

On the Gaussian functions of two discrete variables

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Abstract

A remarkable discrete counterpart of the Gaussian function of one continuous variable can be defined by using a Jacobi theta function, that is, as the sum of a convergent series. We extend this approach to Gaussian functions of two variables, and investigate the Fourier transform and Wigner function of the functions of discrete variable defined in this way.

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I. INTRODUCTION

The Gaussian functions play a fundamental role in mathematics and its applications. By using the (non-normalized) Gaussian function of continuous variable $g_\kappa : \mathbb{R} \rightarrow \mathbb{R}$,

$$g_\kappa(q) = e^{-\frac{\kappa}{2}q^2}, \quad \text{where } \kappa \in (0, \infty) \text{ is a parameter,} \quad (1)$$

one defines [1-5] the periodic Gaussian function (Fig. 1) of discrete variable $\mathbf{g}_\kappa : \mathbb{Z} \rightarrow \mathbb{R}$,

$$\mathbf{g}_\kappa(n) = \sum_{\alpha=-\infty}^{\infty} g_\kappa\left((n+\alpha d)\sqrt{\frac{2\pi}{d}}\right) = \sum_{\alpha=-\infty}^{\infty} e^{-\frac{\kappa\pi}{d}(n+\alpha d)^2} \quad (2)$$

The function $\mathbf{g}_\kappa$, obtained by using a method similar to Weil [6] or Zak [7] transform, is a generalization of Mehta's function f_0 [1]. In this article, we investigate only the case when $d=2j+1$ is a positive odd integer. The function $\mathbf{g}_\kappa$ can be written as

$$\mathbf{g}_\kappa(n) = \frac{1}{\sqrt{\kappa d}} \theta_3\left(\frac{n}{d}, \frac{i}{\kappa d}\right), \quad (3)$$

where θ_3 is the Jacobi function

$$\theta_3(z, \tau) = \sum_{\alpha=-\infty}^{\infty} e^{i\pi\tau\alpha^2} e^{2\pi i\alpha z} \quad (4)$$

having several remarkable properties among which we mention

$$\theta_3(z, i\tau) = \frac{1}{\sqrt{\tau}} e^{-\frac{\pi z^2}{\tau}} \theta_3\left(\frac{z}{i\tau}, \frac{i}{\tau}\right). \quad (5)$$

In the continuous case, the Fourier transform of g_κ computed with the usual definition

$$\mathcal{F}[\psi](p) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipq} \psi(q) dq \quad (6)$$

satisfies the relation

$$\mathcal{F}[g_\kappa] = \frac{1}{\sqrt{\kappa}} g_{\kappa^{-1}}. \quad (7)$$

The discrete Fourier transform of $\mathbf{g}_\kappa$, computed by using the definition

$$\mathbf{F}[\psi](k) \stackrel{\text{def}}{=} \frac{1}{\sqrt{d}} \sum_{n=-j}^j e^{-\frac{2\pi i}{d}kn} \psi(n), \quad (8)$$

satisfies a similar relation, namely [3, 5]

$$\mathbf{F}[\mathbf{g}_\kappa] = \frac{1}{\sqrt{\kappa}} \mathbf{g}_{\kappa^{-1}}. \quad (9)$$

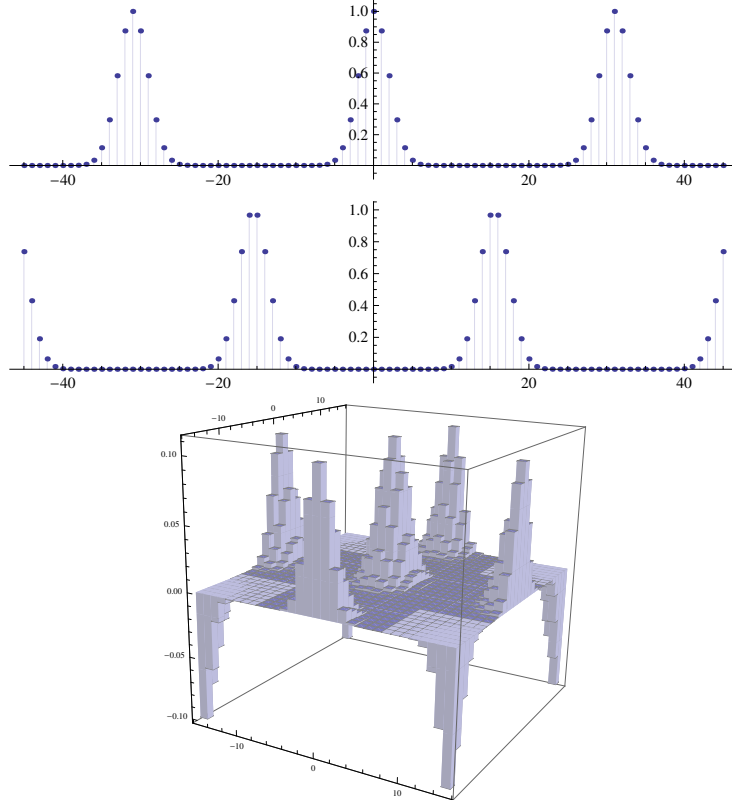


FIG. 1. The functions $\mathbf{g}_\kappa$, $\mathbf{g}_\kappa^+$ and $\mathbf{W}_{\mathbf{g}_\kappa}$ in the case $\kappa = \frac{4}{3}$, $d = 31$.

The Wigner function of g_κ , computed with the usual definition

$$\mathcal{W}_\psi(q, p) \stackrel{\text{def}}{=} \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-2ipx} \overline{\psi(q-x)} \psi(q+x) dx, \quad (10)$$

is a product of Gaussian functions,

$$\mathcal{W}_{g_\kappa}(q, p) = \frac{1}{\sqrt{\kappa\pi}} g_{2\kappa}(q) g_{2\kappa^{-1}}(p). \quad (11)$$

The discrete Wigner function of $\mathbf{g}_\kappa$, computed [5] by using the definition [8]

$$\mathbf{W}_\psi(n, k) \stackrel{\text{def}}{=} \frac{1}{d} \sum_{m=-j}^j e^{-\frac{4\pi i}{d} km} \overline{\psi(n-m)} \psi(n+m) \quad (12)$$

is a sum of four products of Gaussian like functions (Fig 1)

$$\begin{aligned} \mathbf{W}_{\mathbf{g}_\kappa}(n, k) = & \frac{1}{\sqrt{2\kappa d}} \mathbf{g}_{2\kappa}(n) \left[\mathbf{g}_{2\kappa^{-1}}(k) + \mathbf{g}_{2\kappa^{-1}}^+(k) \right] \\ & + \frac{1}{\sqrt{2\kappa d}} \mathbf{g}_{2\kappa}^+(n) \left[\mathbf{g}_{2\kappa^{-1}}(k) - \mathbf{g}_{2\kappa^{-1}}^+(k) \right], \end{aligned} \quad (13)$$

where the periodic function of discrete variable $\mathbf{g}_v^+ : \mathbb{Z} \rightarrow \mathbb{R}$,

$$\mathbf{g}_v^+(n) = \sum_{\alpha=-\infty}^{\infty} g_v \left((n + (\alpha + \frac{1}{2})d) \sqrt{\frac{2\pi}{d}} \right) = \sum_{\alpha=-\infty}^{\infty} e^{-\frac{v\pi}{d}(n + (\alpha + \frac{1}{2})d)^2} \quad (14)$$

is a kind of translated Gaussian function (Fig. 1). The relation (13) can be written as

$$\mathbf{W}_{\mathbf{g}_\kappa}(n, k) = C_\kappa \sum_{\alpha, \beta=-\infty}^{\infty} (-1)^{\alpha\beta} \mathcal{W}_{g_\kappa} \left((n + \alpha \frac{d}{2}) \sqrt{\frac{2\pi}{d}}, (k + \beta \frac{d}{2}) \sqrt{\frac{2\pi}{d}} \right), \quad (15)$$

where C_κ is a constant. Thus, there exists a simple relation between the discrete Wigner function $\mathbf{W}_{\mathbf{g}_\kappa}$ of a Gaussian function $\mathbf{g}_\kappa$ of discrete variable and the Wigner function $\mathcal{W}_{g_\kappa}$ of the corresponding Gaussian function g_κ of continuous variable.

Our purpose is to present a version for functions of two variables of these results.

II. GAUSSIAN FUNCTIONS OF TWO DISCRETE VARIABLES

Let $\sigma = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ be a matrix with real entries and such that $\sigma > 0$.

By using the Gaussian function of two continuous variables $g_\sigma : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$,

$$g_\sigma(q_1, q_2) = e^{-\frac{1}{2}(q_1 \ q_2) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}} = e^{-\frac{1}{2}(aq_1^2 + 2bq_1q_2 + cq_2^2)} \quad (16)$$

we define the periodic *Gaussian function of two discrete variables* $\mathbf{g}_\sigma : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{R}$,

$$\mathbf{g}_\sigma(n_1, n_2) = \sum_{\alpha_1, \alpha_2=-\infty}^{\infty} e^{-\frac{\pi}{d}(n_1 + \alpha_1 d \ n_2 + \alpha_2 d) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} n_1 + \alpha_1 d \\ n_2 + \alpha_2 d \end{pmatrix}} \quad (17)$$

and other three complementary periodic Gaussian like functions

$$\begin{aligned} \mathbf{g}_\sigma^{+0}(n_1, n_2) &= \sum_{\alpha_1, \alpha_2=-\infty}^{\infty} e^{-\frac{\pi}{d}(n_1 + (\alpha_1 + \frac{1}{2})d \ n_2 + \alpha_2 d) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} n_1 + (\alpha_1 + \frac{1}{2})d \\ n_2 + \alpha_2 d \end{pmatrix}}, \\ \mathbf{g}_\sigma^{0+}(n_1, n_2) &= \sum_{\alpha_1, \alpha_2=-\infty}^{\infty} e^{-\frac{\pi}{d}(n_1 + \alpha_1 d \ n_2 + (\alpha_2 + \frac{1}{2})d) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} n_1 + \alpha_1 d \\ n_2 + (\alpha_2 + \frac{1}{2})d \end{pmatrix}}, \\ \mathbf{g}_\sigma^{++}(n_1, n_2) &= \sum_{\alpha_1, \alpha_2=-\infty}^{\infty} e^{-\frac{\pi}{d}(n_1 + (\alpha_1 + \frac{1}{2})d \ n_2 + (\alpha_2 + \frac{1}{2})d) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} n_1 + (\alpha_1 + \frac{1}{2})d \\ n_2 + (\alpha_2 + \frac{1}{2})d \end{pmatrix}}. \end{aligned} \quad (18)$$

III. DISCRETE FOURIER TRANSFORM

In the continuous case, by using the definition

$$\mathcal{F}[\psi](p_1, p_2) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(p_1 q_1 + p_2 q_2)} \psi(q_1, q_2) dq_1 dq_2, \quad (19)$$

we get the known relation

$$\mathcal{F}\left[e^{-\frac{1}{2}\begin{pmatrix} q_1 & q_2 \end{pmatrix}\sigma\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}}\right](p_1, p_2) = \frac{1}{\sqrt{\det\sigma}} e^{-\frac{1}{2}\begin{pmatrix} p_1 & p_2 \end{pmatrix}\sigma^{-1}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}}, \quad (20)$$

that is, we have

$$\mathcal{F}[g_\sigma] = \frac{1}{\sqrt{\det\sigma}} g_{\sigma^{-1}}. \quad (21)$$

In the discrete case, the Fourier transform is usually defined as

$$\mathbf{F}[\psi](k_1, k_2) \stackrel{\text{def}}{=} \frac{1}{d} \sum_{n_1=-j}^j \sum_{n_2=-j}^j e^{-\frac{2\pi i}{d}(k_1 n_1 + k_2 n_2)} \psi(n_1, n_2). \quad (22)$$

Lemma 1. *We have*

$$\begin{aligned} \mathbf{F}[g_\sigma](k_1, k_2) &= \frac{1}{\sqrt{\det\sigma}} \sum_{\beta_1, \beta_2=-\infty}^{\infty} g_{\sigma^{-1}}\left((k_1 + \beta_1 d)\sqrt{\frac{2\pi}{d}}, (k_2 + \beta_2 d)\sqrt{\frac{2\pi}{d}}\right), \\ \mathbf{F}[g_\sigma^{+0}](k_1, k_2) &= \frac{(-1)^{k_1}}{\sqrt{\det\sigma}} \sum_{\beta_1, \beta_2=-\infty}^{\infty} (-1)^{\beta_1} g_{\sigma^{-1}}\left((k_1 + \beta_1 d)\sqrt{\frac{2\pi}{d}}, (k_2 + \beta_2 d)\sqrt{\frac{2\pi}{d}}\right), \\ \mathbf{F}[g_\sigma^{0+}](k_1, k_2) &= \frac{(-1)^{k_2}}{\sqrt{\det\sigma}} \sum_{\beta_1, \beta_2=-\infty}^{\infty} (-1)^{\beta_2} g_{\sigma^{-1}}\left((k_1 + \beta_1 d)\sqrt{\frac{2\pi}{d}}, (k_2 + \beta_2 d)\sqrt{\frac{2\pi}{d}}\right), \\ \mathbf{F}[g_\sigma^{++}](k_1, k_2) &= \frac{(-1)^{k_1+k_2}}{\sqrt{\det\sigma}} \sum_{\beta_1, \beta_2=-\infty}^{\infty} (-1)^{\beta_1+\beta_2} g_{\sigma^{-1}}\left((k_1 + \beta_1 d)\sqrt{\frac{2\pi}{d}}, (k_2 + \beta_2 d)\sqrt{\frac{2\pi}{d}}\right). \end{aligned} \quad (23)$$

Proof. The periodic function of two continuous variables

$$\mathcal{G}_\sigma(q_1, q_2) = \sum_{\alpha_1, \alpha_2=-\infty}^{\infty} e^{-\frac{\pi}{d}(a(q_1 + \alpha_1 d)^2 + 2b(q_1 + \alpha_1 d)(q_2 + \alpha_2 d) + c(q_2 + \alpha_2 d)^2)} \quad (24)$$

can be expended into a Fourier series

$$\mathcal{G}_\sigma(q_1, q_2) = \sum_{m_1, m_2=-\infty}^{\infty} a_{m_1 m_2} e^{\frac{2\pi i}{d}(m_1 q_1 + m_2 q_2)}, \quad (25)$$

where

$$a_{m_1 m_2} = \frac{1}{d^2} \int_0^d \int_0^d e^{-\frac{2\pi i}{d}(m_1 q_1 + m_2 q_2)} \sum_{\alpha_1, \alpha_2=-\infty}^{\infty} e^{-\frac{\pi}{d}(a(q_1 + \alpha_1 d)^2 + 2b(q_1 + \alpha_1 d)(q_2 + \alpha_2 d) + c(q_2 + \alpha_2 d)^2)} dq_1 dq_2.$$

By using (20) and the change of variables $q_1 = y_1 \sqrt{\frac{d}{2\pi}} - \alpha_1 d$, $q_2 = y_2 \sqrt{\frac{d}{2\pi}} - \alpha_2 d$, we get

$$\begin{aligned} a_{m_1 m_2} &= \frac{1}{2\pi d} \sum_{\alpha_1, \alpha_2=-\infty}^{\infty} \int_{\alpha_1 \sqrt{2\pi d}}^{(\alpha_1+1)\sqrt{2\pi d}} \int_{\alpha_2 \sqrt{2\pi d}}^{(\alpha_2+1)\sqrt{2\pi d}} e^{-\frac{2\pi i}{d}(m_1(y_1 \sqrt{\frac{d}{2\pi}} - \alpha_1 d) + m_2(y_2 \sqrt{\frac{d}{2\pi}} - \alpha_2 d))} e^{-\frac{1}{2}(ay_1^2 + 2by_1 y_2 + cy_2^2)} dy_1 dy_2 \\ &= \frac{1}{2\pi d} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{2\pi i}{d}(m_1 y_1 \sqrt{\frac{d}{2\pi}} + m_2 y_2 \sqrt{\frac{d}{2\pi}})} e^{-\frac{1}{2}(ay_1^2 + 2by_1 y_2 + cy_2^2)} dy_1 dy_2 = \frac{1}{d\sqrt{\det\sigma}} e^{-\frac{\pi}{d}\begin{pmatrix} m_1 & m_2 \end{pmatrix}\sigma^{-1}\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}}. \end{aligned}$$

Consequently, we have

$$\mathcal{G}_\sigma(q_1, q_2) = \frac{1}{d\sqrt{\det \sigma}} \sum_{m_1, m_2 = -\infty}^{\infty} e^{\frac{2\pi i}{d}(m_1 q_1 + m_2 q_2)} e^{-\frac{\pi}{d}(m_1 \ m_2) \sigma^{-1} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}}, \quad (26)$$

and the relation

$$\begin{aligned} \mathbf{g}_\sigma(n_1, n_2) &= \frac{1}{d\sqrt{\det \sigma}} \sum_{m_1, m_2 = -\infty}^{\infty} e^{\frac{2\pi i}{d}(m_1 n_1 + m_2 n_2)} e^{-\frac{\pi}{d}(m_1 \ m_2) \sigma^{-1} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}} \\ &= \frac{1}{d\sqrt{\det \sigma}} \sum_{k_1, k_2 = -j}^j \sum_{\beta_1, \beta_2 = -\infty}^{\infty} e^{\frac{2\pi i}{d}((k_1 + \beta_1 d)n_1 + (k_2 + \beta_2 d)n_2)} e^{-\frac{\pi}{d}(k_1 + \beta_1 d \ k_2 + \beta_2 d) \sigma^{-1} \begin{pmatrix} k_1 + \beta_1 d \\ k_2 + \beta_2 d \end{pmatrix}} \\ &= \frac{1}{d\sqrt{\det \sigma}} \sum_{k_1, k_2 = -j}^j e^{\frac{2\pi i}{d}(k_1 n_1 + k_2 n_2)} \mathbf{g}_{\sigma^{-1}}(k_1, k_2) = \frac{1}{\sqrt{\det \sigma}} \mathbf{F}^{-1}[\mathbf{g}_{\sigma^{-1}}](n_1, n_2) \end{aligned}$$

equivalent to $\mathbf{F}[\mathbf{g}_\sigma](k_1, k_2) = \frac{1}{\sqrt{\det \sigma}} \sum_{\beta_1, \beta_2 = -\infty}^{\infty} g_{\sigma^{-1}}\left((k_1 + \beta_1 d)\sqrt{\frac{2\pi}{d}}, (k_2 + \beta_2 d)\sqrt{\frac{2\pi}{d}}\right)$.

The other three relations can be obtained in a similar way. For details see [9, 10]. $\square$

Theorem 1. *The discrete Fourier transform of $\mathbf{g}_\sigma$ satisfies the relation*

$$\mathbf{F}[\mathbf{g}_\sigma] = \frac{1}{\sqrt{\det \sigma}} \mathbf{g}_{\sigma^{-1}}. \quad (27)$$

Proof. This is just the first relation from Lemma 1, written in a different way. $\square$

Lemma 2. *We have*

$$\begin{aligned} \mathbf{F}[\mathbf{g}_{2\sigma}](2k_1, 2k_2) &= \frac{1}{2\sqrt{\det \sigma}} \left(\mathbf{g}_{2\sigma^{-1}}(k_1, k_2) + \mathbf{g}_{2\sigma^{-1}}^{0+}(k_1, k_2) + \mathbf{g}_{2\sigma^{-1}}^{0+}(k_1, k_2) + \mathbf{g}_{2\sigma^{-1}}^{++}(k_1, k_2) \right), \\ \mathbf{F}[\mathbf{g}_{2\sigma}^{0+}](2k_1, 2k_2) &= \frac{1}{2\sqrt{\det \sigma}} \left(\mathbf{g}_{2\sigma^{-1}}(k_1, k_2) - \mathbf{g}_{2\sigma^{-1}}^{0+}(k_1, k_2) + \mathbf{g}_{2\sigma^{-1}}^{0+}(k_1, k_2) - \mathbf{g}_{2\sigma^{-1}}^{++}(k_1, k_2) \right), \\ \mathbf{F}[\mathbf{g}_{2\sigma}^{0+}](2k_1, 2k_2) &= \frac{1}{2\sqrt{\det \sigma}} \left(\mathbf{g}_{2\sigma^{-1}}(k_1, k_2) + \mathbf{g}_{2\sigma^{-1}}^{0+}(k_1, k_2) - \mathbf{g}_{2\sigma^{-1}}^{0+}(k_1, k_2) - \mathbf{g}_{2\sigma^{-1}}^{++}(k_1, k_2) \right), \\ \mathbf{F}[\mathbf{g}_{2\sigma}^{++}](2k_1, 2k_2) &= \frac{1}{2\sqrt{\det \sigma}} \left(\mathbf{g}_{2\sigma^{-1}}(k_1, k_2) - \mathbf{g}_{2\sigma^{-1}}^{0+}(k_1, k_2) - \mathbf{g}_{2\sigma^{-1}}^{0+}(k_1, k_2) + \mathbf{g}_{2\sigma^{-1}}^{++}(k_1, k_2) \right). \end{aligned} \quad (28)$$

Proof. The relations

$$\begin{aligned}
\mathbf{F}[\mathbf{g}_{2\sigma}](2k_1, 2k_2) &= \frac{1}{2\sqrt{\det \sigma}} \sum_{\beta_1, \beta_2 = -\infty}^{\infty} g_{(2\sigma)^{-1}} \left((2k_1 + \beta_1 d) \sqrt{\frac{2\pi}{d}}, (2k_2 + \beta_2 d) \sqrt{\frac{2\pi}{d}} \right), \\
\mathbf{F}[\mathbf{g}_{2\sigma}^{+0}](2k_1, 2k_2) &= \frac{1}{2\sqrt{\det \sigma}} \sum_{\beta_1, \beta_2 = -\infty}^{\infty} (-1)^{\beta_1} g_{(2\sigma)^{-1}} \left((2k_1 + \beta_1 d) \sqrt{\frac{2\pi}{d}}, (2k_2 + \beta_2 d) \sqrt{\frac{2\pi}{d}} \right), \\
\mathbf{F}[\mathbf{g}_{2\sigma}^{0+}](2k_1, 2k_2) &= \frac{1}{2\sqrt{\det \sigma}} \sum_{\beta_1, \beta_2 = -\infty}^{\infty} (-1)^{\beta_2} g_{(2\sigma)^{-1}} \left((2k_1 + \beta_1 d) \sqrt{\frac{2\pi}{d}}, (2k_2 + \beta_2 d) \sqrt{\frac{2\pi}{d}} \right), \\
\mathbf{F}[\mathbf{g}_{2\sigma}^{++}](2k_1, 2k_2) &= \frac{1}{2\sqrt{\det \sigma}} \sum_{\beta_1, \beta_2 = -\infty}^{\infty} (-1)^{\beta_1 + \beta_2} g_{(2\sigma)^{-1}} \left((2k_1 + \beta_1 d) \sqrt{\frac{2\pi}{d}}, (2k_2 + \beta_2 d) \sqrt{\frac{2\pi}{d}} \right)
\end{aligned}$$

can be written as

$$\begin{aligned}
\mathbf{F}[\mathbf{g}_{2\sigma}](2k_1, 2k_2) &= \frac{1}{2\sqrt{\det \sigma}} \sum_{\beta_1, \beta_2 = -\infty}^{\infty} g_{2\sigma^{-1}} \left((k_1 + \beta_1 \frac{d}{2}) \sqrt{\frac{2\pi}{d}}, (k_2 + \beta_2 \frac{d}{2}) \sqrt{\frac{2\pi}{d}} \right), \\
\mathbf{F}[\mathbf{g}_{2\sigma}^{+0}](2k_1, 2k_2) &= \frac{1}{2\sqrt{\det \sigma}} \sum_{\beta_1, \beta_2 = -\infty}^{\infty} (-1)^{\beta_1} g_{2\sigma^{-1}} \left((k_1 + \beta_1 \frac{d}{2}) \sqrt{\frac{2\pi}{d}}, (k_2 + \beta_2 \frac{d}{2}) \sqrt{\frac{2\pi}{d}} \right), \\
\mathbf{F}[\mathbf{g}_{2\sigma}^{0+}](2k_1, 2k_2) &= \frac{1}{2\sqrt{\det \sigma}} \sum_{\beta_1, \beta_2 = -\infty}^{\infty} (-1)^{\beta_2} g_{2\sigma^{-1}} \left((k_1 + \beta_1 \frac{d}{2}) \sqrt{\frac{2\pi}{d}}, (k_2 + \beta_2 \frac{d}{2}) \sqrt{\frac{2\pi}{d}} \right), \\
\mathbf{F}[\mathbf{g}_{2\sigma}^{++}](2k_1, 2k_2) &= \frac{1}{2\sqrt{\det \sigma}} \sum_{\beta_1, \beta_2 = -\infty}^{\infty} (-1)^{\beta_1 + \beta_2} g_{2\sigma^{-1}} \left((k_1 + \beta_1 \frac{d}{2}) \sqrt{\frac{2\pi}{d}}, (k_2 + \beta_2 \frac{d}{2}) \sqrt{\frac{2\pi}{d}} \right).
\end{aligned}$$

By separating the even case from the odd one for β_1 and β_2 , we get (28). $\square$

IV. DISCRETE WIGNER FUNCTION

The Wigner function of g_σ , computed by using the formula

$$\mathcal{W}_\psi(q_1, q_2, p_1, p_2) \stackrel{\text{def}}{=} \frac{1}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2i(p_1 x_1 + p_2 x_2)} \overline{\psi(q_1 - x_1, q_2 - x_2)} \psi(q_1 + x_1, q_2 + x_2) dx_1 dx_2, \quad (29)$$

is a product of Gaussian functions,

$$\mathcal{W}_{g_\sigma}(q_1, q_2, p_1, p_2) = \frac{1}{\pi \sqrt{\det \sigma}} g_{2\sigma}(q_1, q_2) g_{2\sigma^{-1}}(p_1, p_2). \quad (30)$$

Lemma 3. *If the function $f: \mathbb{R} \rightarrow [0, \infty)$ is such that the series are convergent, then*

$$\sum_{\alpha, \beta = -\infty}^{\infty} f(\alpha, \beta) = \sum_{\mu, \eta = -\infty}^{\infty} f(\mu + \eta, \mu - \eta) + \sum_{\mu, \eta = -\infty}^{\infty} f(\mu + \eta + 1, \mu - \eta). \quad (31)$$

Proof. After separating the sum as

$$\sum_{\alpha, \beta = -\infty}^{\infty} f(\alpha, \beta) = \sum_{\substack{\alpha, \beta \\ \text{both even} \\ \text{or} \\ \text{both odd}}} f(\alpha, \beta) + \sum_{\substack{\alpha, \beta \\ \text{one even} \\ \text{and} \\ \text{other odd}}} f(\alpha, \beta), \quad (32)$$

we use the substitutions $(\alpha, \beta) = (\mu + \eta, \mu - \eta)$ and $(\alpha, \beta) = (\mu + \eta + 1, \mu - \eta)$. $\square$

In the discrete case, we use for the Wigner function the definition

$$\mathbf{W}_{\psi}(n_1, n_2, k_1, k_2) \stackrel{\text{def}}{=} \frac{1}{d^2} \sum_{m_1 = -j}^j \sum_{m_2 = -j}^j e^{-\frac{4\pi i}{d}(k_1 m_1 + k_2 m_2)} \overline{\psi(n_1 - m_1, n_2 - m_2)} \psi(n_1 + m_1, n_2 + m_2). \quad (33)$$

Theorem 2. *The Wigner function $\mathbf{W}_{\mathbf{g}_{\sigma}}$ of $\mathbf{g}_{\sigma}$ is an algebraic sum of 16 products of Gaussian like functions, namely*

$$\begin{aligned} \mathbf{W}_{\mathbf{g}_{\sigma}}(n_1, n_2, k_1, k_2) &= \frac{1}{2d\sqrt{\det \sigma}} \mathbf{g}_{2\sigma}(n_1, n_2) \left[\mathbf{g}_{2\sigma^{-1}}(k_1, k_2) + \mathbf{g}_{2\sigma^{-1}}^{+0}(k_1, k_2) + \mathbf{g}_{2\sigma^{-1}}^{0+}(k_1, k_2) + \mathbf{g}_{2\sigma^{-1}}^{++}(k_1, k_2) \right] \\ &+ \frac{1}{2d\sqrt{\det \sigma}} \mathbf{g}_{2\sigma}^{+0}(n_1, n_2) \left[\mathbf{g}_{2\sigma^{-1}}(k_1, k_2) - \mathbf{g}_{2\sigma^{-1}}^{+0}(k_1, k_2) + \mathbf{g}_{2\sigma^{-1}}^{0+}(k_1, k_2) - \mathbf{g}_{2\sigma^{-1}}^{++}(k_1, k_2) \right] \\ &+ \frac{1}{2d\sqrt{\det \sigma}} \mathbf{g}_{2\sigma}^{0+}(n_1, n_2) \left[\mathbf{g}_{2\sigma^{-1}}(k_1, k_2) + \mathbf{g}_{2\sigma^{-1}}^{+0}(k_1, k_2) - \mathbf{g}_{2\sigma^{-1}}^{0+}(k_1, k_2) - \mathbf{g}_{2\sigma^{-1}}^{++}(k_1, k_2) \right] \\ &+ \frac{1}{2d\sqrt{\det \sigma}} \mathbf{g}_{2\sigma}^{++}(n_1, n_2) \left[\mathbf{g}_{2\sigma^{-1}}(k_1, k_2) - \mathbf{g}_{2\sigma^{-1}}^{+0}(k_1, k_2) - \mathbf{g}_{2\sigma^{-1}}^{0+}(k_1, k_2) + \mathbf{g}_{2\sigma^{-1}}^{++}(k_1, k_2) \right]. \end{aligned} \quad (34)$$

Proof. By using Lemma 3, we obtain

$$\begin{aligned} \mathbf{W}_{\mathbf{g}_{\sigma}}(n_1, n_2, k_1, k_2) &= \frac{1}{d^2} \sum_{m_1, m_2 = -j}^j e^{-\frac{4\pi i}{d}(k_1 m_1 + k_2 m_2)} \\ &\times \sum_{\alpha_1, \beta_1 = -\infty}^{\infty} e^{-\frac{a\pi}{d}(n_1 - m_1 + \alpha_1 d)^2} e^{-\frac{2b\pi}{d}(n_1 - m_1 + \alpha_1 d)(n_2 - m_2 + \beta_1 d)} e^{-\frac{c\pi}{d}(n_2 - m_2 + \beta_1 d)^2} \\ &\times \sum_{\alpha_2, \beta_2 = -\infty}^{\infty} e^{-\frac{a\pi}{d}(n_1 + m_1 + \alpha_2 d)^2} e^{-\frac{2b\pi}{d}(n_1 + m_1 + \alpha_2 d)(n_2 + m_2 + \beta_2 d)} e^{-\frac{c\pi}{d}(n_2 + m_2 + \beta_2 d)^2} \\ &= \frac{1}{d^2} \sum_{m_1, m_2 = -j}^j e^{-\frac{4\pi i}{d}(k_1 m_1 + k_2 m_2)} \sum_{\alpha_1, \alpha_2 = -\infty}^{\infty} \sum_{\beta_1, \beta_2 = -\infty}^{\infty} e^{-\frac{a\pi}{d}(n_1 - m_1 + \alpha_1 d)^2} e^{-\frac{a\pi}{d}(n_1 + m_1 + \alpha_2 d)^2} \\ &\times e^{-\frac{2b\pi}{d}(n_1 - m_1 + \alpha_1 d)(n_2 - m_2 + \beta_1 d)} e^{-\frac{2b\pi}{d}(n_1 + m_1 + \alpha_2 d)(n_2 + m_2 + \beta_2 d)} e^{-\frac{c\pi}{d}(n_2 - m_2 + \beta_1 d)^2} e^{-\frac{c\pi}{d}(n_2 + m_2 + \beta_2 d)^2} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{d} \mathbf{g}_{2\sigma}(n_1, n_2) \mathbf{F}[\mathbf{g}_{2\sigma}](2k_1, 2k_2) + \frac{1}{d} \mathbf{g}_{2\sigma}^{+0}(n_1, n_2) \mathbf{F}[\mathbf{g}_{2\sigma}^{+0}](2k_1, 2k_2) \\
&\quad + \frac{1}{d} \mathbf{g}_{2\sigma}^{0+}(n_1, n_2) \mathbf{F}[\mathbf{g}_{2\sigma}^{0+}](2k_1, 2k_2) + \frac{1}{d} \mathbf{g}_{2\sigma}^{++}(n_1, n_2) \mathbf{F}[\mathbf{g}_{2\sigma}^{++}](2k_1, 2k_2).
\end{aligned}$$

By using Lemma 2, we get (34). $\square$

V. CONTINUOUS-DISCRETE CORRESPONDENCE

The function $\mathbf{g}_\sigma$ corresponds to g_σ , namely

$$\mathbf{g}_\sigma(n_1, n_2) = \sum_{\alpha_1, \alpha_2 = -\infty}^{\infty} g_\sigma \left((n_1 + \alpha_1 d) \sqrt{\frac{2\pi}{d}}, (n_2 + \alpha_2 d) \sqrt{\frac{2\pi}{d}} \right). \quad (35)$$

The relation (27) can be written as

$$\mathbf{F}[\mathbf{g}_\sigma](k_1, k_2) = \sum_{\beta_1, \beta_2 = -\infty}^{\infty} \mathcal{F}[g_\sigma] \left((k_1 + \beta_1 d) \sqrt{\frac{2\pi}{d}}, (k_2 + \beta_2 d) \sqrt{\frac{2\pi}{d}} \right), \quad (36)$$

that is, the discrete Fourier transform $\mathbf{F}[\mathbf{g}_\sigma]$ of $\mathbf{g}_\sigma$ is the Gaussian function of two discrete variables corresponding to the continuous Fourier transform $\mathcal{F}[g_\sigma]$ of g_σ .

By using (17), (18) and (30), the relation (34) can be written as

$$\begin{aligned}
\mathbf{W}_{\mathbf{g}_\sigma}(n_1, n_2, k_1, k_2) &= C_\sigma \sum_{\alpha_1, \alpha_2 = -\infty}^{\infty} \sum_{\beta_1, \beta_2 = -\infty}^{\infty} (-1)^{\alpha_1 \beta_1 + \alpha_2 \beta_2} \\
&\quad \times \mathcal{W}_{g_\sigma} \left((n_1 + \alpha_1 \frac{d}{2}) \sqrt{\frac{2\pi}{d}}, (n_2 + \alpha_2 \frac{d}{2}) \sqrt{\frac{2\pi}{d}}, (k_1 + \beta_1 \frac{d}{2}) \sqrt{\frac{2\pi}{d}}, (k_2 + \beta_2 \frac{d}{2}) \sqrt{\frac{2\pi}{d}} \right),
\end{aligned} \quad (37)$$

where C_σ is a constant. Thus, the discrete Wigner function $\mathbf{W}_{\mathbf{g}_\sigma}$ of $\mathbf{g}_\sigma$ can be obtained directly from the corresponding continuous Wigner function $\mathcal{W}_{g_\sigma}$ of g_σ .

VI. CONCLUDING REMARKS

Some remarkable discrete versions of the Gaussian functions, the corresponding Fourier transform and Wigner function can be defined as the sum of a convergent series involving the continuous counterpart. We have investigated the Gaussian functions of two variables, but the definitions and the obtained results can easily be extended to three or more variables.

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