COMPLEX ANALYSIS - Solved Problems 2

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PROBLEMS

Problem 1

Compute:

1a $(2-3\mathrm{i})^2+(\overline{1+\mathrm{i}})+\frac{2-\mathrm{i}}{1+\mathrm{i}}+\mathrm{i}^9+|3+4\mathrm{i}|$.

1b $\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}+\mathrm{e}^{2-\pi\mathrm{i}/3}+\cos(2\mathrm{i})$.

1c $(z^4+\frac{1}{z^2}-\cos z+z^3\mathrm{e}^{-\mathrm{i}z})'$.

1d $\log(2-2\,\mathrm{i})$.

Problem 2

Compute the residues of the functions

2a $f:\mathbb{C}\backslash\{0,\mathrm{i}\}\to\mathbb{C},\quad f(z)=\frac{1}{z(z-\mathrm{i})^2}$

2b $g:\mathbb{C}\backslash\{-\pi\}\to\mathbb{C},\quad g(z)=(z+\mathrm{i})\mathrm{e}^{\frac{1}{z-\pi}}$

in the corresponding singular points.

Problem 3

Compute the integrals

3a $I_0=\int\limits_{\gamma_0}(1-2\bar{z})dz$,

where $\gamma_0:[0,1]\to\mathbb{C},\quad \gamma_0(t)=1-t+t\mathrm{i}$.

3b $I_1=\int\limits_{\gamma_1}\frac{1}{z(z-\pi\mathrm{i})^2}dz$,

where $\gamma_1:[0,2\pi]\to\mathbb{C},\quad \gamma_1(t)=3\cos t+2\mathrm{i}\sin t$.

Problem 4

Compute the integral $\quad I=\int\limits_{-\infty}^{\infty}\frac{1}{1+x^2}dx$

4a - directly,

4b - by using the theorem of residues.

SOME DEFINITIONS AND THEOREMS

D1 Definition

$(x_1+y_1\mathrm{i})(x_2+y_2\mathrm{i})=(x_1x_2-y_1y_2)+(x_1y_2+x_2y_1)\mathrm{i}$

$\overline{x+y\mathrm{i}}=x-y\mathrm{i}$,

$|x+y\mathrm{i}|=\sqrt{x^2+y^2}$,

$|z_1-z_2|=$ distance between z_1 and z_2,

$|z|=$ distance between z and 0,

$\mathrm{e}^{\mathrm{i}t}\overset{\text{def}}{=}\cos t+\mathrm{i}\sin t$ (Euler's formula),

$\mathrm{e}^{x+y\mathrm{i}}=\mathrm{e}^x\cos y+\mathrm{i}\,\mathrm{e}^x\sin y$,

$z=|z|\,\mathrm{e}^{\mathrm{i}\arg z}$, where $-\pi<\arg z\le\pi$,

$\cos z=\frac{1}{2}(\mathrm{e}^{\mathrm{i}z}+\mathrm{e}^{-\mathrm{i}z})$,

$\sin z=\frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}z}-\mathrm{e}^{-\mathrm{i}z})$,

$\log z=\ln|z|+\mathrm{i}\arg z,\qquad \log_k z=\ln|z|+\mathrm{i}(\arg z+2k\pi)$.

D2 Definition Let $D\subseteq\mathbb{C}$ be an open set, and $z_0\in D$.

A function $f:D\longrightarrow\mathbb{C}$ is **complex-differentiable** ($\mathbb{C}$-differentiable) at z_0 $\overset{\text{def}}{\Longleftrightarrow}$ there exists and is finite $f'(z_0)\overset{\text{def}}{=}\lim\limits_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$.

D3 Definition

$f:D\to\mathbb{C}$ defined on an open set D is called $\mathbb{C}$-*differentiable* (or *holomorphic function*) if f is $\mathbb{C}$-differentiable at any point $z_0\in D$.

T1 Theorem

$(f\pm g)'=f'\pm g'\qquad (z^n)'=nz^{n-1}$

$(fg)'=f'g+fg'\qquad (\mathrm{e}^z)'=\mathrm{e}^z$

$\left(\frac{f}{g}\right)'=\frac{f'g-fg'}{g^2}\qquad (\sin z)'=\cos z$

$(f(\varphi(z)))'=f'(\varphi(z))\,\varphi'(z)\qquad (\cos z)'=-\sin z$.

T2 Theorem

$\frac{1}{1-z}=1+z+z^2+z^3+\ldots$ for $|z|<1$

$\mathrm{e}^z=1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\ldots$ for any z

D4 Definition. Let $D\subseteq\mathbb{C}$ be an open set, $f:D\longrightarrow\mathbb{C}$ be a continuous function, $\gamma:[a,b]\longrightarrow D$ be a path of class C^1.

The complex line integral of f along γ is $\int\limits_\gamma f(z)\,dz=\int\limits_a^b f(\gamma(t))\,\gamma'(t)dt$

T3 Theorem

$g:D\to\mathbb{C}$ is a primitive of f (that is $g'=f$) $\Rightarrow$ $\int_\gamma f(z)\,dz=g(\gamma(b))-g(\gamma(a))$

If $f:D\to\mathbb{C}$ admits a primitive and γ is closed, then $\int\limits_\gamma f(z)\,dz=0$.

D5 Definition Let D be an open set and $f:D\to\mathbb{C}$ holomorphic.

$z_0\in\mathbb{C}\backslash D$ is an *isolated singular point* if there exists $r>0$ such that $\{\,z\mid 0<|z-z_0|<r\,\}\subset D$.

$z_0\in D$ is *a zero of multiplicity k* if $f(z_0)=f'(z_0)=\ldots=f^{(k-1)}(z_0)=0$ and $f^{(k)}(z_0)\neq 0$.

$z_0\in\mathbb{C}\backslash D$ is *a pole of order k of f* if z_0 is a zero of multiplicity k of $\frac{1}{f}$.

T4 Theorem Let D be an open set and $f:D\longrightarrow\mathbb{C}$ holomorphic.

z_0 is a an isolated singular point and $\{\,z\mid 0<|z-z_0|<r\,\}\subset D$ $\Rightarrow$ there exists a unique Laurent series such that $f(z)=\sum\limits_{n=-\infty}^{\infty}a_n\,(z-z_0)^n$, for z satisfying $0<|z-z_0|<r$.

The number $\mathbf{Res}_{z_0}f\overset{\text{def}}{=}a_{-1}$ is the **residue** of f in z_0.

T5 Theorem

If z_0 pole of order k, then around z_0 the function f admits an expansion of the form

$$f(z)=\frac{a_{-k}}{(z-z_0)^k}+\ldots+\frac{a_{-1}}{(z-z_0)}+a_0+a_1(z-z_0)+\ldots$$

and

$$\mathbf{Res}_{z_0}f=\frac{1}{(k-1)!}\lim_{z\to z_0}\left((z-z_0)^k\,f(z)\right)^{(k-1)}$$

D6 Definition (*Index* of a point z_0 with respect to a path).

For a closed path γ not passing through z_0,

$n(z_0,\gamma)\overset{\text{def}}{=}\frac{1}{2\pi\mathrm{i}}\int_\gamma\frac{1}{z-z_0}dz$ shows how many turns around z_0, γ makes.

D7 Definition. Let D be an open set .

A closed path $\gamma:[a,b]\to D$ is called *homotopic to zero in D* if γ can be continuosly deformed inside D up to a constant path (a path having as image just a point).

T6 Theorem of Residues. For an open set $D\subseteq\mathbb{C}$:

$f:D\longrightarrow\mathbb{C}$ holomorphic function, S set of isolated singular points, γ path homotopic to zero in $D\cup S$ $\Rightarrow$ $\int_\gamma f(z)dz==2\pi\mathrm{i}\sum\limits_{z\in S}n(z,\gamma)\mathbf{Res}_z f$

T7 Lemma 1. For $\gamma_r:[\alpha,\beta|\to\mathbb{C},\ \gamma_r(t)=r\,\mathrm{e}^{\mathrm{i}t}$

$\lim\limits_{z\to\infty}z\,f(z)=0\ \Rightarrow\ \lim\limits_{r\to\infty}\int_{\gamma_r}f(z)\,dz=0$

SOLUTIONS

Problem 1

$\boxed{1a}$ By using D1, we get

$$(2-3\mathrm{i})^2=4-12\mathrm{i}-9=-5-12\mathrm{i},$$
$$\overline{(1+\mathrm{i})}=1-\mathrm{i},$$
$$\frac{2-\mathrm{i}}{1+\mathrm{i}}=\frac{(1-\mathrm{i})(2-\mathrm{i})}{(1-\mathrm{i})(1+\mathrm{i})}=\frac{1-3\mathrm{i}}{2},$$
$$\mathrm{i}^9=\mathrm{i}^8\,\mathrm{i}=(\mathrm{i}^4)^2\,\mathrm{i}=\mathrm{i},$$
$$|3+4\mathrm{i}|=\sqrt{3^2+4^2}=5,$$

whence

$$(2-3\mathrm{i})^2+\overline{(1+\mathrm{i})}+\frac{1-3\mathrm{i}}{1+\mathrm{i}}+\mathrm{i}^9+|3+4\mathrm{i}|=\frac{3-27\mathrm{i}}{2}.$$

$\boxed{1b}$ By using D1, we get

$$\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}=\cos\frac{\pi}{4}+\mathrm{i}\,\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}+\mathrm{i}\frac{\sqrt{2}}{2},$$
$$\mathrm{e}^{2-\mathrm{i}\frac{\pi}{3}}=\mathrm{e}^2\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}=\mathrm{e}^2\,(\cos\frac{\pi}{3}-\mathrm{i}\sin\frac{\pi}{3})=\frac{\mathrm{e}^2}{2}-\mathrm{i}\frac{\mathrm{e}^2\sqrt{3}}{2},$$
$$\cos(2\mathrm{i})=\frac{\mathrm{e}^{-2}+\mathrm{e}^2}{2},$$

whence

$$\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}+\mathrm{e}^{2-\mathrm{i}\frac{\pi}{3}}+\cos(2\mathrm{i})=\frac{\sqrt{2}+2\mathrm{e}^2-\mathrm{e}^{-2}}{2}+\mathrm{i}\frac{\sqrt{2}-\mathrm{e}^2\sqrt{3}}{2}.$$

$\boxed{1c}$ By using D1, we obtain

$$(z^4+\frac{1}{z^2}-\cos z+z^3\mathrm{e}^{-\mathrm{i}z})'=4z^3-\frac{2}{z^3}+\sin z+3z^2\mathrm{e}^{-\mathrm{i}z}-\mathrm{i}z^3\mathrm{e}^{-\mathrm{i}z}.$$

$\boxed{1d}$ By using D1, we get

$$\log(2-2\,\mathrm{i})=\ln|2-2\,\mathrm{i}|+\mathrm{i}\arg(2-2\,\mathrm{i})=\ln(2\sqrt{2})-\mathrm{i}\frac{\pi}{4}.$$

Problem 2

$\boxed{2a}$ We use T5 and D1. The singular points are 0 and i.

$z_0=0$ is a pole of order 1, and consequently

$$\mathbf{Res}_0 f=\frac{1}{0!}\lim_{z\to 0} z f(z)=\lim_{z\to 0} z\frac{1}{z(z-\mathrm{i})^2}$$
$$=\lim_{z\to 0}\frac{1}{(z-\mathrm{i})^2}=\frac{1}{(-\mathrm{i})^2}=-1.$$

$z_1=\mathrm{i}$ is a pole of order 2, and consequently

$$\mathbf{Res}_{\mathrm{i}} f=\frac{1}{1!}\lim_{z\to \mathrm{i}}\left((z-\mathrm{i})^2 f(z)\right)'=\lim_{z\to \mathrm{i}}\left((z-\mathrm{i})^2\frac{1}{z(z-\mathrm{i})^2}\right)'$$
$$=\lim_{z\to \mathrm{i}}\left(\frac{1}{z}\right)'=\lim_{z\to \mathrm{i}}\left(-\frac{1}{z^2}\right)=-\frac{1}{\mathrm{i}^2}=1.$$

$\boxed{2b}$ We use T2 and T4. The only singular point is π.
It is not a pole. So, we have to use the Laurent series of g.
By looking for a representation of $z+\mathrm{i}$ of the form

$$z+\mathrm{i}=\alpha(z-\pi)+\beta$$

we get

$$z+\mathrm{i}=(z-\pi)+\mathrm{i}+\pi.$$

$\mathbf{Res}_\pi g$ is the coefficient of $\frac{1}{z-\pi}$ from the Laurent expansion of g around the point π.
By direct computation, we get

$$g(z)=(z+\mathrm{i})\mathrm{e}^{\frac{1}{z-\pi}}=[(z-\pi)+\mathrm{i}+\pi]$$
$$\times\left[1+\frac{1}{1!}\frac{1}{z-\pi}+\frac{1}{2!}\frac{1}{(z-\pi)^2}+\frac{1}{3!}\frac{1}{(z-\pi)^3}+\frac{1}{4!}\frac{1}{(z-\pi)^4}+...\right]$$
$$=...+\left(\frac{1}{2!}+\mathrm{i}+\pi\right)\frac{1}{z-\pi}+...$$

Consequently, $\mathbf{Res}_\pi g=\frac{1}{2}+\pi+\mathrm{i}$.

Problem 3

$\boxed{3a}$ By using the definition D4 of the complex integral we get

$$I_0=\int_{\gamma_0}(1-2\bar{z})dz=\int_0^1[1-2(\overline{1-t+t\mathrm{i}})](1-t+t\mathrm{i})'dt$$
$$=(\mathrm{i}-1)\int_0^1(-1+2t+2t\mathrm{i})dt=(\mathrm{i}-1)(-t+t^2+t^2\mathrm{i})|_0^1$$
$$=(\mathrm{i}-1)(-1+1+\mathrm{i})=-1-\mathrm{i}.$$

$\boxed{3b}$ Through the identification $\mathbb{C}\to\mathbb{R}^2:\ x+y\mathrm{i}\mapsto(x,y)$, the image of the path γ_1 corresponds to the ellipsis

$$\frac{x^2}{9}+\frac{y^2}{4}=1$$

The singular points are 0 and πi, but only 0 is inside the ellipsis, and it is a pole of order 1.
By using the theorem of residues T6, we get.

$$I_1=\int_{\gamma_1}\frac{1}{z(z-\pi\mathrm{i})^2}dz=2\pi\mathrm{i}\mathbf{Res}_0\frac{1}{z(z-\pi\mathrm{i})^2}$$
$$=2\pi\mathrm{i}\lim_{z\to 0} z\frac{1}{z(z-\pi\mathrm{i})^2}=2\pi\mathrm{i}\lim_{z\to 0}\frac{1}{(z-\pi\mathrm{i})^2}=-\frac{2}{\pi}\mathrm{i}.$$

Problem 4

$\boxed{4a}$ We have

$$I=\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx=\arctan x\Big|_{-\infty}^{\infty}=\frac{\pi}{2}-(-\frac{\pi}{2})=\pi.$$

$\boxed{4b}$ The integral is convergent: at infinity the function $x\mapsto\frac{1}{1+x^2}$ has the same behaviour as $x\mapsto\frac{1}{x^2}$ because

$$\lim_{x\to\infty}\frac{\frac{1}{1+x^2}}{\frac{1}{x^2}}=1.$$

We consider the the function

$$f:\mathbb{C}\backslash\{\mathrm{i},-\mathrm{i}\}\to\mathbb{C},\quad f(z)=\frac{1}{1+z^2}$$

whose singular isolated points are $z_0=\mathrm{i}$ and $z_1=-\mathrm{i}$.
We use the theorem of residues T6 and the lemma T7.
By integrating the function f along the closed path obtained by composing

$$\Gamma_r:[0,\pi]\longrightarrow\mathbb{C},\ \Gamma_r(t)=r\,\mathrm{e}^{\mathrm{i}t}$$

and

$$\gamma_r:[-r,r]\longrightarrow\mathbb{C},\ \gamma_r(t)=t,$$

we get for $r>1$ the relation

$$\int_{\Gamma_r}f(z)dz+\int_{-r}^r f(x)dx=2\pi\mathrm{i}\,\mathbf{Res}_{\mathrm{i}}f.\qquad(*)$$

It is known that, for any complex numbers z_1 and z_2, we have the inequality

$$|\,|z_1|-|z_2|\,|\le|z_1-z_2|.$$

From the relation

$$0\le|z\,f(z)|=\frac{|z|}{|1+z^2|}=\frac{|z|}{|1-(-z^2)|}\le\frac{|z|}{|\,|1|-|-z^2|\,|}=\frac{|z|}{|\,1-|z|^2\,|}$$

and

$$\lim_{z\to\infty}\frac{|z|}{|\,1-|z|^2\,|}=0$$

it follows that

$$\lim_{z\to\infty}z\,f(z)=0$$

and in view of lemma 1,

$$\lim_{r\to\infty}\int_{\Gamma_r}f(z)dz=0.$$

Consequently, for $r\to\infty$ the relation (*) becomes

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx=2\pi\mathrm{i}\,\mathbf{Res}_{\mathrm{i}}f.$$

The point $z_0=\mathrm{i}$ is a pole of order 1. By using T5, we obtain

$$\mathbf{Res}_{\mathrm{i}}f=\mathbf{Res}_{\mathrm{i}}\frac{1}{1+z^2}=\lim_{z\to\mathrm{i}}(z-\mathrm{i})\frac{1}{(z-\mathrm{i})(z+\mathrm{i})}=\frac{1}{2\mathrm{i}},$$

and

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx=\pi.$$