0} = \lim_{t\to t_0} \frac{\varphi(t)-\varphi(t_0)}{t-t_0} + \lim_{t\to t_0} \frac{\psi(t)-\psi(t_0)}{t-t_0}\,\mathrm{i}.$$

10.4.5 Definition. Let $D \subseteq \mathbb{C}$. A *path of class* C^1 in D is a differentiable mapping

$$\gamma : [a, b] \longrightarrow D$$

with continuous derivative $\gamma' : [a, b] \longrightarrow \mathbb{C}$.

10.4.6 Examples.

a) For any $z \in \mathbb{C}$, theconstant mapping

$$\gamma : [0, 1] \longrightarrow \mathbb{C}, \qquad \gamma(t) = z$$

is a path of class C^1 in $\mathbb{C}$ (called a *constant path*).

b) For any complex numbers z_1 and z_2 the mapping

$$\gamma : [0, 1] \longrightarrow \mathbb{C}, \qquad \gamma(t) = (1 - t)\, z_1 + t\, z_2$$

is a path of class C^1 in $\mathbb{C}$ (called the *linear path* connecting z_1 with z_2).

c) For any $z_0 = x_0 + y_0\mathrm{i} \in \mathbb{C}$ and $r > 0$ teh mapping

$$\gamma : [0, 2\pi] \longrightarrow \mathbb{C}, \qquad \gamma(t) = z_0 + r\mathrm{e}^{\mathrm{i}t} = x_0 + r\cos t + (y_0 + r\sin t)\mathrm{i}$$

is a path of class C^1 in $\mathbb{C}$ (called the *circular path* of radius r and center z_0).

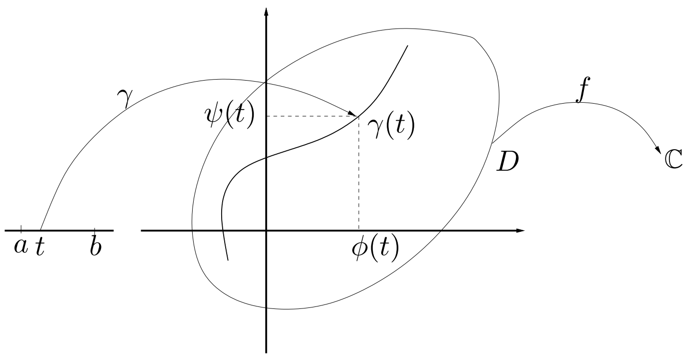

Figure 10.15: Complex line integral.

10.4.7 Definition. Let $f : D \to \mathbb{C}$ be a continuous function and $\gamma : [a, b] \longrightarrow D$ a path of class C^1 in D. The *complex line integral* of the function f along the path γ (see Figure 10.16) is the number

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\, \gamma'(t)\, dt.$$

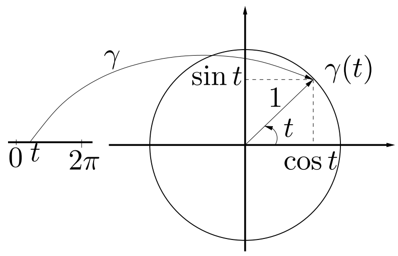

Figure 10.16: The path $\gamma(t) = e^{it} = \cos t + i \sin t$.

10.4.8 Exercise. Consider the function

$$f : \mathbb{C}^* \longrightarrow \mathbb{C}, \qquad f(z) = \frac{1}{z}$$

where $\mathbb{C}^* = \mathbb{C} - \{0\}$ and the path of class C^1

$$\gamma : [0, 2\pi] \longrightarrow \mathbb{C}^*, \qquad \gamma(t) = e^{it} = \cos t + i \sin t.$$

Compute

$$\int_\gamma f(z)dz.$$

Solution. Since $f(\gamma(t)) = \frac{1}{\gamma(t)} = e^{-it}$ and $\gamma'(t) = ie^{it}$ we get

$$\int_\gamma f(z)dz = \int_0^{2\pi} f(\gamma(t))\,\gamma'(t)\,dt = \int_0^{2\pi} e^{-it}\,i\,e^{it}dt = 2\pi i.$$

10.4.9 In the case of a constant path $\gamma(t) = z$ we have $\gamma'(t) = 0$ and hence

$$\int_\gamma f(z)\,dz = 0$$

for any function f.

10.4.10 If $f(x + yi) = u(x, y) + v(x, y)i$ and $\gamma(t) = \varphi(t) + \psi(t)\,i$ then

$$\int_\gamma f(z)dz = \int_a^b [u(\varphi(t), \psi(t))\,\varphi'(t) - v(\varphi(t), \psi(t))\,\psi'(t)]\,dt$$
$$+i \int_a^b [u(\varphi(t), \psi(t))\,\psi'(t) + v(\varphi(t), \psi(t))\,\varphi'(t)]\,dt.$$

10.4.11 Exercise. Compute

$$\int_\gamma \bar{z}\,dz$$

where γ is the linear path connecting $z_1 = 1$ with $z_2 = i$.

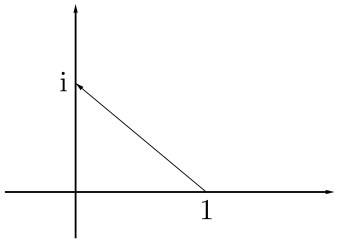

Figure 10.17: The linear path connecting 1 with i.

Solution. Since

$$\gamma : [0,1] \longrightarrow \mathbb{C}, \qquad \gamma(t) = (1-t)1 + t\mathrm{i}$$

we get the relations $f(\gamma(t)) = \overline{\gamma(t)} = 1 - t - t\mathrm{i}$ and $\gamma'(t) = -1 + \mathrm{i}$ whence

$$\int_\gamma \bar{z}\, dz = \int_0^1 (1-t-t\mathrm{i})(-1+\mathrm{i})dt = \int_0^1 (-1+2t)dt + \mathrm{i}\int_0^1 dt = \mathrm{i}.$$

10.4.12 MATHEMATICA: Integral along a closed polygonal path

```
In[1]:=Integrate[Conjugate[z], {z, 1, I}]           ↦  Out[1]=ⅈ
In[2]:=Integrate[1/z, {z, 1, I, -1, -I, 1}]         ↦  Out[2]=2ⅈπ
```

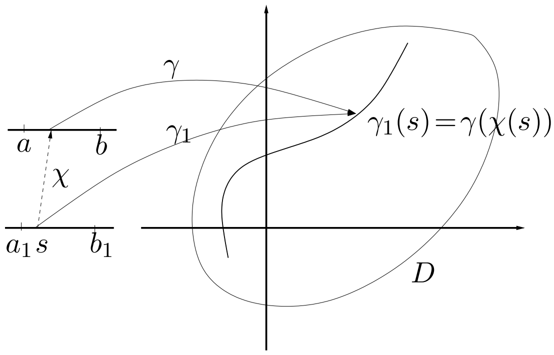

Figure 10.18: Equivalent paths.

10.4.13 Definition. Let $D \subseteq \mathbb{C}$ be a subset. Two paths of class C^1

$$\gamma : [a,b] \longrightarrow D \qquad \text{and} \qquad \gamma_1 : [a_1,b_1] \longrightarrow D$$

are *equivalent* if there exists a bijective, differentiable and increasing mapping

$$\chi : [a_1, b_1] \longrightarrow [a, b]$$

such that

$$\gamma_1(s) = \gamma(\chi(s)), \qquad \text{for any} \quad s \in [a_1, b_1].$$

10.4.14 The defined relation is an equivalence relation wich allows to divide the set of all the paths into equivalence classes. Each equivalence class represents a *curve*. The elements of the class are all the possible *parametrizations* of the considered curve.

10.4.15 Proposition. *If*

$$f : D \longrightarrow \mathbb{C}$$

is a continuous function and if the paths of class C^1

$$\gamma : [a, b] \longrightarrow D, \qquad \gamma_1 : [a_1, b_1] \longrightarrow D$$

are equivalent then

$$\int_\gamma f(z)\, dz = \int_{\gamma_1} f(z)\, dz$$

that is, the value of the integral depends on the curve, but not on the particular parametrization we choose.

Demonstrație. By using a change of variable we get

$$\int_{\gamma_1} f(z)\, dz = \int_{a_1}^{b_1} f(\gamma_1(s))\, \gamma_1'(s)\, ds$$
$$= \int_{a_1}^{b_1} f(\gamma(\chi(s)))\, \gamma'(\chi(s))\, \chi'(s)\, ds = \int_a^b f(\gamma(t))\, \gamma'(t)\, dt = \int_\gamma f(z)\, dz.$$

10.4.16 Any path

$$\gamma : [a, b] \longrightarrow D$$

is equivalent to a path defined on $[0, 1]$, namely,

$$\gamma_0 : [0, 1] \longrightarrow D, \qquad \gamma_0(t) = \gamma((1-t)a + tb).$$

10.4.17 Definition. Let $\gamma : [a, b] \longrightarrow D$ be a path of class C^1. The path

$$\tilde{\gamma} : [a, b] \longrightarrow D, \qquad \tilde{\gamma}(t) = \gamma(a + b - t)$$

is the *inverse paths* corresponding to γ.

10.4.18 Proposition. *If*

$$f : D \longrightarrow \mathbb{C}$$

is a continuous function and

$$\gamma : [a, b] \longrightarrow D$$

a path of class C^1 in D then

$$\int_{\tilde{\gamma}} f(z)\, dz = -\int_{\gamma} f(z)\, dz.$$

Proof. By using the change of variable $s = a + b - t$ we get

$$\begin{aligned}\textstyle\int_{\tilde{\gamma}} f(z)\, dz &= \textstyle\int_a^b f(\tilde{\gamma}(t))\, \tilde{\gamma}'(t)\, dt = -\int_a^b f(\gamma(a+b-t))\, \gamma'(a+b-t)\, dt \\ &= \textstyle\int_b^a f(\gamma(s))\, \gamma'(s)\, ds = -\int_{\gamma} f(z)\, dz.\end{aligned}$$

10.4.19 Definition. Let $D \subseteq \mathbb{C}$. A *path piecewise of class* C^1 in D is a continuous mapping

$$\gamma : [a, b] \longrightarrow D$$

such that there exists a subdivision $a = t_0 < t_1 < \ldots < t_n = b$ with:

1) the restrictions $\gamma|_{(t_{i-1}, t_i)}$ are differentiable and with continuous derivative

2) there exist and are finite the limits

$$\lim_{t \searrow a} \gamma'(t), \qquad \lim_{t \searrow t_j} \gamma'(t), \qquad \lim_{t \nearrow t_j} \gamma'(t), \qquad \lim_{t \searrow b} \gamma'(t)$$

for any $j \in \{1, 2, \ldots, n-1\}$.

10.4.20 The considered path is composed by the paths of class C^1

$$\gamma_1 : [t_0, t_1] \longrightarrow D, \qquad \gamma_1 = \gamma|_{[t_0, t_1]}$$

$$\gamma_2 : [t_1, t_2] \longrightarrow D, \qquad \gamma_2 = \gamma|_{[t_1, t_2]}$$

....

$$\gamma_n : [t_{n-1}, t_n] \longrightarrow D, \qquad \gamma_n = \gamma|_{[t_{n-1}, t_n]}$$

and for any continuous function

$$f : D \longrightarrow \mathbb{C}$$

we define the *complex line integral* of the function f along the path γ as

$$\int_{\gamma} f(z) dz = \sum_{j=1}^{n} \int_{\gamma_j} f(z) dz = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} f(\gamma(t))\, \gamma'(t)\, dt.$$

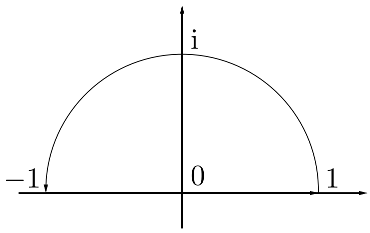

Figure 10.19: Path piecewise of class C^1.

We consider only paths piecewise of class C^1, and call them simply paths.

10.4.21 Example. The mapping (Figure 10.19)

$$\gamma : [0,2] \longrightarrow \mathbb{C}, \qquad \gamma(t) = \begin{cases} e^{\pi i t} & \text{dacă} \quad t \in [0,1] \\ 2t-3 & \text{dacă} \quad t \in (1,2] \end{cases}$$

is a path in $\mathbb{C}$, and for any continuous function

$$f : \mathbb{C} \longrightarrow \mathbb{C}$$

we have

$$\int_\gamma f(z)dz = \int_0^1 f(e^{\pi i t})\, \pi i e^{\pi i t} dt + \int_1^2 f(2t-3)\, 2dt.$$

10.4.22 Antiderivatives of some real functions of a real variable $f : I \to \mathbb{R}$
(I is an interval contained in the domain of derivability of the antiderivatives)

Function	Set of all the antiderivatives	Interval	Conditions
$f(x)=1$	$\int dx = x+\mathcal{C}$	$I\subseteq\mathbb{R}$	
$f(x)=x^n$	$\int x^n dx = \frac{1}{n+1}\,x^{n+1}+\mathcal{C}$	$I\subseteq\mathbb{R}$	$n\in\mathbb{N}$
$f(x)=x^\alpha$	$\int x^\alpha dx = \frac{1}{\alpha+1}\,x^{\alpha+1}+\mathcal{C}$	$I\subseteq(0,\infty)$	$\alpha\in\mathbb{R}-\{-1\}$
$f(x)=\frac{1}{x}$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert+\mathcal{C}$	$I\subseteq\mathbb{R}-\{0\}$	
$f(x)=\mathrm{e}^x$	$\int \mathrm{e}^x dx = \mathrm{e}^x+\mathcal{C}$	$I\subseteq\mathbb{R}$	
$f(x)=a^x$	$\int a^x dx = \frac{1}{\ln a}\,a^x+\mathcal{C}$	$I\subseteq\mathbb{R}$	$0<a\neq 1$
$f(x)=\sin x$	$\int \sin x\,dx = -\cos x+\mathcal{C}$	$I\subseteq\mathbb{R}$	
$f(x)=\cos x$	$\int \cos x\,dx = \sin x+\mathcal{C}$	$I\subseteq\mathbb{R}$	
$f(x)=\frac{1}{\cos^2 x}$	$\int \frac{1}{\cos^2 x}\,dx = \tan x+\mathcal{C}$	$I\subseteq\mathbb{R}-\left(\frac{\pi}{2}+\mathbb{Z}\pi\right)$	
$f(x)=\frac{1}{\sin^2 x}$	$\int \frac{1}{\sin^2 x}\,dx = -\cot x+\mathcal{C}$	$I\subseteq\mathbb{R}-\mathbb{Z}\pi$	
$f(x)=\frac{1}{\sqrt{a^2-x^2}}$	$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\frac{x}{a}+\mathcal{C}$	$I\subseteq(-a,a)$	$a\neq 0$
$f(x)=\frac{1}{\sqrt{x^2-a^2}}$	$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left\lvert x+\sqrt{x^2-a^2}\right\rvert+\mathcal{C}$	$I\subseteq\mathbb{R}-[-a,a]$	$a>0$
$f(x)=\frac{1}{\sqrt{x^2+a^2}}$	$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)+\mathcal{C}$	$I\subseteq\mathbb{R}$	$a\neq 0$
$f(x)=\frac{1}{a^2+x^2}$	$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\frac{x}{a}+\mathcal{C}$	$I\subseteq\mathbb{R}$	$a\neq 0$
$f(x)=\frac{1}{x^2-a^2}$	$\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert+\mathcal{C}$	$I\subseteq\mathbb{R}-\{\pm a\}$	$a\neq 0$
$f(x)=\mathrm{sh}\,x$	$\int \mathrm{sh}\,x\,dx = \mathrm{ch}\,x+\mathcal{C}$	$I\subseteq\mathbb{R}$	
$f(x)=\mathrm{ch}\,x$	$\int \mathrm{ch}\,x\,dx = \mathrm{sh}\,x+\mathcal{C}$	$I\subseteq\mathbb{R}$	

10.4.23 Definition. We say that the function defined on an open set D

$$f : D \longrightarrow \mathbb{C}$$

admits antiderivatives in D if there exists a holomorphic function

$$g : D \longrightarrow \mathbb{C}$$

such that

$$g'(z) = f(z), \qquad \text{for any} \quad z \in D.$$

10.4.24 Examples.

a) If $k\in\{0,1,2,\ldots\}$ then the function

$$f : \mathbb{C} \longrightarrow \mathbb{C}, \qquad f(z) = z^k = \underbrace{z\cdot z\cdots z}_{k \text{ ori}}$$

admits in $\mathbb{C}$ the antiderivative

$$g : \mathbb{C} \longrightarrow \mathbb{C}, \qquad g(z) = \frac{z^{k+1}}{k+1}$$

because

$$\left(\frac{z^{k+1}}{k+1}\right)' = z^k, \qquad \text{for any} \quad z\in\mathbb{C}.$$

b) If $k\in\{2,3,4,\ldots\}$ then the function

$$f:\mathbb{C}^*\longrightarrow\mathbb{C}, \qquad f(z)=z^{-k}=\frac{1}{z^k}$$

admits in $\mathbb{C}^*=\mathbb{C}-\{0\}$ the antiderivative

$$g:\mathbb{C}^*\longrightarrow\mathbb{C}, \qquad g(z)=\frac{z^{1-k}}{1-k}=-\frac{1}{(k-1)z^{k-1}}$$

because

$$\left(\frac{z^{1-k}}{1-k}\right)' = z^{-k}, \qquad \text{for any} \quad z\in\mathbb{C}^*.$$

c) The exponential function

$$f:\mathbb{C}\longrightarrow\mathbb{C}, \qquad f(z)=\mathrm{e}^z$$

admits in $\mathbb{C}$ the antiderivative

$$g:\mathbb{C}\longrightarrow\mathbb{C}, \qquad g(z)=\mathrm{e}^z$$

because

$$(\mathrm{e}^z)'=\mathrm{e}^z, \qquad \text{for any} \quad z\in\mathbb{C}.$$

d) The function

$$\cos:\mathbb{C}\longrightarrow\mathbb{C}, \qquad f(z)=\cos z$$

admits in $\mathbb{C}$ the antiderivative

$$g:\mathbb{C}\longrightarrow\mathbb{C}, \qquad g(z)=\sin z$$

because

$$(\sin z)'=\cos z, \qquad \text{for any} \quad z\in\mathbb{C}.$$

e) The function

$$\sin:\mathbb{C}\longrightarrow\mathbb{C}, \qquad f(z)=\sin z$$

admits in $\mathbb{C}$ the antiderivative

$$g:\mathbb{C}\longrightarrow\mathbb{C}, \qquad g(z)=-\cos z$$

because

$$(-\cos z)'=\sin z, \qquad \text{for any} \quad z\in\mathbb{C}.$$

10.4.25 Proposition. *If the continuous function*

$$f : D \longrightarrow \mathbb{C}$$

admits in D the antiderivative

$$g : D \longrightarrow \mathbb{C}$$

and if

$$\gamma : [a, b] \longrightarrow D$$

is a path contained in D then

$$\int_\gamma f(z)dz = g(z)|_{\gamma(a)}^{\gamma(b)} = g(\gamma(b)) - g(\gamma(a)).$$

Proof. By using the change of variable formula we get

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\, \gamma'(t)\, dt = \int_a^b g'(\gamma(t))\, \gamma'(t)\, dt$$

$$= \int_a^b \tfrac{d}{dt} g(\gamma(t))\, dt = g(\gamma(t))|_{t=a}^{t=b} = g(z)|_{z=\gamma(a)}^{z=\gamma(b)}.$$

10.4.26 If a continuous function

$$f : D \longrightarrow \mathbb{C}$$

admits an antiderivative in D, then the integral along a path

$$\gamma : [a, b] \longrightarrow D$$

contained in D depends only on the endpoints $\gamma(a)$ and $\gamma(b)$ of the path. If

$$\gamma : [a, b] \longrightarrow D \qquad \text{and} \qquad \gamma_1 : [a, b] \longrightarrow D$$

are two paths in D such that $\gamma(a) = \gamma_1(a)$ and $\gamma(b) = \gamma_1(b)$ then

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz.$$

10.4.27 Exercise. Compute the integrals

$$\int_\gamma z^3\, dz, \qquad \int_\gamma \frac{1}{z^2}\, dz, \qquad \int_\gamma \mathrm{e}^z\, dz, \qquad \int_\gamma (2z^3 + \frac{5}{z^2} - \mathrm{e}^z)\, dz$$

where γ is a path in $\mathbb{C}^*$ with the starting point 1 and end point i (Figure 10.20).

Solution. Let $\gamma : [a, b] \longrightarrow \mathbb{C}^*$ be a path with the starting point $z_1 = 1$ and end point $z_2 = \mathrm{i}$, that is, such that $\gamma(a) = 1$ and $\gamma(b) = \mathrm{i}$. We have

$$\int_\gamma z^3\, dz = \left.\frac{z^4}{4}\right|_{z=\gamma(a)}^{z=\gamma(b)} = \left.\frac{z^4}{4}\right|_{z=1}^{z=\mathrm{i}} = \frac{\mathrm{i}^4}{4} - \frac{1^4}{4} = 0,$$

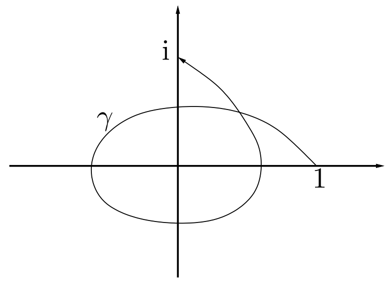

Figure 10.20: A paths γ with the starting point 1 and end point i.

$$\int_\gamma \frac{1}{z^2}\, dz = -\frac{1}{z}\Big|_{z=\gamma(a)}^{z=\gamma(b)} = -\frac{1}{z}\Big|_{z=1}^{z=\mathrm{i}} = -\frac{1}{\mathrm{i}}+1 = 1+\mathrm{i},$$

$$\int_\gamma \mathrm{e}^z\, dz = \mathrm{e}^z|_{z=\gamma(a)}^{z=\gamma(b)} = \mathrm{e}^z|_{z=1}^{z=\mathrm{i}} = \mathrm{e}^{\mathrm{i}} - \mathrm{e} = \cos 1 + \mathrm{i}\sin 1 - \mathrm{e},$$

$$\int_\gamma (2z^3 + \tfrac{5}{z^2} - \mathrm{e}^z)\, dz = 2\int_\gamma z^3\, dz + 5\int_\gamma \tfrac{1}{z^2}\, dz - \int_\gamma \mathrm{e}^z\, dz$$

$$= 5 + \mathrm{e} - \cos 1 + (5 - \sin 1)\mathrm{i}.$$

10.4.28 Definition. A path $\gamma : [a, b] \longrightarrow D$ is called a *closed path* if

$$\gamma(a) = \gamma(b)$$

that is, if the starting point $\gamma(a)$ and the end point $\gamma(b)$ coincide.

10.4.29 Proposition. *If the continuous function*

$$f : D \longrightarrow \mathbb{C}$$

admits in D an antiderivative

$$g : D \longrightarrow \mathbb{C}$$

and if

$$\gamma : [a, b] \longrightarrow D$$

is a closed path contained in D then

$$\int_\gamma f(z)dz = 0.$$

Proof. Since $\gamma(a) = \gamma(b)$ we have

$$\int_\gamma f(z)dz = g(z)|_{\gamma(a)}^{\gamma(b)} = g(\gamma(b)) - g(\gamma(a)) = 0.$$

10.4.30 Exercise. Consider the circular path

$$\gamma : [0, 2\pi] \longrightarrow \mathbb{C}, \qquad \gamma(t) = \mathrm{e}^{\mathrm{i}t} = \cos t + \mathrm{i} \sin t.$$

a) Prove that we have

$$\int_\gamma z^k \, dz = 0$$

for any $k \in \mathbb{Z} - \{-1\} = \{\ldots, -3, -2, 0, 1, 2, 3, \ldots\}$, but

$$\int_\gamma z^{-1} \, dz = \int_\gamma \frac{1}{z} \, dz = 2\pi\mathrm{i}.$$

b) Prove that

$$\int_\gamma \left(\frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 \, z + a_2 \, z^2 \right) dz = 2\pi\mathrm{i} a_{-1}$$

for any $a_{-2}, \; a_{-1}, \; a_0, \; a_1, \; a_2 \in \mathbb{C}$.

Solution. a) The path γ is contained in the open set $\mathbb{C}^* = \mathbb{C} - \{0\}$ and the function

$$f : \mathbb{C}^* \longrightarrow \mathbb{C}, \qquad f(z) = z^k$$

admits in $\mathbb{C}^*$ the antiderivative

$$g : \mathbb{C}^* \longrightarrow \mathbb{C}, \qquad g(z) = \frac{z^{k+1}}{k+1}$$

for any $k \in \mathbb{Z} - \{-1\}$.
b) By using the definition of the complex line integral we get

$$\int_\gamma \frac{1}{z} \, dz = \int_0^{2\pi} \frac{1}{\gamma(t)} \, \gamma'(t) \, dt = \int_0^{2\pi} \frac{1}{\mathrm{e}^{\mathrm{i}t}} \, \mathrm{i} \, \mathrm{e}^{\mathrm{i}t} \, dt = \mathrm{i} \int_0^{2\pi} dt = 2\pi\mathrm{i}.$$

10.4.31 From the previous exercice it follows that the holomorphic function

$$f : \mathbb{C}^* \longrightarrow \mathbb{C}, \qquad f(z) = \frac{1}{z}$$

does not admit any antiderivative in $\mathbb{C}^*$.

10.4.32 Exercise. Consider the circular path

$$\gamma : [0, 2\pi] \longrightarrow \mathbb{C}, \qquad \gamma(t) = z_0 + r\mathrm{e}^{\mathrm{i}t}$$

a) Prove that

$$\int_\gamma (z - z_0)^k \, dz = 0 \qquad \text{for any} \quad k \in \mathbb{Z} - \{-1\}$$

but

$$\int_\gamma (z - z_0)^{-1} \, dz = \int_\gamma \frac{1}{z - z_0} \, dz = 2\pi\mathrm{i}.$$

b) Prove that

$$\int_\gamma \left(\frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1\,(z-z_0) + a_2\,(z-z_0)^2 \right) dz = 2\pi \mathrm{i} a_{-1}$$

for any $a_{-2},\ a_{-1},\ a_0,\ a_1,\ a_2 \in \mathbb{C}$.

Solution. a) The path γ is contained in the open set $\mathbb{C} - \{z_0\}$ and the function

$$f : \mathbb{C}-\{z_0\} \longrightarrow \mathbb{C}, \qquad f(z) = (z-z_0)^k$$

admits in $\mathbb{C}^*$ the antiderivative

$$g : \mathbb{C}-\{z_0\} \longrightarrow \mathbb{C}, \qquad g(z) = \frac{(z-z_0)^{k+1}}{k+1}$$

for any $k \in \mathbb{Z}-\{-1\}$.

b) By using the definition of the complex line integral we get

$$\int_\gamma \frac{1}{z-z_0}\, dz = \int_0^{2\pi} \frac{1}{\gamma(t)-z_0}\, \gamma'(t)\, dt = \int_0^{2\pi} \frac{1}{r\mathrm{e}^{\mathrm{i}t}}\, \mathrm{i}\, r\, \mathrm{e}^{\mathrm{i}t}\, dt = \mathrm{i} \int_0^{2\pi} dt = 2\pi \mathrm{i}.$$

10.4.33 From the previous exercice it follows that the holomorphic function

$$f : \mathbb{C}-\{z_0\} \longrightarrow \mathbb{C}, \qquad f(z) = (z-z_0)^{-1} = \frac{1}{z-z_0}$$

does not admit any antiderivative in $\mathbb{C} - \{z_0\}$.

10.4.34 Definition. A *connected set* is a set which cannot be partitioned into two nonempty subsets such that each subset has no points in common with the set closure of the other.

A set D is called a *path-connected set* if for any two points z_1, z_2 from D there exists a path contained in D connecting z_1 and z_2.

10.4.35 Any path-connected set is a connected set.

An open connected set is called a *domain*.

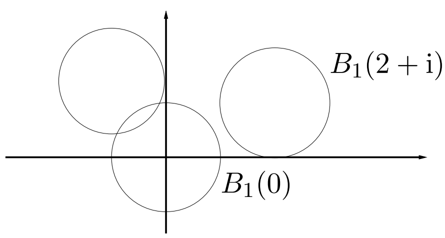

Figure 10.21: The disks $B_1(0)$, $B_1(2+\mathrm{i})$ and $B_1(-1+\mathrm{i}\sqrt{2})$.

10.4.36 Example. The set $B_1(0) \cup B_1(-1+\mathrm{i}\sqrt{2})$ is a domain, but $B_1(0) \cup B_1(2+\mathrm{i})$ is not a doain (Figure 10.21).

10.4.37 We know that any path $\gamma : [a,b] \longrightarrow D$ is equivalent with a path

$$[0,1] \longrightarrow D : \ t \mapsto \gamma((1-t)a+tb).$$

Without loss of generality, we can use only paths defined on the interval $[0,1]$.

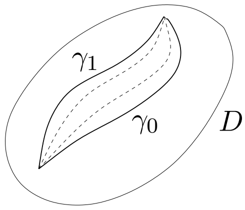

Figure 10.22: Paths homotopic in the domain D.

10.4.38 Definition. We say that the paths γ_0 and γ_1 from D having the same end points are *homotopic in* D if one can be continuously deformed into the other inside D, that is, if there exists a continuous mapping

$$h : [0,1] \times [0,1] \longrightarrow D : (s,t) \mapsto h(s,t)$$

satisfying the following conditions

a) $h(0,t) = \gamma_0(t)$, for any oricare ar fi $t \in [0,1]$,

b) $h(1,t) = \gamma_1(t)$, for any $t \in [0,1]$,

c) $h(s,0) = \gamma_0(0) = \gamma_1(0)$, for any $s \in [0,1]$,

d) $h(s,1) = \gamma_0(1) = \gamma_1(1)$, for any $s \in [0,1]$.

10.4.39 Example. The paths $\gamma_0, \ \gamma_1 : [0,1] \longrightarrow \mathbb{C}$,

$$\gamma_0(t) = \mathrm{e}^{2\pi \mathrm{i}t}, \qquad \gamma_1(t) = \frac{1}{2} + \frac{1}{2}\mathrm{e}^{2\pi \mathrm{i}t}$$

are homotopic in $D = \mathbb{C} - \bar{B}_{\frac{1}{4}}(\frac{1}{2})$. In this case we can choose (Figure 10.23)

$$h(s,t) = (1-s)\,\gamma_0(t) + s\,\gamma_1(t).$$

Generally, we shall 'decide' whether two paths are homotopic directly by using the corresponding figure.

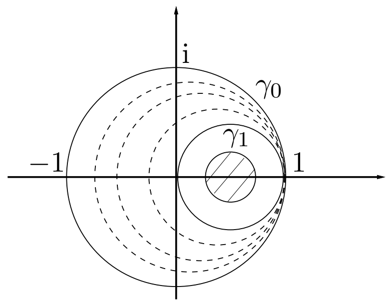

Figure 10.23: Homotopic paths.

10.4.40 Example. The circular path

$$\gamma_0 : [0,1] \longrightarrow \mathbb{C}, \qquad \gamma_0(t) = 3\mathrm{e}^{2\pi \mathrm{i} t} = 3\cos 2\pi t + 3\mathrm{i}\sin 2\pi t$$

is homotopic in $\mathbb{C}^*$ with the elliptic path

$$\gamma_1 : [0,1] \longrightarrow \mathbb{C}, \qquad \gamma_1(t) = 3\cos 2\pi t + \mathrm{i}\sin 2\pi t$$

but the two paths are not homotopic in $D = \mathbb{C} - \{2\mathrm{i}\}$ (Figure 10.24).

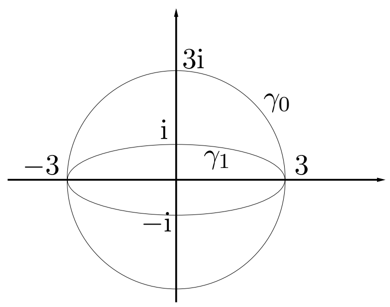

Figure 10.24: Circular path homotopic to an ellyptic one.

10.4.41 Example. The paths $\gamma_0,\ \gamma_1 : [0,1] \longrightarrow \mathbb{C}$,

$$\gamma_0(t) = 1 - 2t, \qquad \gamma_1(t) = \mathrm{e}^{\pi \mathrm{i} t}$$

are homotopic in $\mathbb{C}$, but they are not homotopic in $\mathbb{C} - \{\frac{1}{2}\mathrm{i}\}$ (Figure 10.25).

10.4.42 Definition. We say that the closed path

$$\gamma : [a,b] \longrightarrow \mathbb{C}$$

is *homotopic to zero in D* if it is homotopic in D with a constant path

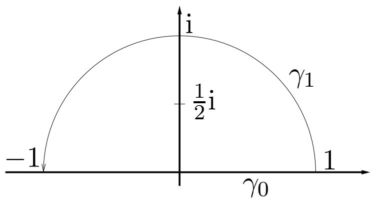

Figure 10.25: The paths $\gamma_0(t) = 1 - 2t$ and $\gamma_1(t) = \mathrm{e}^{\pi \mathrm{i} t}$.

$$[a, b] \longrightarrow D : \; t \mapsto \gamma(a).$$

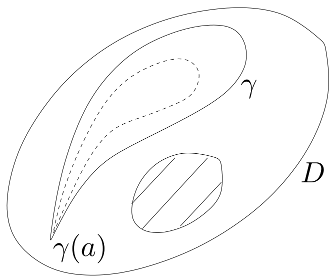

Figure 10.26: Path homotopic to zero in D.

10.4.43 Example. The circular path

$$\gamma : [0, 1] \longrightarrow \mathbb{C}, \qquad \gamma(t) = \mathrm{e}^{2\pi \mathrm{i} t}$$

is homotopic to zero in $D = \mathbb{C} - \{2\mathrm{i}\}$, but it is not homotopic to zero in $\mathbb{C}^*$.

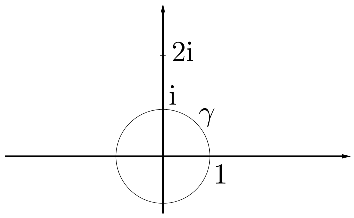

Figure 10.27: The path $\gamma : [0, 1] \longrightarrow \mathbb{C}$, $\gamma(t) = \mathrm{e}^{2\pi \mathrm{i} t}$.

10.4.44 Theorem (Cauchy) *If $D \subseteq \mathbb{C}$ is an open set,*

$$f : D \longrightarrow \mathbb{C}$$

is a holomorphic function and

$$\gamma : [a, b] \longrightarrow D$$

is a closed path homotopic to zero in D then

$$\int_\gamma f(z)\, dz = 0.$$

A proof can be found in [**?**].

10.4.45 Proposition. *If $D \subseteq \mathbb{C}$ is an open set,*

$$f : D \longrightarrow \mathbb{C}$$

is a holomorphic function and

$$\gamma_0 : [a, b] \longrightarrow D, \qquad \gamma_1 : [a, b] \longrightarrow D$$

are two paths homotopic in D then

$$\int_{\gamma_0} f(z)\, dz = \int_{\gamma_1} f(z)\, dz. \tag{10.1}$$

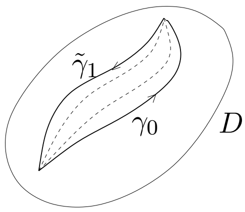

Figure 10.28: The paths γ_0 and $\tilde{\gamma}_1$ form a closed path.

Proof. The path obtained by composing γ_0 with the inverse $\tilde{\gamma}_1$ of γ_1 is a closed path homotopic to zero in D. By using Cauchy' theorem we get the relation

$$\int_{\gamma_0} f(z)\, dz + \int_{\tilde{\gamma}_1} f(z)\, dz = 0.$$

echivalentă cu (10.1).

10.4.46 Let k be a positive integer. The path

$$\gamma : [0, 1] \longrightarrow \mathbb{C}, \qquad \gamma(t) = z_0 + \mathrm{e}^{2k\pi \mathrm{i} t}$$

winds k times around z_0 in the direct sense (counterclockwise) and

$$\frac{1}{2\pi \mathrm{i}}\int_\gamma \frac{1}{z-z_0}\,dz = k.$$

The path

$$\gamma : [0,1] \longrightarrow \mathbb{C}, \qquad \gamma(t) = z_0 + \mathrm{e}^{-2k\pi i t}$$

winds k times around z_0 in the inverse sense (clockwise) and

$$\frac{1}{2\pi \mathrm{i}}\int_\gamma \frac{1}{z-z_0}\,dz = -k.$$

The path γ from Figure 10.29 is homotopic in $\mathbb{C}-\{z_0\}$ to the path

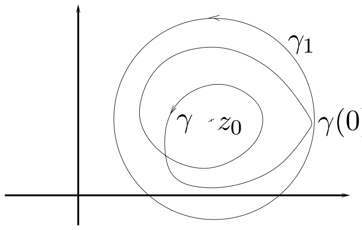

Figure 10.29: Drumul γ are indexul 2 faţă de z_0.

$$\gamma_1 : [0,1] \longrightarrow \mathbb{C}, \qquad \gamma_1(t) = z_0 + r\mathrm{e}^{4\pi i t}$$

and hence

$$\frac{1}{2\pi \mathrm{i}}\int_\gamma \frac{1}{z-z_0}\,dz = \frac{1}{2\pi \mathrm{i}}\int_{\gamma_1} \frac{1}{z-z_0}\,dz = 2.$$

Generally, if γ is a closed path not passing through z_0 the number

$$n(z_0,\gamma) = \frac{1}{2\pi \mathrm{i}}\int_\gamma \frac{1}{z-z_0}\,dz$$

called the *index* of z_0 with respect to γ, shows how many counterclockwise turns around z_0 the path γ makes. A proof can be found in [**?**].

10.4.47 A closed path γ determines a partition of the set of all the points not lying on γ into connected sets. All the points belonging to the same component of the partition have the same index with respect to γ (Figure 10.30).

10.4.48 Theorem (Cauchy's formula). *Any holomorphic function*

$$f : D \longrightarrow \mathbb{C}$$

defined on an open set D is indefinitely differentiable, and for any path

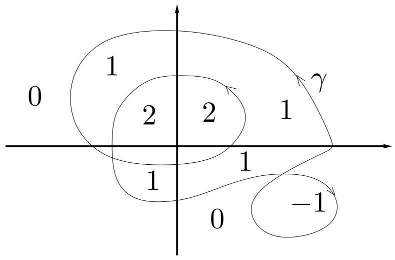

Figure 10.30: The index with respect to γ of the points not lying on γ.

$$\gamma : [0,1] \longrightarrow D$$

homotopic to zero in D we have

$$n(z,\gamma)\, f^{(k)}(z) = \frac{k!}{2\pi \mathrm{i}} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{k+1}}\, d\zeta$$

for any $k \in \mathbb{N}$ and any $z \in D - \{\ \gamma(t)\ |\ \ t \in [0,1]\ \}$.
A proof can be found in [**?**].

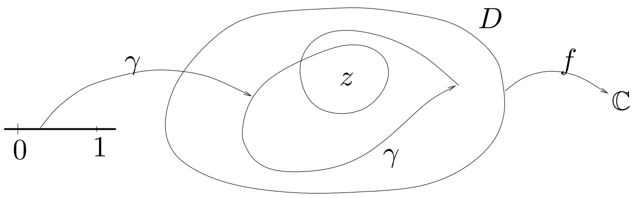

Figure 10.31: $f^{(k)}(z)$ satisfies Cauchy's formula.

10.5 Laurent series

10.5.1 Definition. Let $D \subseteq \mathbb{C}$ be a subset and

$$f_n : D \longrightarrow \mathbb{C}, \qquad n \in \mathbb{N}$$

be functions defined on D. The series of complex functions

$$\sum_{n=0}^{\infty} f_n$$

is said to be *convergent* (respectively, *uniformly convergent*) if the sequence of partial sums $(s_k)_{k\geq 0}$, where

$$s_k = \sum_{n=0}^{k} f_n$$

is convergent (respectively, uniformly convergent). The limit of this sequence

$$\sum_{n=0}^{\infty} f_n = \lim_{k\to\infty} s_k = \lim_{k\to\infty} \sum_{n=0}^{k} f_n = \lim_{k\to\infty} (f_0 + f_1 + \cdots f_k)$$

is the *series sum*. The considered series is said to be *absolutely convergent* if the series of real functions

$$\sum_{n=0}^{\infty} |f_n|$$

is convergent.

10.5.2 Proposition. *If $|z| < 1$ then the geometric series*

$$\sum_{n=0}^{\infty} z^n$$

is convergent and its sum is $\frac{1}{1-z}$, that is,

$$|z| < 1 \qquad \Longrightarrow \qquad \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.$$

Proof. If $|z| < 1$ then

$$\lim_{k\to\infty} \sum_{n=0}^{k} z^n = \lim_{k\to\infty} (1 + z + z^2 + \cdots + z^k) = \lim_{k\to\infty} \frac{1 - z^{k+1}}{1-z} = \frac{1}{1-z}.$$

10.5.3 Theorem (Weierstrass). *Let $D \subseteq \mathbb{C}$ be a subset and*

$$f_n : D \longrightarrow \mathbb{C}, \qquad n \in \mathbb{N}$$

be functions defined on D. If there exists a convergent series of real numbers

$$\sum_{n=0}^{\infty} \alpha_n$$

such that

$$|f_n(z)| \leq \alpha_n, \qquad for\ any\ \ z \in D,\ n \in \mathbb{N}$$

then the series of complex functions

$$\sum_{n=0}^{\infty} f_n$$

is absolutely and uniformly convergent.

10.5.4 Definition. A *power series* about z_0 (or centered at z_0) is a series of the form

$$\sum_{n=0}^{\infty} a_n\,(z-z_0)^n$$

where the coefficients a_0, a_1, a_2 ,... are complex numbers.
It can also be written in the form

$$a_0 + a_1\,(z-z_0) + a_2\,(z-z_0)^2 + \cdots \ .$$

10.5.5 Any power series is a series of functions

$$\sum_{n=0}^{\infty} f_n$$

with the functions f_n having the particular form

$$f_n : D \longrightarrow \mathbb{C}, \qquad f_n(z) = a_n\,(z-z_0)^n.$$

10.5.6 Definition. Let $D \subseteq \mathbb{C}$ be an open set and

$$f : D \longrightarrow \mathbb{C}, \qquad f_n : D \longrightarrow \mathbb{C}, \qquad n \in \mathbb{N}$$

be functions defined on D. We say that the sequence of functions $(f_n)_{n\geq 0}$ converges *uniformly on compact sets* to f if the sequence of the restrictions $f_n|_K$ converges uniformly to $f|_K$, for any compact subset $K \subset D$.

10.5.7 Theorem (Weierstrass). *Let $D \subseteq \mathbb{C}$ be an open set and*

$$f : D \longrightarrow \mathbb{C}, \qquad f_n : D \longrightarrow \mathbb{C}, \qquad n \in \mathbb{N}$$

be functions defined on D. If the functions f_n are holomorphic and $(f_n)_{n\geq 0}$ converges uniformly on compact sets to f then f is a holomorphic function and

$$\lim_{n\to\infty} f_n^{(k)} = f^{(k)}, \qquad for\ any \quad k \in \mathbb{N}.$$

A proof can be found in [**?**].

10.5.8 Theorem (Weierstrass). *If D is an open set and the series*

$$\sum_{n=0}^{\infty} f_n$$

of holomorphic functions converges uniformly on compact sets in D then its sum

$$S : D \longrightarrow \mathbb{C}, \qquad S(z) = \sum_{n=0}^{\infty} f_n(z)$$

is a holomorphic function and

$$S^{(k)} = \sum_{n=0}^{\infty} f_n^{(k)}, \qquad \textit{for any} \quad k \in \mathbb{N}.$$

Proof. The statement follows from the previous theorem.

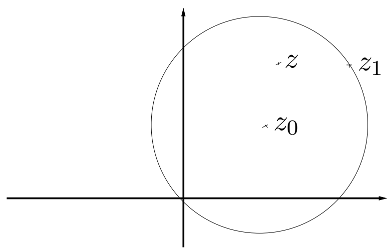

Figure 10.32: The disk of center z_0 and radius $|z_1 - z_0|$.

10.5.9 Theorem (Abel). *If the power series*

$$\sum_{n=0}^{\infty} a_n\, (z - z_0)^n$$

is convergent for $z = z_1 \neq z_0$ then it is convergent in the disk

$$\{\ z \mid |z - z_0| < |z_1 - z_0|\ \}$$

of center z_0 and radius $|z_1 - z_0|$.

Proof. The series $\sum_{n=0}^{\infty} a_n(z_1 - z_0)^n$ being convergent, we have

$$\lim_{n\to\infty} a_n(z_1 - z_0)^n = 0$$

and therefore there exists $n_0 \in \mathbb{N}$ such that

$$|a_n\, (z_1 - z_0)^n| < 1, \qquad \text{for any} \ \ n \geq n_0$$

that is,

$$|a_n| < \frac{1}{|z_1 - z_0|^n}, \qquad \text{for any} \;\; n \geq n_0.$$

From the relation

$$|a_n\,(z - z_0)^n| < \left(\frac{|z - z_0|}{|z_1 - z_0|}\right)^n, \qquad \text{for any} \;\; n \geq n_0$$

and from the convergence of the geometric series

$$\sum_{n=0}^{\infty} \left(\frac{|z - z_0|}{|z_1 - z_0|}\right)^n$$

for $|z - z_0| < |z_1 - z_0|$ it follows the convergence of the seris $\sum_{n=0}^{\infty} |a_n\,(z - z_0)^n|$. The normed space $(\mathbb{C}, |\;|)$ being complete, any absolutely convergent series is convergent.

10.5.10 Consider the power series

$$\sum_{n=0}^{\infty} a_n\,(z - z_0)^n.$$

If z is such that there exists

$$\lim_{n\to\infty} \sqrt[n]{|a_n(z - z_0)^n|} < 1$$

that is, such that

$$|z - z_0| < \frac{1}{\lim_{n\to\infty} \sqrt[n]{|a_n|}}$$

then, in view of the root test, the considered series is convergent.

10.5.11 Theorem (Cauchy-Hadamard). *In the case of a power series*

$$\sum_{n=0}^{\infty} a_n\,(z - z_0)^n$$

there exists

$$R = \begin{cases} 0 & \text{if} \;\; \overline{\lim}_{n\to\infty} \sqrt[n]{|a_n|} = \infty \\ \frac{1}{\overline{\lim}_{n\to\infty} \sqrt[n]{|a_n|}} & \text{if} \;\; \overline{\lim}_{n\to\infty} \sqrt[n]{|a_n|} \notin \{0, \infty\} \\ \infty & \text{if} \;\; \overline{\lim}_{n\to\infty} \sqrt[n]{|a_n|} = 0 \end{cases}$$

(called the radius of convergence*) such that:*

a) In the disk (called the convergence disk*)*

$$B_R(z_0) = \{\, z \mid \; |z - z_0| < R \,\}$$

the series converges absolutely and uniformly on compact sets.

b) In $\mathbb{C} - \bar{B}_R(z_0) = \{\, z \mid \ |z - z_0| > R \}$ *the series is divergent.*

c) The sum series

$$S : B_R(z_0) \longrightarrow \mathbb{C}, \qquad S(z) = \sum_{n=0}^{\infty} a_n \, (z - z_0)^n$$

is a holomorphic function.

d) The derivative series is a power series with the same radius of convergence and

$$S'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}, \qquad for\ any \quad k \in B_R(z_0).$$

A proof can be found in [**?**].

10.5.12 If there exists the limit

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$$

then

$$\overline{\lim}_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}.$$

10.5.13 Examples.

a) The radius of convergence of the geometric series

$$\sum_{n=0}^{\infty} z^n$$

is $R = 1$ because in this case $a_n = 1$, for any $n \in \mathbb{N}$.

b) The radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

is $R = \lim_{n\to\infty} \frac{1/n!}{1/(n+1)!} = \lim_{n\to\infty}(n+1) = \infty$.

10.5.14 By assuming that f is the sum of a power series centered at z_0

$$f(z) = \sum_{n=0}^{\infty} a_n \, (z - z_0)^n$$

with a non-null radius of convergence, from Cauchy-Hadamard theorem we get

$$f^{(k)}(z) = \sum_{n=0}^{\infty} [a_n \, (z - z_0)^n]^{(k)}, \qquad \text{for any} \quad k \in \mathbb{N}.$$

This relation leads to

$$a_k = \frac{f^{(k)}(z_0)}{k!}.$$

10.5.15 Theorem (Taylor series expansion). *If the function*

$$f : B_r(z_0) \longrightarrow \mathbb{C}$$

is holomorphic and R is the radius of convergence of the associated Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}\,(z-z_0)^n$$

then $R \geq r$ and

$$\begin{aligned} f(z) &= \textstyle\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}\,(z-z_0)^n \\ &= f(z_0) + \tfrac{f'(z_0)}{1!}\,(z-z_0) + \tfrac{f''(z_0)}{2!}\,(z-z_0)^2 + \cdots \end{aligned}$$

for any $z \in B_r(z_0)$.

A proof can be found in [**?**].

10.5.16 Examples. By using the previous theorem we get

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots \qquad \text{for} \quad |z| < 1$$

$$\mathrm{e}^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots \qquad \text{for any} \quad z \in \mathbb{C}$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots \qquad \text{for any} \quad z \in \mathbb{C}.$$

By differentiation or an adequate substitution we get other Taylor series expansions

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n = 1 - z + z^2 - \cdots \qquad \text{for} \quad |z| < 1$$

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} n z^{n-1} = 1 + 2z + 3z^2 + \cdots \qquad \text{for} \quad |z| < 1$$

$$\frac{1}{(1+z)^2} = \sum_{n=0}^{\infty} n(-1)^{n-1} z^{n-1} = 1 - 2z + 3z^2 - \cdots \qquad \text{for} \quad |z| < 1$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \qquad \text{for any} \quad z \in \mathbb{C}.$$

10.5.17 MATHEMATICA: Series[f[z], {z, z0, n}]

In[1]:=Series[1/(1−z), {z, 0, 5}] $\mapsto$ Out[1]=$1+z+z^2+z^3+z^4+z^5+O[z]^6$

In[2]:=Series[Exp[z], {z, 0, 6}] $\mapsto$ Out[2]=$1+z+\frac{z^2}{2}+\frac{z^3}{6}+\frac{z^4}{24}+\frac{z^5}{120}+\frac{z^6}{720}+O[z]^7$

In[3]:=Series[Exp[z], {z, 1, 3}] $\mapsto$ Out[3]=$\mathrm{e}+\mathrm{e}(z-1)+\frac{1}{2}\mathrm{e}(z-1)^2+\frac{1}{6}\mathrm{e}(z-1)^3+O[z-1]^4$

In[4]:=Series[Exp[z], {z, I, 3}] $\mapsto$ Out[4]=$\mathrm{e}^{\mathrm{i}}+\mathrm{e}^{\mathrm{i}}(z-\mathrm{i})+\frac{1}{2}\mathrm{e}^{\mathrm{i}}(z-\mathrm{i})^2+\frac{1}{6}\mathrm{e}^{\mathrm{i}}(z-\mathrm{i})^3+O[z-\mathrm{i}]^4$

In[5]:=Series[Cos[z], {z, 0, 6}] $\mapsto$ Out[5]=$1-\frac{z^2}{2}+\frac{z^4}{24}-\frac{z^6}{720}+O[z]^7$

10.5.18 Definition. A *Laurent series* centered at z_0 is a series of the form

$$\sum_{n=-\infty}^{\infty} a_n\,(z-z_0)^n$$

where a_n are complex numbers. It can also be written in the form

$$\cdots+\frac{a_{-2}}{(z-z_0)^2}+\frac{a_{-1}}{z-z_0}+a_0+a_1\,(z-z_0)+a_2\,(z-z_0)^2+\cdots\ .$$

10.5.19 Theorem (Annulus of convergence). *Consider a Laurent series*

$$\sum_{n=-\infty}^{\infty} a_n\,(z-z_0)^n$$

and define

$$r=\overline{\lim}_{n\to\infty}\sqrt[n]{|a_{-n}|}$$

$$R=\begin{cases} 0 & \textit{if}\quad \overline{\lim}_{n\to\infty}\sqrt[n]{|a_n|}=\infty\\ \frac{1}{\overline{\lim}_{n\to\infty}\sqrt[n]{|a_n|}} & \textit{if}\quad \overline{\lim}_{n\to\infty}\sqrt[n]{|a_n|}\not\in\{0,\infty\}\\ \infty & \textit{if}\quad \overline{\lim}_{n\to\infty}\sqrt[n]{|a_n|}=0. \end{cases}$$

If $r<R$ then:

a) In the circular annulus (called the annulus of convergence*)*

$$D=\{\ z\ |\ \ r<|z-z_0|<R\ \}$$

Laurent series converges absolutely and uniformly on compact sets.

b) Laurent series diverges in $\{\ z\ |\ \ |z-z_0|<r\ \}\cup\{\ z\ |\ \ |z-z_0|>R\ \}$.

c) The sum of the Laurent series $S:D\longrightarrow\mathbb{C}$,

$$S(z)=\sum_{n=-\infty}^{\infty} a_n\,(z-z_0)^n=\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}+\sum_{n=0}^{\infty} a_n(z-z_0)^n$$

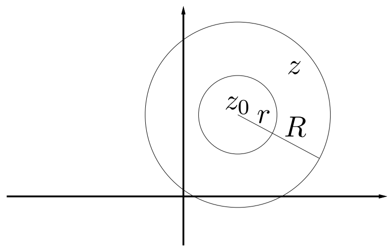

Figure 10.33: The circular annulus $\{\ z\ |\ \ r<|z-z_0|<R\ \}$.

is a holomorphic function.

A proof can be found in [**?**].

10.5.20 Theorem (Laurent series expansion). *If the function defined on the annulus*

$$f : D = \{\ z\ |\ \ r<|z-z_0|<R\ \} \longrightarrow \mathbb{C}$$

is holomorphic then there exists a unique Laurent series

$$\sum_{n=-\infty}^{\infty} a_n\,(z-z_0)^n$$

with the annulus of convergence including D and such that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n\,(z-z_0)^n \qquad for\ any \quad z\in D.$$

10.5.21 Examples.

a) The holomorphic function

$$f : D = \{\ z\ |\ \ 0<|z|<1\ \} \longrightarrow \mathbb{C}, \qquad f(z) = \frac{1}{z^2(1-z)}$$

admits in the annulus D the Laurent series expansion centered at 0

$$f(z)=\frac{1}{z^2}\frac{1}{1-z}=\frac{1}{z^2}(1+z+z^2+\cdots)=\frac{1}{z^2}+\frac{1}{z}+1+z+z^2+\cdots \tag{10.2}$$

b) The holomorphic function

$$f : D = \{\ z\ |\ \ 0<|z-\mathrm{i}|<\infty\ \} \longrightarrow \mathbb{C}, \qquad f(z) = \frac{\mathrm{e}^z}{(z-\mathrm{i})^2}$$

admits in the annulus D the Laurent series expansion centered at i

$$\begin{aligned} f(z) &= \tfrac{\mathrm{e}^z}{(z-\mathrm{i})^2} = \tfrac{\mathrm{e}^{\mathrm{i}}}{(z-\mathrm{i})^2}\,\mathrm{e}^{z-\mathrm{i}} = \tfrac{\mathrm{e}^{\mathrm{i}}}{(z-\mathrm{i})^2}\left(1+\tfrac{z-\mathrm{i}}{1!}+\tfrac{(z-\mathrm{i})^2}{2!}+\cdots\right) \\ &= \tfrac{\mathrm{e}^{\mathrm{i}}}{(z-\mathrm{i})^2}+\tfrac{\mathrm{e}^{\mathrm{i}}}{z-\mathrm{i}}+\tfrac{\mathrm{e}^{\mathrm{i}}}{2!}+\tfrac{\mathrm{e}^{\mathrm{i}}}{3!}(z-\mathrm{i})+\cdots \end{aligned} \tag{10.3}$$

c) The holomorphic function

$$f : D = \{\ z \mid \ 0 < |z| < \infty\ \} \longrightarrow \mathbb{C}, \qquad f(z) = z^2\, \mathrm{e}^{\frac{1}{z}}$$

admits in the annulus D the Laurent series expansion centered at 0

$$\begin{aligned} f(z) &= z^2\, \mathrm{e}^{\frac{1}{z}} = z^2 \left(1 + \tfrac{1}{1!}\,\tfrac{1}{z} + \tfrac{1}{2!}\,\tfrac{1}{z^2} + \cdots\right) \\ &= \cdots + \tfrac{1}{4!}\,\tfrac{1}{z^2} + \tfrac{1}{3!}\,\tfrac{1}{z} + \tfrac{1}{2!} + \tfrac{1}{1!}\,z + z^2 + 0\,z^3 + 0\,z^4 + \cdots \end{aligned} \tag{10.4}$$

10.5.22 Definition. Let $f : D \longrightarrow \mathbb{C}$ be a holomorphic function defined on an open set D. We say that the point $z_0 \in \mathbb{C} - D$ is an *isolated singular point* of the function f if there exists $r > 0$ such that the annulus $\{\ z \mid \ 0 < |z - z_0| < r\,\}$ is included into D. The coefficient a_{-1} from the Laurent expansion

$$f(z) = \cdots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{z - z_0} + a_0 + a_1\,(z - z_0) + a_2\,(z - z_0)^2 + \cdots$$

of f in this annulus is called the *residue* of f at the isolated singular point z_0 and is denoted by $\mathbf{Rez}_{z_0} f$, that is,

$$\mathbf{Rez}_{z_0} f = a_{-1}.$$

10.5.23 Examples.

a) The only isolated singular point of the function

$$f : D = \{\ z \mid \ 0 < |z| < 1\ \} \longrightarrow \mathbb{C}, \qquad f(z) = \frac{1}{z^2(1 - z)}$$

is $z = 0$ and from (10.2) we get $\mathbf{Rez}_0 = 1$.

b) The only isolated singular point of the function

$$f : D = \{\ z \mid \ 0 < |z - \mathrm{i}| < \infty\ \} \longrightarrow \mathbb{C}, \qquad f(z) = \frac{\mathrm{e}^z}{(z - \mathrm{i})^2}$$

is $z = i$ and from (10.3) we get $\mathbf{Rez}_i f = \mathrm{e}^{\mathrm{i}}$.

c) The only isolated singular point of the function

$$f : D = \{\ z \mid \ 0 < |z| < \infty\ \} \longrightarrow \mathbb{C}, \qquad f(z) = z^2\, \mathrm{e}^{\frac{1}{z}}$$

is $z = 0$ and from (10.4) we get $\mathbf{Rez}_0 f = \frac{1}{3!} = \frac{1}{6}$.

10.5.24 MATHEMATICA: `Series[f[z], {z, a, n}]` , `Residue[f[z], {z, a}]`

`In[1]:=Series[1/(z^2(1-z)), {z, 0, 4}]` $\mapsto$ Out[1]$=\frac{1}{z^2}+\frac{1}{z}+1+z+z^2+z^3+z^4+O[z]^5$

`In[2]:=Residue[1/(z^2(1-z)), {z, 0}]` $\mapsto$ Out[2]$=1$

`In[3]:=Series[1/(z^2(1-z)), {z, 1, 2}]` $\mapsto$ Out[3]$=-\frac{1}{z-1}+2-3(z-1)+4(z-1)^2+O[z]^3$

`In[4]:=Residue[1/(z^2(1-z)), {z, 1}]` $\mapsto$ Out[4]$=-1$

`In[5]:=Series[Exp[z]/(z-I)^2, {z, I, 1}]` $\mapsto$ Out[5]$=\frac{e^{i}}{(z-i)^2}+\frac{e^{i}}{z-i}+\frac{e^{i}}{2}+\frac{1}{6}e^{i}(z-i)+O[z-i]^2$

`In[6]:=Residue[Exp[z]/(z-I)^2, {z, I}]` $\mapsto$ Out[6]$=e^{i}$.

10.5.25 Definition. Let D be an open set and

$$f : D \longrightarrow \mathbb{C}$$

a holomorphic function. A *zero of multiplicity* n of f is a point $z_0 \in D$ satisfying the following conditions

$$f(z_0) = f'(z_0) = \ldots = f^{(n-1)}(z_0) = 0 \qquad \text{and} \qquad f^{(n)}(z_0) \neq 0.$$

An isolated singular point z_0 of f is called a *pole of order* n if it is a zero of multiplicity n for the function $\frac{1}{f}$.
A pole of order 1 is called a *simple pole.*

10.5.26 Theorem. *If the isolated singular point z_0 of the holomorphic function $f : D \longrightarrow \mathbb{C}$ is a pole of order n then there is $r > 0$ such that the annulus*

$$\{\ z \mid 0 < |z - z_0| < r\,\}$$

is contained in D and f admits in this annulus a Laurent expansion of the form

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \cdots + \frac{a_{-1}}{(z - z_0)} + a_0 + a_1\,(z - z_0) + a_2\,(z - z_0)^2 + \cdots$$

10.5.27 a) A function f with a first order pole z_0 admits around z_0 the expansion

$$f(z) = \frac{a_{-1}}{(z - z_0)} + a_0 + a_1\,(z - z_0) + a_2\,(z - z_0)^2 + \cdots$$

By multiplying with $(z - z_0)$ we get the relation

$$(z - z_0)\, f(z) = a_{-1} + a_0\,(z - z_0) + a_1\,(z - z_0)^2 + a_2\,(z - z_0)^3 + \cdots$$

leading to

$$\mathbf{Rez}_{z_0} f = a_{-1} = \lim_{z \to z_0} (z - z_0)\, f(z).$$

b) A function f with a second order pole z_0 admits locally around z_0 the expansion

$$f(z) = \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{(z - z_0)} + a_0 + a_1\,(z - z_0) + a_2\,(z - z_0)^2 + \cdots$$

By multiplying with $(z-z_0)^2$ and then differentiating the obtained relation we get

$$[(z-z_0)^2\, f(z)]' = a_{-1} + 2a_0\,(z-z_0) + 3a_1\,(z-z_0)^2 + \cdots$$

whence

$$\mathbf{Rez}_{z_0} f = a_{-1} = \lim_{z\to z_0} [(z-z_0)^2\, f(z)]'.$$

c) A function f with a third order pole z_0 admits locally around z_0 the expansion

$$f(z) = \frac{a_{-3}}{(z-z_0)^3} + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1\,(z-z_0) + \cdots$$

By multiplying with $(z-z_0)^3$ and then differentiating twice we get the relation

$$[(z-z_0)^3\, f(z)]'' = 2!\, a_{-1} + 6a_0\,(z-z_0) + 12a_1\,(z-z_0)^2 + \cdots$$

leading to

$$\mathbf{Rez}_{z_0} f = a_{-1} = \frac{1}{2!} \lim_{z\to z_0} [(z-z_0)^3\, f(z)]''.$$

d) If z_0 is a pole of order n then

$$\mathbf{Rez}_{z_0} f = \frac{1}{(n-1)!} \lim_{z\to z_0} [(z-z_0)^n\, f(z)]^{(n-1)}.$$

10.5.28 Example. The function

$$f : \mathbb{C}-\{0,\, 1\} \longrightarrow \mathbb{C}, \qquad f(z) = \frac{1}{z^2(1-z)}$$

has two isolated singular points 0 and 1. The point 0 is a second order pole and

$$\mathbf{Rez}_0 f = \lim_{z\to 0}[z^2\, f(z)]' = \lim_{z\to 0}\left[\frac{1}{1-z}\right]' = \lim_{z\to 0}\frac{1}{(1-z)^2} = 1. \tag{10.5}$$

The point $z=1$ is a first order pole and

$$\mathbf{Rez}_1 f = \lim_{z\to 1}(z-1)\, f(z) = \lim_{z\to 1}\frac{-1}{z^2} = -1. \tag{10.6}$$

10.6 The residue theorem and some applications

10.6.1 If

$$\gamma : [a,b] \longrightarrow \mathbb{C}-\{z_0\}$$

is a closed path not passing through z_0 then

$$\begin{aligned}&\int_\gamma \left(\tfrac{a_{-2}}{(z-z_0)^2}+\tfrac{a_{-1}}{(z-z_0)}+a_0+a_1\,(z-z_0)+a_2\,(z-z_0)^2\right)dz\\ &\qquad\qquad = a_{-1}\int_\gamma \tfrac{dz}{z-z_0}=2\pi \mathrm{i} a_{-1}\,n(z_0,\gamma)\end{aligned} \tag{10.7}$$

for any $a_{-2},\ a_{-1},\ a_0,\ a_1,\ a_2\in\mathbb{C}$. The point z_0 is an isolated singular point (second order pole) for the function $f:\mathbb{C}-\{z_0\}\longrightarrow\mathbb{C}$,

$$f(z)=\frac{a_{-2}}{(z-z_0)^2}+\frac{a_{-1}}{(z-z_0)}+a_0+a_1\,(z-z_0)+a_2\,(z-z_0)^2$$

and $\mathrm{Rez}_{z_0}f=a_{-1}$. The relation (10.7) can be written as

$$\int_\gamma f(z)\,dz=2\pi\mathrm{i}\ n(z_0,\gamma)\ \mathbf{Rez}_{z_0}f.$$

10.6.2 Theorem (Residue Theorem). *If $D\subseteq\mathbb{C}$ is an open set,*

$$f:D\longrightarrow\mathbb{C}$$

is a holomorphic function, S is the set of all the singular isolated points of f and if

$$\gamma:[a,b]\longrightarrow D$$

is a path homotopic to zero in $\tilde{D}=D\cup S$ then

$$\int_\gamma f(z)\,dz=2\pi\mathrm{i}\sum_{z\in S}n(z,\gamma)\,\mathbf{Rez}_z f.$$

A proof can be found in [**?**].

10.6.3 Exercise. Compute the integral

$$\int_\gamma \frac{4\,dz}{(z^2+1)(z-3)^2}$$

where

$$\gamma:[0,1]\longrightarrow\mathbb{C},\qquad \gamma(t)=2\,\mathrm{e}^{2\pi\mathrm{i}t}.$$

Solution. Consider the set $D=\mathbb{C}-\{3,\ \mathrm{i},\ -\mathrm{i}\}$ and the holomorphic function

$$f:D\longrightarrow\mathbb{C},\qquad f(z)=\frac{4}{(z^2+1)(z-3)^2}.$$

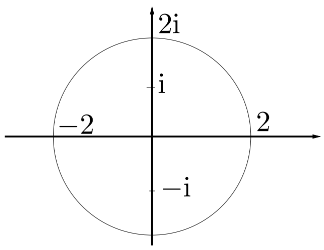

Figure 10.34: The path $\gamma : [0,1] \longrightarrow \mathbb{C}$, $\gamma(t) = 2\,\mathrm{e}^{2\pi\mathrm{i}t}$.

The set of all the isolated singular points of f is $S = \{3,\ \mathrm{i},\ -\mathrm{i}\}$ and the path γ is homotopic to zero in $D \cup S = \mathbb{C}$. In view of the residue theorem we have

$$\int_\gamma \frac{4\,dz}{(z^2+1)(z-3)^2} = 2\pi\mathrm{i}\,(n(3,\gamma)\,\mathbf{Rez}_3 f + n(\mathrm{i},\gamma)\,\mathbf{Rez}_\mathrm{i} f + n(-\mathrm{i},\gamma)\,\mathbf{Rez}_{-\mathrm{i}} f)\,.$$

Since the path γ winds zero times around 3 and once around i and $-$i we get

$$n(3,\gamma) = 0, \qquad n(\mathrm{i},\gamma) = n(-\mathrm{i},\gamma) = 1$$

whence (Figure 10.34)

$$\int_\gamma \frac{4\,dz}{(z^2+1)(z-3)^2} = 2\pi\mathrm{i}\,(\mathbf{Rez}_\mathrm{i} f + \mathbf{Rez}_{-\mathrm{i}} f)\,.$$

The singular points i and $-$i being simple poles we have

$$\mathbf{Rez}_\mathrm{i} f = \lim_{z\to\mathrm{i}}(z-\mathrm{i})f(z) = \lim_{z\to\mathrm{i}} \frac{4}{(z-3)^2(z+\mathrm{i})} = \frac{4}{2\mathrm{i}(\mathrm{i}-3)^2} = \frac{3}{25} - \frac{4}{25}\mathrm{i}$$

$$\mathbf{Rez}_{-\mathrm{i}} f = \lim_{z\to-\mathrm{i}}(z+\mathrm{i})f(z) = \lim_{z\to-\mathrm{i}} \frac{4}{(z-3)^2(z-\mathrm{i})} = \frac{4}{-2\mathrm{i}(\mathrm{i}+3)^2} = \frac{3}{25} + \frac{4}{25}\mathrm{i}$$

and

$$\int_\gamma \frac{4\,dz}{(z^2+1)(z-3)^2} = \frac{12}{25}\pi\mathrm{i}.$$

10.6.4 MATHEMATICA: `Residue[f[z], {z, a}]`

`In[1]:=Residue[4/((z^2+1)(z-3)^2), {z, I}]` $\mapsto$ Out[1]=$\frac{3}{25}-\frac{4\mathrm{i}}{25}$

`In[2]:=Residue[4/((z^2+1)(z-3)^2), {z, -I}]` $\mapsto$ Out[2]=$\frac{3}{25}+\frac{4\mathrm{i}}{25}$

10.6.5 Exerciţiu. Compute the integral

$$\int_\gamma \frac{\mathrm{e}^z}{z^3}\,dz$$

where

$$\gamma : [0,1] \longrightarrow \mathbb{C}, \qquad \gamma(t) = \mathrm{e}^{-4\pi\mathrm{i}t}.$$

Solution. Consider the holomorphic function

$$f : \mathbb{C}^* \longrightarrow \mathbb{C}, \qquad f(z) = \frac{\mathrm{e}^z}{z^3}$$

defined on the open set $\mathbb{C}^* = \mathbb{C} - \{0\}$. The singular point $z = 0$ is a third order pole. In order to compute the residue at 0 of f we can use the Laurent expansion around 0

$$f(z) = \tfrac{\mathrm{e}^z}{z^3} = \tfrac{1}{z^3}\left(1 + \tfrac{z}{1!} + \tfrac{z^2}{2!} + \tfrac{z^3}{3!} + \cdots\right)$$

$$= \tfrac{1}{z^3} + \tfrac{1}{1!}\tfrac{1}{z^2} + \tfrac{1}{2!}\tfrac{1}{z} + \tfrac{1}{3!} + \tfrac{1}{4!}z + \cdots$$

or the relation

$$\mathbf{Rez}_0 f = \frac{1}{2!}\lim_{z\to 0}(z^3\, f(z))'' = \frac{1}{2}.$$

By remarking that γ winds twice around 0 clockwise or by using the formula

$$n(0,\gamma) = \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{dz}{z} = -2$$

we get

$$\int_\gamma \frac{\mathrm{e}^z}{z^3}\, dz = 2\pi\mathrm{i}\, n(0,\gamma)\, \mathbf{Rez}_0 f = -2\pi\mathrm{i}.$$

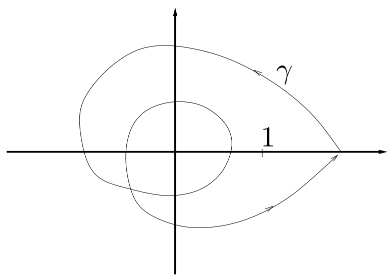

Figure 10.35: The path γ.

10.6.6 Exercise. Compute the integral

$$\int_\gamma \frac{1}{z^2(1-z)}\, dz$$

where γ is the path shown in Figure 10.35 .

Rezolvare. The singular points of the holomorphic function

$$f : \mathbb{C} - \{0,1\} \longrightarrow \mathbb{C}, \qquad f(z) = \frac{1}{z^2(1-z)}$$

are $z = 0$ şi $z = 1$. We know that $\mathbf{Rez}_0 f = 1$ (see (10.5)) and $\mathbf{Rez}_1 f = -1$ (see (10.6)). By remarking that γ winds twice about 0 and once about 1, in view of the residue theorem we have

$$\int_\gamma \frac{1}{z^2(1-z)}\,dz = 2\pi\mathrm{i}\,(2\,\mathbf{Rez}_0 f + \mathbf{Rez}_1 f) = 2\pi\mathrm{i}.$$

10.6.7 Exercise. Compute the integral

$$I = \int_0^{2\pi} \frac{1}{a+\cos t}\,dt \qquad \text{where} \quad a \in (1,\infty)$$

Solution. The integral can be regarded as a complex line integral and computed by using the residue theorem. We have

$$\begin{aligned} I &= \int_0^{2\pi} \frac{1}{a+\frac{\mathrm{e}^{\mathrm{i}t}+\mathrm{e}^{-\mathrm{i}t}}{2}}dt = \int_0^{2\pi} \frac{1}{\mathrm{i}\mathrm{e}^{\mathrm{i}t}} \frac{2}{2a+\mathrm{e}^{\mathrm{i}t}+\mathrm{e}^{-\mathrm{i}t}} (\mathrm{e}^{\mathrm{i}t})'\,dt \\ &= -\mathrm{i}\int_\gamma \frac{1}{z}\frac{2}{2a+z+\frac{1}{z}}\,dz = -\mathrm{i}\int_\gamma \frac{2}{z^2+2az+1}dz \end{aligned}$$

where $\gamma : [0,2\pi] \longrightarrow \mathbb{C}$, $\gamma(t) = \mathrm{e}^{\mathrm{i}t}$. The function

$$f : \mathbb{C}-\{z_1,z_2\} \longrightarrow \mathbb{C}, \qquad f(z) = \frac{2}{z^2+2az+1}$$

where

$$z_1 = -a+\sqrt{a^2-1}, \qquad z_2 = -a-\sqrt{a^2-1}$$

are the roots of the polynomial $z^2+2az+1$, has two isolated singular points (first order poles) z_1 and z_2.

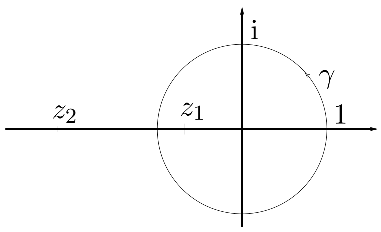

Figure 10.36: The path $\gamma : [0,2\pi] \longrightarrow \mathbb{C}$, $\gamma(t) = \mathrm{e}^{\mathrm{i}t}$.

Since z_1, z_2 are real numbers, $-1 < z_1 < 0$ and $z_2 < -1$ we get $n(z_1,\gamma) = 1$ and $n(z_2,\gamma) = 0$ (Figure 10.36). In view of the residue theorem

$$\begin{aligned} I &= -\mathrm{i}\int_\gamma \frac{2}{z^2+2az+1}dz = 2\pi\mathbf{Rez}_{z_1} f = 2\pi \lim_{z\to z_1}(z-z_1)f(z) \\ &= 2\pi \lim_{z\to z_1}(z-z_1)\frac{2}{(z-z_1)(z-z_2)} = \frac{4\pi}{z_1-z_2} = \frac{2\pi}{\sqrt{a^2-1}}. \end{aligned}$$

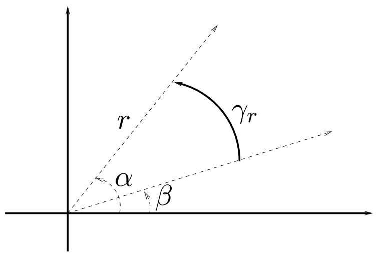

Figure 10.37: The paths γ_r.

10.6.8 Proposition. *Consider $\alpha<\beta$ and a continuous function*

$$f:D\longrightarrow\mathbb{C}$$

defined on a domaine D containing the images of the paths (Figure 10.37)

$$\gamma_r:[\alpha,\beta]\longrightarrow\mathbb{C},\qquad \gamma_r(t)=r\mathrm{e}^{\mathrm{i}t}$$

for any $r>0$. If

$$\lim_{z\to\infty} z\,f(z)=0$$

then

$$\lim_{r\to\infty}\int_{\gamma_r} f(z)\,dz=0.$$

Proof. For any $\varepsilon>0$ there exists $r_\varepsilon>0$ such that

$$|z|>r_\varepsilon\qquad\Longrightarrow\qquad |z\,f(z)|<\varepsilon.$$

Particularly, for $r>r_\varepsilon$ we have

$$\left|\int_{\gamma_r} f(z)\,dz\right|=\left|\int_\alpha^\beta f(r\mathrm{e}^{\mathrm{i}t})\,r\mathrm{i}\,\mathrm{e}^{\mathrm{i}t}\,dt\right|\leq\int_\alpha^\beta |f(r\mathrm{e}^{\mathrm{i}t})\,r\mathrm{i}\,\mathrm{e}^{\mathrm{i}t}|\,dt<\varepsilon\int_\alpha^\beta dt=(\beta-\alpha)\varepsilon.$$

10.6.9 From the relations

$$|z_1|=|z_1-z_2+z_2|\leq|z_1-z_2|+|z_2|,$$
$$|z_2|=|z_2-z_1+z_1|\leq|z_1-z_2|+|z_1|$$

we get

$$-\,|z_1-z_2|\leq|z_1|-|z_2|\leq|z_1-z_2|$$

that is,

$$|\;|z_1|-|z_2|\;|\leq|z_1-z_2|.$$

10.6.10 Exercise. Compute the integral

$$I=\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}dx$$

Solution. The integral is a real improper integral. The interval of integration is unbounded, but the considered function is bounded. Since

$$\lim_{x\to\infty}\frac{\frac{x^2}{(x^2+1)(x^2+4)}}{\frac{1}{x^2}}=1$$

the integrals

$$\int_1^\infty \frac{x^2}{(x^2+1)(x^2+4)}dx \qquad \text{and} \qquad \int_1^\infty \frac{1}{x^2}dx$$

are either both convergent or both divergent. We know that the improper integral

$$\int_1^\infty \frac{1}{x^\lambda}dx$$

is convergent for $\lambda>1$. Therefore, the considered integral

$$I=\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}dx=\int_0^1 \frac{x^2}{(x^2+1)(x^2+4)}dx+\int_1^\infty \frac{x^2}{(x^2+1)(x^2+4)}\,dx$$

is convergent.

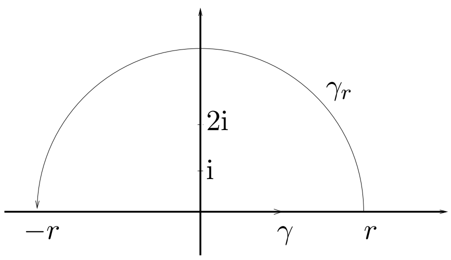

Figure 10.38: The paths γ_r.

In order to compute the value of the integral we consider the holomorphic function

$$f:\mathbb{C}-\{-2\mathrm{i},\,-\mathrm{i},\,\mathrm{i},\,2\mathrm{i}\}\longrightarrow\mathbb{C},\qquad f(z)=\frac{z^2}{(z^2+1)(z^2+4)}$$

and the path of integration shown in Figure 10.38 composed by

$$\gamma_r:[0,\pi]\longrightarrow\mathbb{C},\qquad \gamma_r(t)=r\,\mathrm{e}^{\mathrm{i}t}$$

and

$$\gamma:[-r,r]\longrightarrow\mathbb{C},\qquad \gamma(t)=t.$$

In view of the residue theorem, for any $r>2$ we have the relation

$$\int_{\gamma_r} f(z)dz+\int_{-r}^{r} f(x)dx=2\pi\mathrm{i}\,(\mathbf{Rez}_{\mathrm{i}}f+\mathbf{Rez}_{2\mathrm{i}}f)$$

leading to

$$\lim_{r\to\infty}\int_{\gamma_r} f(z)dz+\int_{-\infty}^{\infty} f(x)dx=2\pi\mathrm{i}\,(\mathbf{Rez}_{\mathrm{i}}f+\mathbf{Rez}_{2\mathrm{i}}f). \qquad (10.8)$$

Since

$$|z\,f(z)|=\frac{|z^3|}{|z^2+1|\cdot|z^2+4|}=\frac{|z^3|}{|z^2-(-1)|\cdot|z^2-(-4)|}\leq\frac{|z|^3}{|\,|z|^2-1|\cdot|\,|z|^2-4|}$$

we have

$$\lim_{z\to\infty} z\,f(z)=0$$

and in view of the result presented at pag. 56-**8**

$$\lim_{r\to\infty}\int_{\gamma_r} f(z)dz=0.$$

Since $f(-x)=f(x)$, from the relation (10.8) we obtain

$$\int_0^{\infty} f(x)dx=\pi\mathrm{i}\,(\mathbf{Rez}_{\mathrm{i}}f+\mathbf{Rez}_{2\mathrm{i}}f).$$

But

$$\mathbf{Rez}_{\mathrm{i}}=\lim_{z\to\mathrm{i}}(z-\mathrm{i})\,f(z)=\lim_{z\to\mathrm{i}}\frac{z^2}{(z+\mathrm{i})(z^2+4)}=\frac{\mathrm{i}}{6}$$

$$\mathbf{Rez}_{2\mathrm{i}}=\lim_{z\to2\mathrm{i}}(z-2\mathrm{i})\,f(z)=\lim_{z\to2\mathrm{i}}\frac{z^2}{(z^2+1)(z+2\mathrm{i})}=-\frac{\mathrm{i}}{3}$$

and hence

$$\int_0^{\infty} f(x)dx=\pi\mathrm{i}\left(\frac{\mathrm{i}}{6}-\frac{\mathrm{i}}{3}\right)=\frac{\pi}{6}.$$

10.6.11 MATHEMATICA: `Residue[f[z], {z, a}]`, `Integrate[f[x], {x, a, b}]`

```
In[1]:=Residue[z^2/((z^2 + 1) (z^2 + 4)), {z, I}]
```
$\mapsto$ Out[1]=$\frac{\mathrm{i}}{6}$

```
In[2]:=Residue[z^2/((z^2 + 1) (z^2 + 4)), {z, I}]
```
$\mapsto$ Out[2]=$-\frac{\mathrm{i}}{3}$

```
In[3]:=Integrate[x^2/((x^2 + 1) (x^2 + 4)), {x, 0, Infinity}]
```
$\mapsto$ Out[3]=$\frac{\mathrm{i}}{6}$

10.6.12 Exercise. Prove that

$$1\geq\frac{\sin t}{t}\geq\frac{2}{\pi}\qquad\text{for any}\quad t\in\left[0,\frac{\pi}{2}\right].$$

Solution. The function

$$\varphi:\left[0,\frac{\pi}{2}\right]\longrightarrow\mathbb{R},\qquad\varphi(t)=\frac{\sin t}{t}$$

is a decreasing function because

$$\varphi'(t) = \frac{t\,\cos t - \sin t}{t^2} \leq 0.$$

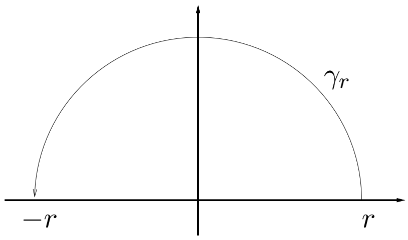

Figure 10.39: The paths γ_r.

10.6.13 Proposition (Jordan's lemma). *If the continuous function*

$$f : \{\ z = x + y\mathrm{i}\ |\ \ y \geq 0\ \} \longrightarrow \mathbb{C}$$

is such that

$$\lim_{z\to\infty} f(z) = 0 \tag{10.9}$$

and

$$\gamma_r : [0,\pi] \longrightarrow \mathbb{C}, \qquad \gamma_r(t) = r\,\mathrm{e}^{\mathrm{i}t}$$

(Figure 10.39) then

$$\lim_{r\to\infty} \int_{\gamma_r} f(z)\,\mathrm{e}^{\mathrm{i}z}\,dz = 0$$

Proof. Let $\varepsilon > 0$. From the relation (10.9) it follows that there is $\varepsilon > 0$ such that

$$r > r_\varepsilon \qquad \Longrightarrow \qquad |f(r\,\mathrm{e}^{\mathrm{i}t})| < \frac{2\varepsilon}{\pi}$$

and

$$\begin{aligned}\left|\int_{\gamma_r} f(z)\,\mathrm{e}^{\mathrm{i}z}\,dz\right| &= \left|\int_0^\pi f(r\,\mathrm{e}^{\mathrm{i}t})\,\mathrm{e}^{\mathrm{i}r(\cos t + \mathrm{i}\sin t)}\mathrm{i}r\mathrm{e}^{\mathrm{i}t}dt\right| \\ &\leq \int_0^\pi |f(r\,\mathrm{e}^{\mathrm{i}t})|\,\mathrm{e}^{-r\sin t}\,r\,dt \leq \tfrac{2\varepsilon}{\pi}r\int_0^\pi \mathrm{e}^{-r\sin t}\,dt \\ &\leq \tfrac{2\varepsilon}{\pi}r\int_0^\pi \mathrm{e}^{-r\frac{2}{\pi}t}\,dt = \tfrac{2\varepsilon}{\pi}r\,\tfrac{-\pi}{2r}\,\mathrm{e}^{-r\frac{2}{\pi}t}\Big|_0^{\frac{\pi}{2}} = \varepsilon(1-\mathrm{e}^{-r}) \leq \varepsilon.\end{aligned}$$

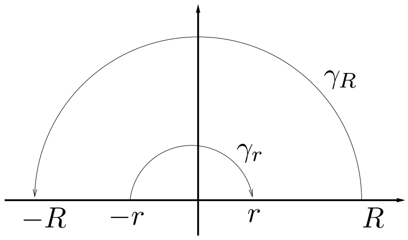

Figure 10.40: The path used in the case of Poisson's integral.

10.6.14 Exercise (Poisson's integral). Prove that

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2} \tag{10.10}$$

Solution. Let $0 < r < R$ and the paths (Figure 10.40)

$$\gamma_R : [0,\pi] \longrightarrow \mathbb{C}, \qquad \gamma_R(t) = R\,\mathrm{e}^{\mathrm{i}t}$$
$$\gamma_r : [0,\pi] \longrightarrow \mathbb{C}, \qquad \gamma_r(t) = r\,\mathrm{e}^{\mathrm{i}(\pi - t)}.$$

From the residue theorem (or Cauchy's theorem) it follows the relation

$$\int_{\gamma_R} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\,dz + \int_{-R}^{-r} \frac{\mathrm{e}^{\mathrm{i}x}}{x}\,dx + \int_{\gamma_r} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\,dz + \int_r^R \frac{\mathrm{e}^{\mathrm{i}x}}{x}\,dx = 0$$

which can also be written as

$$\int_{\gamma_R} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\,dz + \int_{\gamma_r} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\,dz + \int_r^R \frac{\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x}}{x}\,dx = 0$$

or

$$\int_{\gamma_R} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\,dz + \int_{\gamma_r} \frac{1}{z}\,dz + \int_{\gamma_r} \frac{\mathrm{e}^{\mathrm{i}z} - 1}{z}\,dz + 2\mathrm{i}\int_r^R \frac{\sin x}{x}\,dx = 0$$

By using the relation

$$\int_{\gamma_r} \frac{1}{z}\,dz = -\pi\mathrm{i}$$

and denoting by g an antiderivative of the function $f(z) = \frac{\mathrm{e}^{\mathrm{i}z}-1}{z}$ we get

$$\int_{\gamma_R} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\,dz - \pi\mathrm{i} + (g(r) - g(-r)) + 2\mathrm{i}\int_r^R \frac{\sin x}{x}\,dx = 0.$$

Since, in view of Jordan's lemma,

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{\mathrm{e}^{\mathrm{i}z}}{z} = 0$$

for $R \to \infty$ and $r \to 0$ we get the relation

$$2\mathrm{i}\int_0^\infty \frac{\sin x}{x}\,dx = \pi\mathrm{i}.$$

10.6.15 MATHEMATICA: `Integrate[f[x], {x, a, b}]`

`In[1]:=Integrate[Sin[x]/x, {x, 0, Infinity}]` $\mapsto$ Out[1]=$\frac{\pi}{2}$

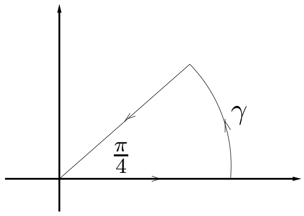

Figure 10.41: The path used in the case of Fresnel's integrals.

10.6.16 Fresnel's integrals. By integrating the function

$$f(z) = \mathrm{e}^{\mathrm{i}z^2}$$

along the path shown in Figure 10.41 one can prove [?] that

$$\int_0^\infty \cos x^2\, dx = \int_0^\infty \sin x^2\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

10.6.17 MATHEMATICA: `Integrate[f[x], {x, a, b}]`

`In[1]:=Integrate[Sin[x^2], {x, 0, Infinity}]` $\mapsto$ Out[1]=$\frac{\sqrt{\frac{\pi}{2}}}{2}$

`In[2]:=Integrate[Cos[x^2], {x, 0, Infinity}]` $\mapsto$ Out[2]=$\frac{\sqrt{\frac{\pi}{2}}}{2}$

10.6.18 Definition. Let $\varphi : \mathbb{R} \longrightarrow \mathbb{C}$. The function (if exists)

$$\mathcal{F}[\varphi] : \mathbb{R} \longrightarrow \mathbb{C}, \qquad \mathcal{F}[\varphi](\xi) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\xi x}\varphi(x)dx$$

is called the *Fourier transform* of φ.

10.6.19 Exercise. Prouve that

$$\mathcal{F}[\mathrm{e}^{-ax^2}](\xi) = \sqrt{\frac{\pi}{a}}\ \mathrm{e}^{-\frac{\xi^2}{4a}}$$

for any $a \in (0, \infty)$.

Solution. We have

$$\mathcal{F}[\mathrm{e}^{-ax^2}](\xi) = \int_{-\infty}^{\infty} \mathrm{e}^{-ax^2}\mathrm{e}^{\mathrm{i}\xi x}dx = \int_{-\infty}^{\infty} \mathrm{e}^{-ax^2+\mathrm{i}\xi x}dx = \mathrm{e}^{-\frac{\xi^2}{4a}} \int_{-\infty}^{\infty} \mathrm{e}^{-a\left(x-\mathrm{i}\frac{\xi}{2a}\right)^2} dx.$$

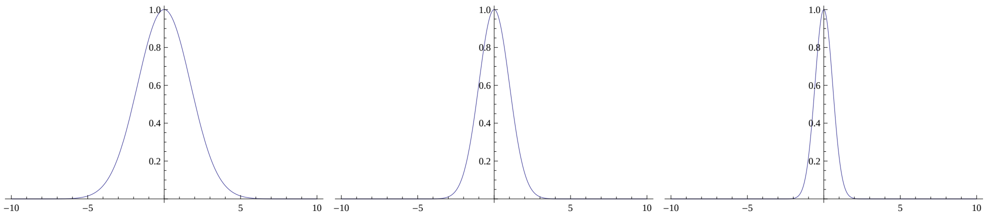

Figure 10.42: The functions $\mathrm{e}^{-\frac{3}{2}x^2}$, $\mathrm{e}^{-\frac{1}{2}x^2}$, $\mathrm{e}^{-\frac{1}{6}x^2}$.

By starting from the integral

$$\int_{-r}^{r} \mathrm{e}^{-at^2}dt + \int_{r}^{r-\mathrm{i}\frac{\xi}{2a}} \mathrm{e}^{-az^2}dz - \int_{-r-\mathrm{i}\frac{\xi}{2a}}^{r-\mathrm{i}\frac{\xi}{2a}} \mathrm{e}^{-az^2}dz + \int_{-r-\mathrm{i}\frac{\xi}{2a}}^{-r} \mathrm{e}^{-az^2}dz = 0$$

of the function

$$f : \mathbb{C} \longrightarrow \mathbb{C}, \qquad f(z) = \mathrm{e}^{-az^2}$$

along the rectangular path shown in Figure 10.43 we prouve that

$$\int_{-\infty}^{\infty} \mathrm{e}^{-a\left(t-\mathrm{i}\frac{\xi}{2a}\right)^2}dt = \int_{-\infty}^{\infty} \mathrm{e}^{-at^2}dt = \frac{1}{\sqrt{a}}\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}dx = \sqrt{\frac{\pi}{a}}.$$

We have

$$\lim_{r\to\infty}\int_{-r}^{r} \mathrm{e}^{-at^2}dt = \int_{-\infty}^{\infty} \mathrm{e}^{-at^2}dt.$$

By choosing for the linear path connecting r with $r-\mathrm{i}\frac{\xi}{2a}$ the parametrization

$$\gamma_1 : [0,1] \longrightarrow \mathbb{C}, \qquad \gamma_1(t) = r - \mathrm{i}t\frac{\xi}{2a}$$

we obtain the relation

$$\int_{r}^{r-\mathrm{i}\frac{\xi}{2a}} \mathrm{e}^{-az^2}dz = \int_0^1 \mathrm{e}^{-a\left(r-\mathrm{i}t\frac{\xi}{2a}\right)^2}(-\mathrm{i})\frac{\xi}{2a}dt = -\mathrm{i}\frac{\xi}{2a}\,\mathrm{e}^{-ar^2}\int_0^1 \mathrm{e}^{\mathrm{i}rt\xi+\frac{t^2\xi^2}{4a}}dt$$

whence

$$\lim_{r\to\infty}\int_{r}^{r-\mathrm{i}\frac{\xi}{2a}} \mathrm{e}^{-az^2}dz = 0.$$

In a similar way one can prove that

$$\lim_{r\to\infty}\int_{-r-\mathrm{i}\frac{\xi}{2a}}^{-r} \mathrm{e}^{-az^2}dz = 0.$$

By choosing for the linear path connecting $-r-\mathrm{i}\frac{\xi}{2a}$ with $r-\mathrm{i}\frac{\xi}{2a}$ the parametrization

$$\gamma_2 : [-r,r] \longrightarrow \mathbb{C}, \qquad \gamma_2(t) = t - \mathrm{i}\frac{\xi}{2a}$$

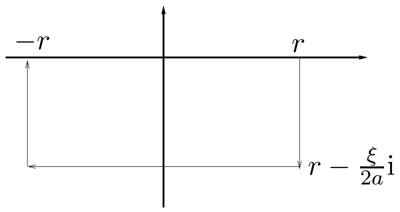

Figure 10.43: The used rectangular path.

we obtain the relation

$$\int_{-r-\mathrm{i}\frac{\xi}{2a}}^{r-\mathrm{i}\frac{\xi}{2a}} \mathrm{e}^{-az^2} dz = \int_{-r}^{r} \mathrm{e}^{-a\left(t-\mathrm{i}\frac{\xi}{2a}\right)^2} dt$$

whence

$$\lim_{r\to\infty} \int_{-r-\mathrm{i}\frac{\xi}{2a}}^{r-\mathrm{i}\frac{\xi}{2a}} \mathrm{e}^{-az^2} dz = \int_{-\infty}^{\infty} \mathrm{e}^{-a\left(t-\mathrm{i}\frac{\xi}{2a}\right)^2} dt.$$

10.6.20 MATHEMATICA: The used definition $\mathcal{F}[\varphi](x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\varphi(t)dt$

`In[1]:=FourierTransform[Exp[-t^2], t, x]` $\mapsto$ Out[1]=$\frac{\mathrm{e}^{-\frac{x^2}{4}}}{\sqrt{2}}$