Continuous versus Discrete Quantum Systems

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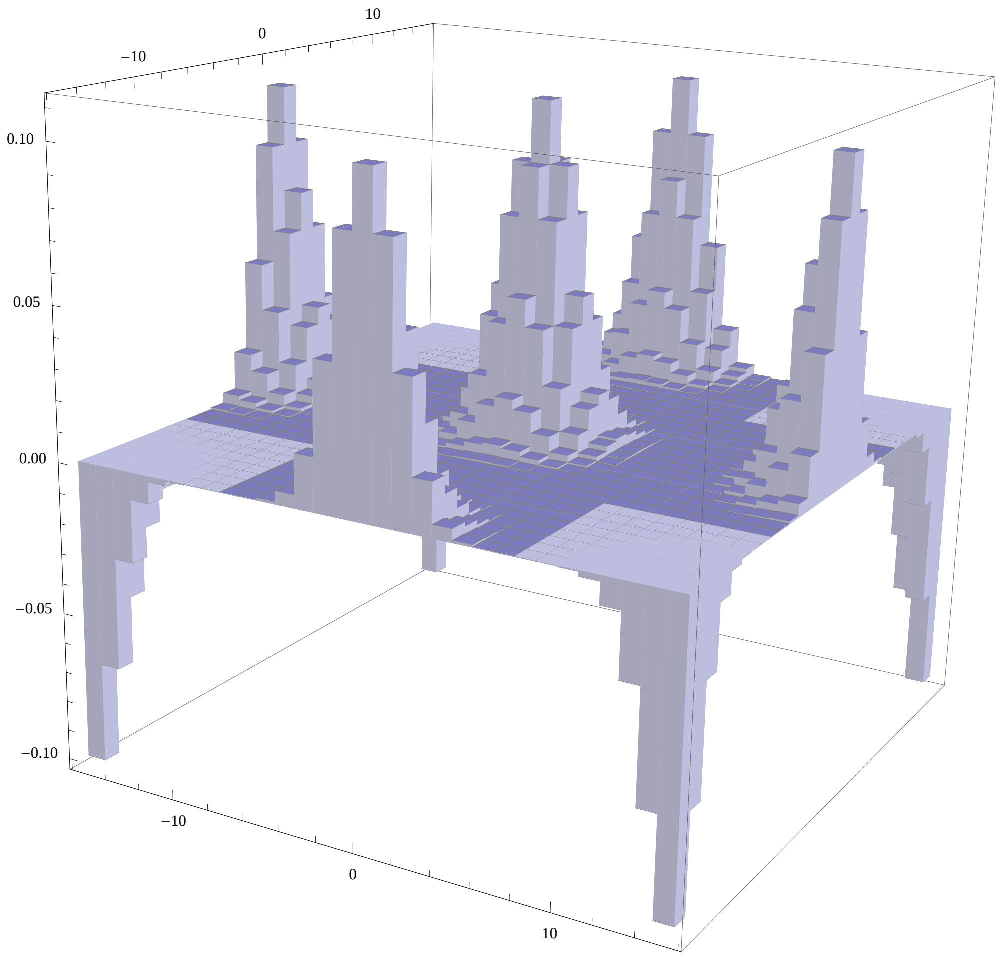
(for future updates see "Documente" at https://unibuc.ro/user/nicolae.cotfas/)



Hilbert space $L^2(\mathbb{R})$

• With the integral in Lebesgue sense,
$$\mathfrak{L}^2(\mathbb{R})=\left\{\psi:\mathbb{R}\longrightarrow\mathbb{C}\ \middle|\ \int_{-\infty}^{\infty}|\psi(q)|^2\,dq<\infty\right\}$$
is a vector space, and
$$\mathfrak{L}^2(\mathbb{R})\times\mathfrak{L}^2(\mathbb{R})\longrightarrow\mathbb{C}:\quad (\varphi,\psi)\mapsto\langle\varphi,\psi\rangle=\int_{-\infty}^{\infty}\overline{\varphi(q)}\,\psi(q)\,dq,$$
satisfies the relations
$$\langle\varphi,\psi\rangle=\overline{\langle\psi,\varphi\rangle},$$
$$\langle\varphi,\alpha\,\psi_1+\beta\,\psi_2\rangle=\alpha\langle\varphi,\psi_1\rangle+\beta\langle\varphi,\psi_2\rangle,$$
but is not a scalar product since
$$\langle\psi,\psi\rangle=0\quad\not\Rightarrow\quad\psi=0.$$
The relation
$$\varphi\sim\psi\quad\Leftrightarrow\quad\int_{-\infty}^{\infty}|\varphi(q)-\psi(q)|\,dq=0.$$
is an equivalence relation on $\mathfrak{L}^2(\mathbb{R})$, and
$$L^2(\mathbb{R})=\left\{\psi:\mathbb{R}\to\mathbb{C}\ \middle|\ \int_{-\infty}^{\infty}|\psi(q)|^2\,dq<\infty\right\}\Big/\sim$$
is a Hilbert space. Each element of $L^2(\mathbb{R})$ is an infinite class of equivalent functions.

• All the functions of the form
$$\psi(q)=\begin{cases}\alpha & \text{for}\quad q=q_0,\\ \frac{1}{\sqrt[4]{\pi}}\,\mathrm{e}^{-\frac{q^2}{2}} & \text{for}\quad q\neq q_0,\end{cases}$$
represent the same element of $L^2(\mathbb{R})$.

• The Hermite-Gauss functions $\Psi_0,\ \Psi_1,\ \Psi_2,\dots$
$$\Psi_n(q)=\frac{1}{\sqrt{2^n\,n!\,\sqrt{\pi}}}\,H_n(q)\,\mathrm{e}^{-\frac{q^2}{2}},$$
form an orthonormal basis in $L^2(\mathbb{R})$.Writing
$$\begin{cases}|\psi\rangle & \text{instead of}\quad\psi,\\ |n\rangle & \text{instead of}\quad\Psi_n,\end{cases}$$
we have $\langle n|k\rangle=\delta_{nk}$, and
$$|\psi\rangle=\sum_{n=0}^{\infty}|n\rangle\langle n|\psi\rangle,\quad\text{for all }|\psi\rangle\in L^2(\mathbb{R}),$$
that is, the resolution of the identity
$$\mathbb{I}=\sum_{n=0}^{\infty}|n\rangle\langle n|.$$

Hilbert space $\mathbb{C}^d$ (odd $d=2j+1$)

• In the case of $\mathbb{C}^d$, the scalar product
$$\langle x,y\rangle=\sum_{n=0}^{d-1}\bar{x}_n\,y_n=(\bar{x}_0\ \bar{x}_1\ \cdots\ \bar{x}_{d-1})\begin{pmatrix}y_0\\y_1\\\vdots\\y_{d-1}\end{pmatrix}$$
can be written as
$$\langle x,y\rangle=\langle x|y\rangle$$
by using the Dirac notations
$$\langle x|=(\bar{x}_0\ \bar{x}_1\ \cdots\ \bar{x}_{d-1}),\qquad |y\rangle=\begin{pmatrix}y_0\\y_1\\\vdots\\y_{d-1}\end{pmatrix}.$$

• If we consider
$$\mathbb{C}^d=\left\{|x\rangle=\begin{pmatrix}x_0\\x_1\\\vdots\\x_{d-1}\end{pmatrix}\ \middle|\ x_0,\ x_1,\ \dots\, x_{d-1}\in\mathbb{C}\right\}$$
then $\langle a|$ can be regarded as the linear map
$$\mathbb{C}^d\longrightarrow\mathbb{C}:\ \ |x\rangle\mapsto\langle a|x\rangle$$
and $|a\rangle\langle b|$ as the linear operator
$$\mathbb{C}^d\longrightarrow\mathbb{C}^d:\ \ |x\rangle\mapsto|a\rangle\langle b|x\rangle.$$

• If $\{|e_0\rangle,|e_1\rangle,\dots,|e_{d-1}\rangle\}$ is an orthonormal basis, then we have
$$\langle e_n|e_k\rangle=\delta_{nk},$$
$$|x\rangle=\sum_{n=0}^{d-1}|e_n\rangle\langle e_n|x\rangle,\quad\text{for all }|x\rangle\in\mathbb{C}^d,$$
that is, the resolution of the identity
$$\mathbb{I}=\sum_{n=0}^{d-1}|e_n\rangle\langle e_n|.$$

• In the case of canonical basis from $\mathbb{C}^2$:
$$\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}(1\ \ 0)+\begin{pmatrix}0\\1\end{pmatrix}(0\ \ 1).$$

Hilbert space $L^2(\mathbb{R})$

• We get the relation

$$\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$$

rapidly decreasing (test) functions — tempered distributions

by identifying $\varphi \in L^2(\mathbb{R})$ with the distribution

$$\varphi : \mathcal{S}(\mathbb{R}) \longrightarrow \mathbb{C}, \quad \langle \varphi, \psi \rangle = \int_{-\infty}^{\infty} \varphi(q)\, \psi(q)\, dq,$$

Two equivalent functions $\varphi \sim \psi$ define the same distribution $\varphi = \psi$ in $\mathcal{S}'(\mathbb{R})$.

• The definition of Dirac distribution

$$\delta_a : \mathcal{S}(\mathbb{R}) \longrightarrow \mathbb{C}, \quad \langle \delta_a, \psi \rangle = \psi(a),$$

can be formally (symbolically) written as

$$\int_{-\infty}^{\infty} dq\, \delta(q-a)\, \psi(q) = \psi(a)$$

or

$$\int_{-\infty}^{\infty} dq\, \delta(a-q)\, \psi(q) = \psi(a)$$

or

$$\int_{-\infty}^{\infty} dq\, \delta_q(a)\, \psi(q) = \psi(a)$$

or

$$\psi = \int_{-\infty}^{\infty} dq\, \psi(q)\, \delta_q. \qquad (*)$$

• Since $\psi(q) = \langle \delta_q, \psi \rangle$, by writing

$$\begin{cases} |\psi\rangle & \text{instead of } \psi, \\ |q\rangle & \text{instead of } \delta_q, \end{cases}$$

the relation (*) becomes

$$|\psi\rangle = \int_{-\infty}^{\infty} dq\, |q\rangle\langle q|\psi\rangle$$

and the system $\{|q\rangle\}_{q=-\infty}^{\infty}$ satisfies the formal (symbolical) relation

$$\mathbb{I} = \int_{-\infty}^{\infty} dq\, |q\rangle\langle q|.$$

• Here, $\langle q|$ represents the linear functional

$$C^0(\mathbb{R}) \longrightarrow \mathbb{C} : |\psi\rangle \mapsto \langle q|\psi\rangle = \psi(q).$$

• Formally, we have

$$\langle a|b\rangle = \delta(a-b) \quad \text{for any } a, b \in \mathbb{R}.$$

Hilbert space $\mathbb{C}^d$ (odd $d=2j+1$)

• Consider $\mathbb{C}^d$ as the space of functions

$$\mathbb{C}^d = \{\, \psi : \{-j, -j+1, ..., j-1, j\} \longrightarrow \mathbb{C}\,\},$$

with the scalar product

$$\langle \varphi, \psi \rangle = \sum_{n=-j}^{j} \overline{\varphi(n)}\, \psi(n).$$

• The functions

$$\delta_{-j}, \delta_{-j+1}, ..., \delta_j : \{-j, -j+1, ..., j-1, j\} \to \mathbb{C},$$

$$\delta_m(k) = \delta_{km} = \begin{cases} 1 & \text{for } k=m, \\ 0 & \text{for } k \neq m. \end{cases}$$

form an orthonormal basis :

$$\langle \delta_m, \delta_n \rangle = \delta_{mn},$$

$$\psi = \sum_{m=-j}^{j} \psi(m)\, \delta_m, \quad \text{for all } \psi.$$

• Since $\psi(m) = \langle \delta_m, \psi \rangle$, by writing

$$\begin{cases} |\psi\rangle & \text{instead of } \psi, \\ |j;m\rangle & \text{instead of } \delta_m, \end{cases}$$

we have

$$\langle j;m|j;n\rangle = \delta_{mn},$$

$$|\psi\rangle = \sum_{m=-j}^{j} |j;m\rangle\langle j;m|\psi\rangle, \quad \text{for all } |\psi\rangle,$$

that is, the resolution of the identity

$$\mathbb{I} = \sum_{m=-j}^{j} |j;m\rangle\langle j;m|.$$

Hilbert space $L^2(\mathbb{R})$

• **Fourier transform of a function**
$$\psi:\mathbb{R}\longrightarrow\mathbb{C}$$
is the function $\quad F[\psi]:\mathbb{R}\longrightarrow\mathbb{C},$
$$F[\psi](p)=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}pq}\,\psi(q)\,dq$$
(if it exists !). The transform
$$F:\mathcal{S}(\mathbb{R})\longrightarrow\mathcal{S}(\mathbb{R})$$
is one-to-one and $F^{-1}=F^{+}$, that is
$$F^{-1}[\psi](p)=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq}\,\psi(q)\,dq.$$

• Particularly, $\psi=F^{-1}[F[\psi]]$, that is
$$\psi(q)=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq}\,F[\psi](p)\,dp.$$

• **Fourier transform of a distribution**
$$\psi:\mathcal{S}(\mathbb{R})\longrightarrow\mathbb{C}$$
is the distribution $\quad F[\psi]:\mathcal{S}(\mathbb{R})\longrightarrow\mathbb{C},$
$$\langle F[\psi],\varphi\rangle=\langle\psi,F[\varphi]\rangle.$$
The transform
$$F:\mathcal{S}'(\mathbb{R})\longrightarrow\mathcal{S}'(\mathbb{R})$$
is one-to-one, and
$$\langle F^{-1}[\psi],\varphi\rangle=\langle\psi,F^{-1}[\varphi]\rangle.$$

• The Hermite-Gauss functions $\Psi_0,\ \Psi_1,\ \Psi_2,\ldots$
$$\Psi_n(q)=\frac{1}{\sqrt{2^n\,n!\,\sqrt{\pi}}}\,H_n(q)\,\mathrm{e}^{-\frac{q^2}{2}},$$
are eigenfunctions of the Fourier transform:
$$F[\Psi_n]=(-\mathrm{i})^n\Psi_n.$$
Writting $|n\rangle$ instead of Ψ_n, we have
$$F|n\rangle=(-\mathrm{i})^n|n\rangle.$$

• **Fourier transform** $F:L^2(\mathbb{R})\longrightarrow L^2(\mathbb{R})$
$$F=\sum_{n=0}^{\infty}(-\mathrm{i})^n|n\rangle\langle n|.$$

• **Fractional Fourier transform**
$$F^{\alpha}=\sum_{n=0}^{\infty}(-\mathrm{i})^{n\alpha}|n\rangle\langle n|.$$

Hilbert space $\mathbb{C}^d$ (odd $d=2j+1$)

• We identify the space
$$\mathbb{C}^d=\{\,\psi:\{-j,-j+1,\ldots,j-1,j\}\longrightarrow\mathbb{C}\,\}$$
with the space of periodic functions
$$\{\,\psi:\mathbb{Z}\longrightarrow\mathbb{C}\mid\psi(n)=\psi(n+d),\ \forall n\}$$
that is, we consider
$$\mathbb{C}^d\equiv\ell^2(\mathbb{Z}_d)=\{\,\psi:\mathbb{Z}_d\longrightarrow\mathbb{C}\,\},$$
where $\qquad\mathbb{Z}_d=$ set of classes modulo d.
We use the scalar product
$$\langle\varphi,\psi\rangle=\sum_{n\in\mathbb{Z}_d}\overline{\varphi(n)}\,\psi(n),$$
the canonical basis $\{|j;m\rangle\equiv\delta_m\}_{n\in\mathbb{Z}_d}$, where
$$\delta_m(n)=\delta^{[d]}_{mn}=\begin{cases}1 & \text{for}\quad n=m\ (\mathrm{mod}\ d),\\ 0 & \text{for}\quad n\neq m\ (\mathrm{mod}\ d),\end{cases}$$

• **Finite Fourier transform** of a function
$$\psi:\mathbb{Z}_d\longrightarrow\mathbb{C}$$
is the function $\qquad\mathbf{F}[\psi]:\mathbb{Z}_d\longrightarrow\mathbb{C},$
$$\mathbf{F}[\psi](k)=\frac{1}{\sqrt{d}}\sum_{n\in\mathbb{Z}_d}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}kn}\,\psi(n).$$
The transform
$$\mathbf{F}:\ell^2(\mathbb{Z}_d)\longrightarrow\ell^2(\mathbb{Z}_d)$$
is one-to-one and $\mathbf{F}^{-1}=\mathbf{F}^{+}$, that is
$$\mathbf{F}^{-1}[\psi](k)=\frac{1}{\sqrt{d}}\sum_{n\in\mathbb{Z}_d}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kn}\psi(n).$$

• Particularly, $\psi=\mathbf{F}^{-1}[\mathbf{F}[\psi]]$, that is
$$\psi(n)=\frac{1}{\sqrt{d}}\sum_{k\in\mathbb{Z}_d}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kn}\,\mathbf{F}[\psi](k).$$

• **Finite fractional Fourier transform**
$$\mathbf{F}^{\alpha}=\sum_{n=0}^{d-1}(-\mathrm{i})^{n\alpha}|\psi_n\rangle\langle\psi_n|$$
is defined by using a finite counterpart $\{\psi_n\}_{n=0}^{d-1}$ of the Hermite-Gauss functions, defined as the eigenfunctions of a Fourier invariant Hamiltonian (finite oscillator).

Hilbert space $L^2(\mathbb{R})$

• **Position and momentum** operators

$$\psi \mapsto Q\psi, \quad (Q\psi)(q) = q\,\psi(q)$$

$$\psi \mapsto P\psi, \quad (P\psi)(q) = -\mathrm{i}\hbar\frac{d\psi}{dq}(q)$$

admit the extensions

$$Q\!:\mathcal{S}'(\mathbb{R}) \longrightarrow \mathcal{S}'(\mathbb{R}) : \psi \mapsto q\,\psi,$$

$$P\!:\mathcal{S}'(\mathbb{R}) \longrightarrow \mathcal{S}'(\mathbb{R}) : \psi \mapsto -\mathrm{i}\hbar\psi',$$

satisfying the relation

$$P = F^{-1}QF,$$

which for $\psi\in\mathcal{S}(\mathbb{R})$ becomes

$$(P\psi)(q) = \frac{1}{2\pi\hbar}\int_{\mathbb{R}^2} x\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}xy}\,\psi(q-y)dx\,dy.$$

• We have

$$q\delta_a = a\delta_a,$$

that is

$$Q|a\rangle = a|a\rangle.$$

• From the relation

$$\langle F^{-1}[\delta_p],\varphi\rangle = \langle\delta_p, F^{-1}[\varphi]\rangle = F^{-1}[\varphi](p)$$

$$= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq}\,\varphi(q)\,dq = \frac{1}{\sqrt{2\pi\hbar}}\langle\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq},\varphi\rangle$$

it follows that

$$F^{-1}[\delta_p] = \frac{1}{\sqrt{2\pi\hbar}}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq}.$$

• The distribution

$$|p\rangle\!\rangle = F^{-1}|p\rangle = F^{-1}[\delta_p]$$

satisfies the relations

$$P|p\rangle\!\rangle = p\,|p\rangle\!\rangle, \qquad \langle q|p\rangle\!\rangle = \frac{1}{\sqrt{2\pi\hbar}}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq}$$

and formally: $\langle\!\langle a|b\rangle\!\rangle = \delta(a-b)$,

$$\mathbb{I} = \int_{-\infty}^{\infty} dq\,|q\rangle\langle q|, \qquad \mathbb{I} = \int_{-\infty}^{\infty} dp\,|p\rangle\!\rangle\langle\!\langle p|,$$

$$Q = \int_{-\infty}^{\infty} dq\,q|q\rangle\langle q|, \qquad P = \int_{-\infty}^{\infty} dp\,p|p\rangle\!\rangle\langle\!\langle p|,$$

$$\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(a-b)q}dq = \delta(a-b).$$

• We have

$$[Q,P] = \mathrm{i}\hbar\mathbb{I}.$$

Hilbert space $\mathbb{C}^d$ (odd $d=2j+1$)

• We consider $\mathbb{C}^d \equiv \ell^2(\mathbb{Z}_d) = \{\,\psi:\mathbb{Z}_d \longrightarrow \mathbb{C}\,\}$, where

$$\mathbb{Z}_d = \text{set of classes modulo } d,$$

and use the scalar product

$$\langle\varphi,\psi\rangle = \sum_{n\in\mathbb{Z}_d}\overline{\varphi(n)}\,\psi(n),$$

the canonical basis $\{|j;m\rangle \equiv \delta_m\}_{n\in\mathbb{Z}_d}$, where

$$\delta_m(n) = \delta^{[d]}_{mn} = \begin{cases} 1 & \text{for } n=m \pmod d, \\ 0 & \text{for } n\neq m \pmod d, \end{cases}$$

and the Fourier transform $\mathbf{F}^{\pm1}\!:\mathbb{C}^d \longrightarrow \mathbb{C}^d$,

$$\mathbf{F}^{\pm1}[\psi](k) = \frac{1}{\sqrt{d}}\sum_{n=-j}^{j}\mathrm{e}^{\mp\frac{2\pi\mathrm{i}}{d}kn}\,\psi(n).$$

• **Position and momentum** operators are

$$\mathbf{Q}\!:\mathbb{C}^d \longrightarrow \mathbb{C}^d, \qquad (\mathbf{Q}\psi)(n) = \tilde{n}\,\psi(n),$$

where

$$\tilde{n}\in\{-j,-j+1,\ldots,j-1,j\},$$

is the representative mod(d) of n, and

$$\mathbf{P}\!:\mathbb{C}^d \longrightarrow \mathbb{C}^d, \qquad \mathbf{P} = \mathbf{F}^{-1}\mathbf{QF},$$

that is

$$\mathbf{P}\psi(n) = \frac{1}{d}\sum_{k,m=-j}^{j}k\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}km}\psi(n-m).$$

• We have

$$\mathbf{Q}\delta_m = \tilde{m}\delta_m,$$

that is

$$\mathbf{Q}|j;m\rangle = \tilde{m}|j;m\rangle.$$

• The function

$$|j;k\rangle\!\rangle = \mathbf{F}^{-1}|j;k\rangle = \mathbf{F}^{-1}[\delta_k]$$

satisfies the relations

$$\mathbf{P}|j;k\rangle\!\rangle = \tilde{k}\,|j;k\rangle\!\rangle, \quad \langle j;m|j;k\rangle\!\rangle = \frac{1}{\sqrt{d}}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}km}$$

and

$$\mathbb{I} = \sum_{m=-j}^{j}|j;m\rangle\langle j;m|, \qquad \mathbb{I} = \sum_{m=-j}^{j}|j;k\rangle\!\rangle\langle\!\langle j;k|,$$

$$\mathbf{Q} = \sum_{m=-j}^{j}m|j;m\rangle\langle j;m|, \qquad \mathbf{P} = \sum_{k=-j}^{j}k|j;k\rangle\!\rangle\langle\!\langle j;k|.$$

• We have

$$[\mathbf{Q},\mathbf{P}]\psi(n) = \frac{1}{d}\sum_{m,k=-j}^{j}m\,k\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}mk}\,\psi(n-m).$$

Hilbert space $L^2(\mathbb{R})$

- **Position and momentum** operators
$$\psi \mapsto Q\psi, \quad (Q\psi)(q) = q\,\psi(q)$$
$$\psi \mapsto P\psi, \quad P = -\mathrm{i}\hbar\tfrac{d}{dq} = F^{-1}QF.$$
- **Translation operators**. For any $q, p\in\mathbb{R}$,
$$\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pQ}\psi(q)=\left(1+\tfrac{1}{1!}\tfrac{\mathrm{i}p}{\hbar}Q+\tfrac{1}{2!}\tfrac{(\mathrm{i}p)^2}{\hbar^2}Q^2+...\right)\psi(q)$$
$$=\left(1+\tfrac{1}{1!}\tfrac{\mathrm{i}p}{\hbar}q+\tfrac{1}{2!}\tfrac{(\mathrm{i}p)^2}{\hbar^2}q^2+...\right)\psi(q)=\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq}\,\psi(q),$$
$$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}qP}\psi(a)=\left(1+\tfrac{-q}{1!}\tfrac{d}{dq}+\tfrac{(-q)^2}{2!}\tfrac{d^2}{dq^2}+...\right)\psi(a)$$
$$=\psi(a)-\tfrac{\psi'(a)}{1!}q+\tfrac{\psi''(a)}{2!}q^2-...=\psi(a-q),$$
that is
$$\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pQ}\,\psi(q)=\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq}\psi(q),$$
$$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}qP}\psi(a)=\psi(a-q).$$
- We have
$$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}qP}|a\rangle=|a+q\rangle,$$
$$\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pQ}|a\rangle=\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pa}|a\rangle$$
$$\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pQ}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}qP}=\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}qP}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pQ}$$
and
$$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}qP}=F^{-1}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}qQ}F = F\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}qQ}\,F^{-1}$$
- **Displacement operators**
$$D(q,p)=\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(qP-pQ)} = \mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}pq}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pQ}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}qP} = \mathrm{e}^{\frac{\mathrm{i}}{2\hbar}pq}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}qP}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pQ}$$
satisfy the relations:
$$D^+(q,p)=D(-q,-p)=D^{-1}(q,p),$$
$$D(q,p)\psi(a)=\mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}pq}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pa}\,\psi(a-q),$$
$$\langle a|D(q,p)|b\rangle=\mathrm{e}^{\frac{\mathrm{i}}{2\hbar}p(a+b)}\,\delta(a-b-q),$$
and
$$D(q,p)\,D(a,b)=\mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}(qb-pa)}\,D(q+a,p+b).$$

Hilbert space $\mathbb{C}^d$ (odd $d=2j+1$)

- **Position and momentum** operators are
$$\mathbf{Q}:\mathbb{C}^d\longrightarrow\mathbb{C}^d, \qquad (\mathbf{Q}\psi)(n)=\tilde{n}\,\psi(n),$$
$$\mathbf{P}:\mathbb{C}^d\longrightarrow\mathbb{C}^d, \qquad \mathbf{P}=\mathbf{F}^{-1}\mathbf{QF}$$
where $\tilde{n}$ is the representative of n satisfying
$$\tilde{n}\in\{-j,-j+1,...,j-1,j\}.$$
- **Translation operators**. For any $n,k\in\mathbb{Z}_d$,
$$\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k\mathbf{Q}}\psi(n)=\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}kn}\psi(n),$$
$$\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}n\mathbf{P}}=\mathbf{F}^{-1}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}n\mathbf{Q}}\mathbf{F}=\mathbf{F}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}n\mathbf{Q}}\mathbf{F}^{-1}$$
satisfy the relations
$$\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}n\mathbf{P}}\psi(m)=\psi(m-n),$$
$$\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}n\mathbf{P}}|j;m\rangle=|j;m+n\rangle,$$
$$\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k\mathbf{Q}}|j;m\rangle=\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}km}|j;m\rangle,$$
$$\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k\mathbf{Q}}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}n\mathbf{P}}=\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}nk}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}n\mathbf{P}}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k\mathbf{Q}},$$
- **Displacement operators**
$$\mathbf{D}(n,k)=\mathrm{e}^{-\frac{\pi\mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k\mathbf{Q}}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}n\mathbf{P}} = \mathrm{e}^{\frac{\pi\mathrm{i}}{d}nk}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}n\mathbf{P}}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k\mathbf{Q}},$$
where
$(n,k)\in\{-j,-j+1,...,j\}\times\{-j,-j+1,...,j\}$,
satisfy the relations:
$$\mathbf{D}^+(n,k)=\mathbf{D}(-n,-k)=\mathbf{D}^{-1}(n,k),$$
$$\mathbf{D}(n,k)\psi(m)=\mathrm{e}^{-\frac{\pi\mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}km}\,\psi(m-n)$$
$$\langle j;m|\mathbf{D}(n,k)|j;\ell\rangle=\mathrm{e}^{\frac{\pi\mathrm{i}}{d}k(m+\ell)}\,\delta_{m,\ell+n}$$
$$\mathrm{tr}(\mathbf{D}^+(n,k)\,\mathbf{D}(m,\ell))=d\,\delta_{nm}\,\delta_{k\ell}$$
$$\frac{1}{d}\sum_{n,k=-j}^{j}\mathbf{D}(n,k)=\sum_{n=-j}^{j}|j;n\rangle\langle j;-n|$$
and
$$\mathbf{D}(n,k)\,\mathbf{D}(m,\ell)=\mathrm{e}^{-\frac{\pi\mathrm{i}}{d}(n\ell-km)}\,\mathbf{D}(n+m,k+\ell).$$
- $\mathbf{D}(n,k)$ is not modulo d invariant:
$$\mathbf{D}(n+d,k)=(-1)^k\,\mathbf{D}(n,k),$$
$$\mathbf{D}(n,k+d)=(-1)^n\,\mathbf{D}(n,k).$$

Hilbert space $L^2(\mathbb{R})$

- By writing

$$\begin{cases} |\psi\rangle \text{ instead of } \psi, \\ |q\rangle \text{ instead of the distribution } \delta_q, \\ |n\rangle \text{ instead of Hermite-Gauss function } \Psi_n, \end{cases}$$

we have

$$|\psi\rangle = \int_{-\infty}^{\infty} dq\, |q\rangle\langle q|\psi\rangle, \qquad |\psi\rangle = \sum_{n=0}^{\infty} |n\rangle\langle n|\psi\rangle,$$

$$\mathbb{I} = \int_{-\infty}^{\infty} dq\, |q\rangle\langle q|, \qquad \mathbb{I} = \sum_{n=0}^{\infty} |n\rangle\langle n|.$$

- If A, B satisfy certain conditions, then

$$\langle A, B\rangle = \mathrm{tr}\,(A^+B) = \sum_{n=0}^{\infty} \langle n|A^+B|n\rangle = \int_{-\infty}^{\infty} dq\, \langle q|A^+B|q\rangle.$$

- **Displacement operators**

$$D(q,p) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(qP-pQ)} = \mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}pq}\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}pQ}\, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}qP} = \mathrm{e}^{\frac{\mathrm{i}}{2\hbar}pq}\, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}qP}\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}pQ}$$

satisfy the relations

$$D(q,p)\psi(a) = \mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}pq}\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}pa}\, \psi(a-q),$$

$$\langle a|D(q,p)|b\rangle = \mathrm{e}^{\frac{\mathrm{i}}{2\hbar}p(a+b)}\, \delta(a-b-q).$$

- **Parity operators**

$$\Pi(q,p) = D(q,p)\, \Pi\, D(q,p)^+$$

where

$$\Pi\psi(q) = \psi(-q),$$

satisfy the relations

$$\Pi(q,p)\psi(a) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}2p(q-a)}\, \psi(2q-a),$$

$$\langle a|\Pi(q,p)|b\rangle = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}2p(q-a)}\, \delta_b(2q-a) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}p(a-b)}\, \delta(2q-a-b).$$

- We have

$$\Pi(q,p) = \frac{1}{2\pi\hbar} \iint_{\mathbb{R}^2} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(qb-pa)} D(a,b)\, da db.$$

Hilbert space $\mathbb{C}^d$ (odd $d=2j+1$)

- If $A, B : \mathbb{C}^d \longrightarrow \mathbb{C}^d$ are two operators, then

$$\langle A, B\rangle = \sum_{n,m=-j}^{j} \overline{\langle j;n|A|j;m\rangle}\, \langle j;n|B|j;m\rangle = \sum_{n,m=-j}^{j} \langle j;m|A^+|j;n\rangle\, \langle j;n|B|j;m\rangle = \sum_{m=-j}^{j} \langle j;m|A^+B|j;m\rangle = \mathrm{tr}\,(A^+B).$$

- **Discrete displacement operators** $\mathbf{D}(n,k)$ are *unitary operators*, and

$$\mathrm{tr}(\mathbf{D}^+(n,k)\, \mathbf{D}(m,\ell)) = d\, \delta_{nm}\, \delta_{k\ell},$$

that is

$$\langle \mathbf{D}(n,k),\, \mathbf{D}(m,\ell)\rangle = d\, \delta_{nm}\, \delta_{k\ell}.$$

Consequently $\left\{ \frac{1}{\sqrt{d}}\mathbf{D}(n,k) \right\}$ is an orthonormal basis in d^2-dimensional *complex Hilbert space* of all the linear operators $C : \mathbb{C}^d \longrightarrow \mathbb{C}^d$, and

$$C = \frac{1}{d} \sum_{n,k=-j}^{j} \mathrm{tr}(C\, \mathbf{D}^+(n,k))\, \mathbf{D}(n,k).$$

- **Discrete parity operators**

$$\Pi(n,k) = \mathbf{D}(n,k)\, \Pi\, \mathbf{D}(n,k)^+$$

where

$$\Pi\psi(n) = \psi(-n),$$

are *self-adjoint operators*, and

$$\Pi(n,k)\psi(m) = \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}2k(n-m)}\, \psi(2n-m),$$

$$\langle j;a|\Pi(n,k)|j;b\rangle = \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k(a-b)}\, \delta_{2n,a+b},$$

$$\mathrm{tr}(\Pi(n,k)\, \Pi(m,\ell)) = d\, \delta_{nm}\, \delta_{k\ell},$$

that is

$$\langle \Pi(n,k),\, \Pi(m,\ell)\rangle = d\, \delta_{nm}\, \delta_{k\ell}.$$

Consequently $\left\{ \frac{1}{\sqrt{d}}\Pi(n,k) \right\}$ is an orthonormal basis in d^2-dimensional *real Hilbert space* of all the self-adjoint operators $A : \mathbb{C}^d \longrightarrow \mathbb{C}^d$, and

$$A = \frac{1}{d} \sum_{n,k=-j}^{j} \mathrm{tr}(A\, \Pi(n,k))\, \Pi(n,k).$$

- We have

$$\Pi(n,k) = \frac{1}{d} \sum_{m,\ell=-j}^{j} \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}(n\ell-km)} \mathbf{D}(m,\ell).$$

Hilbert space $L^2(\mathbb{R})$

- We have:

$$\Pi(q,p)\psi(y) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}2p(q-y)}\,\psi(2q-y),$$

$$\langle A,B\rangle = \mathrm{tr}\,(A^+B) = \int_{-\infty}^{\infty} dq\,\langle q|A^+B|q\rangle.$$

- **Wigner function** of a pure state

$$\psi:\mathbb{R}\longrightarrow\mathbb{C}$$

with the density operator $\hat{\varrho} = |\psi\rangle\langle\psi|$ is

$$\begin{aligned}
W_\psi(q,p) &= \tfrac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dx\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}px}\,\overline{\psi(q-\tfrac{x}{2})}\,\psi(q+\tfrac{x}{2})\\
&= \tfrac{1}{\pi\hbar}\int_{-\infty}^{\infty} dx\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}2px}\,\overline{\psi(q-x)}\,\psi(q+x)\\
&= \tfrac{1}{\pi\hbar}\int_{-\infty}^{\infty} dx\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}2px}\,\overline{\langle q-x|\psi\rangle}\,\langle q+x|\psi\rangle\\
&= \tfrac{1}{\pi\hbar}\int_{-\infty}^{\infty} dx\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}2px}\,\langle q+x|\psi\rangle\langle\psi|q-x\rangle\\
&= \tfrac{1}{\pi\hbar}\int_{-\infty}^{\infty} dx\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}2px}\,\langle q+x|\hat{\varrho}|q-x\rangle.
\end{aligned}$$

We also have:

$$W_\psi(q,p) = \tfrac{1}{\pi\hbar}\int_{-\infty}^{\infty} dy\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}2qy}\,\overline{F[\psi](p-y)}\,F[\psi](p+y),$$

$$\begin{aligned}
W_\psi(q,p) &= \tfrac{1}{\pi\hbar}\int_{-\infty}^{\infty} dx\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}2px}\,\overline{\psi(q-x)}\,\psi(q+x)\\
&= \tfrac{1}{\pi\hbar}\int_{-\infty}^{\infty} dy\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}2p(q-y)}\,\overline{\psi(y)}\,\psi(2q-y)\\
&= \tfrac{1}{\pi\hbar}\int_{-\infty}^{\infty} dy\,\overline{\psi(y)}\,\Pi(q,p)\psi(y)\\
&= \tfrac{1}{\pi\hbar}\langle\psi|\Pi(q,p)|\psi\rangle,
\end{aligned}$$

$$\begin{aligned}
W_\psi(q,p) &= \tfrac{1}{\pi\hbar}\int_{-\infty}^{\infty} dy\,\overline{\psi(y)}\,\Pi(q,p)\psi(y)\\
&= \tfrac{1}{\pi\hbar}\int_{-\infty}^{\infty} dy\,\overline{\langle y|\psi\rangle}\,\langle y|\Pi(q,p)|\psi\rangle\\
&= \tfrac{1}{\pi\hbar}\int_{-\infty}^{\infty} dy\,\langle y|\Pi(q,p)|\psi\rangle\langle\psi|y\rangle\\
&= \tfrac{1}{\pi\hbar}\,\mathrm{tr}(\Pi(q,p)|\psi\rangle\langle\psi|)\\
&= \tfrac{1}{\pi\hbar}\,\mathrm{tr}(\Pi(q,p)\,\hat{\varrho}) = \tfrac{1}{\pi\hbar}\,\mathrm{tr}(\hat{\varrho}\,\Pi(q,p)).
\end{aligned}$$

- $W_{\hat{\varrho}}(q,p) = \frac{1}{\pi\hbar}\,\mathrm{tr}(\hat{\varrho}\,\Pi(q,p))$.

Hilbert space $\mathbb{C}^d$ (odd $d=2j+1$)

- In $\mathbb{C}^d \equiv \{\,\psi:\mathbb{Z}_d\longrightarrow\mathbb{C}\,\}$, we have:

$$\Pi(n,k)\psi(m) = \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}2k(n-m)}\,\psi(2n-m).$$

$$\begin{aligned}
\langle A,B\rangle &= \sum_{n,m=-j}^{j}\overline{\langle j;n|A|j;m\rangle}\,\langle j;n|B|j;m\rangle\\
&= \sum_{n,m=-j}^{j}\langle j;m|A^+|j;n\rangle\,\langle j;n|B|j;m\rangle\\
&= \sum_{m=-j}^{j}\langle j;m|A^+B|j;m\rangle = \mathrm{tr}\,(A^+B).
\end{aligned}$$

- If $\hat{\varrho} = |\psi\rangle\langle\psi|$, then

$$\begin{aligned}
\mathrm{tr}(\,\hat{\varrho}\,A) &= \mathrm{tr}(\,|\psi\rangle\langle\psi|\,A\,)\\
&= \sum_{m=-j}^{j}\langle j;m|\psi\rangle\langle\psi|\,A|j;m\rangle\\
&= \sum_{m=-j}^{j}\langle\psi|\,A|j;m\rangle\langle j;m|\psi\rangle\\
&= \langle\psi|\,A|\psi\rangle.
\end{aligned}$$

A version of discrete Wigner function

Hilbert space $L^2(\mathbb{R})$

• **Wigner function** of a pure state
$$\psi:\mathbb{R}\longrightarrow\mathbb{C}$$
is
$$W_\psi:\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R},$$
$$W_\psi(q,p)=\frac{1}{\pi\hbar}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}2px}\,\overline{\psi(q-x)}\;\psi(q+x)\,dx$$

• We have $\langle F[\delta],\varphi\rangle=\langle\delta,F[\varphi]\rangle=F[\varphi](0)$
$$=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\varphi(q)\,dq=\frac{1}{\sqrt{2\pi\hbar}}\langle 1,\varphi\rangle,$$
whence
$$F[\delta]=\frac{1}{\sqrt{2\pi\hbar}},\quad\text{that is}\quad F^{-1}[1]=\sqrt{2\pi\hbar}.$$
From this relation, written as
$$\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq}\,dq=\delta(p)$$
we get
$$\int_{-\infty}^{\infty}W_\psi(q,p)\,dp=2\int_{-\infty}^{\infty}\delta(-2x)\,\overline{\psi(q-x)}\;\psi(q+x)\,dx$$
$$=2\int_{-\infty}^{\infty}\delta(x)\,\overline{\psi(q-\tfrac{x}{2})}\;\psi(q+\tfrac{x}{2})\,dx=|\psi(q)|^2.$$

• By using the change of variables
$$\begin{cases} q-x=u\\ q+x=v\end{cases}$$
we get
$$|F[\psi](p)|^2=\left|\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}pq}\,\psi(q)\,dq\right|^2$$
$$=\frac{1}{2\pi\hbar}\iint_{\mathbb{R}^2}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pu}\,\overline{\psi(u)}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}pv}\,\psi(v)\,dudv$$
$$=\frac{1}{\pi\hbar}\iint_{\mathbb{R}^2}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}p2x}\,\overline{\psi(q-x)}\,\psi(q+x)\,dxdq$$
$$=\int_{-\infty}^{\infty}W_\psi(q,p)\,dq.$$

• From
$$W_\psi(q,p)=\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}px}\,\overline{\psi(q-\tfrac{x}{2})}\;\psi(q+\tfrac{x}{2})\,dx$$
we obtain
$$\overline{\psi(q-\tfrac{x}{2})}\;\psi(q+\tfrac{x}{2})=\int_{\infty}^{\infty}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}px}\,W_\psi(q,p)\,dp,$$
whence
$$\psi(x)=\frac{1}{\overline{\psi(0)}}\int_{\infty}^{\infty}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}px}\,W_\psi(\tfrac{x}{2},p)\,dp.$$

Hilbert space $\mathbb{C}^d$ *(odd* $d=2j+1$*)*

• **Discrete Wigner function** of a pure state
$$\psi:\mathbb{Z}_d\longrightarrow\mathbb{C}$$
is
$$\mathbf{W}_\psi:\mathbb{Z}_d\times\mathbb{Z}_d\longrightarrow\mathbb{R},$$
$$\boxed{\mathbf{W}_\psi(n,k)=\frac{1}{d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}2km}\,\overline{\psi(n-m)}\;\psi(n+m).}$$
We see that
$$\mathbf{W}_\psi(n+d,k)=\mathbf{W}_\psi(n,k)=\mathbf{W}_\psi(n,k+d)$$

• From $\sum_{k\in\mathbb{Z}_d}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}2km}=d\,\delta_{2m,0}=d\,\delta_{m,0}$

we get
$$\sum_{k\in\mathbb{Z}_d}\mathbf{W}_\psi(n,k)=|\psi(n)|^2.$$

• We have
$$|\mathbf{F}[\psi](k)|^2=\left|\frac{1}{\sqrt{d}}\sum_{n\in\mathbb{Z}_d}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}kn}\,\psi(n)\right|^2$$
$$=\frac{1}{d}\sum_{\alpha,\beta\in\mathbb{Z}_d}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k\alpha}\,\overline{\psi(\alpha)}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}k\beta}\,\psi(\beta)$$
$$=\frac{1}{d}\sum_{\alpha,\beta\in\mathbb{Z}_d}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k(\alpha-\beta)}\,\overline{\psi(\alpha)}\,\psi(\beta).$$
Since
$$\mathbb{Z}_d\times\mathbb{Z}_d\to\mathbb{Z}_d\times\mathbb{Z}_d:(n,m)\mapsto(n-m,n+m)$$
is bijective, we get
$$\sum_{n\in\mathbb{Z}_d}\mathbf{W}_\psi(n,k)=\frac{1}{d}\sum_{n,m\in\mathbb{Z}_d}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}2km}\,\overline{\psi(n-m)}\,\psi(n+m)$$
$$=\frac{1}{d}\sum_{\alpha,\beta\in\mathbb{Z}_d}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k(\alpha-\beta)}\,\overline{\psi(\alpha)}\,\psi(\beta)=|\mathbf{F}[\psi](k)|^2.$$

• From
$$\mathbf{W}_\psi(n,k)=\frac{1}{d}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}2km}\,\overline{\psi(n-m)}\;\psi(n+m)$$
we obtain
$$\overline{\psi(n-m)}\;\psi(n+m)=\sum_{k\in\mathbb{Z}_d}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}2km}\,\mathbf{W}_\psi(n,k),$$
whence
$$\psi(2n)=\frac{1}{\overline{\psi(0)}}\sum_{m\in\mathbb{Z}_d}\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}2kn}\,\mathbf{W}_\psi(n,k).$$
For odd $d=2j+1$, we have
$$\psi(2n+1)=\psi(2n+1+d)=\psi(2(n+j+1)).$$

Wigner function of a discrete Gaussian function

[N. Cotfas and D.Dragoman, J. Phys. A:Math. Theor. **45** (2012) 425305]

Hilbert space $L^2(\mathbb{R})$

• **Wigner function** of a pure state

$$\psi:\mathbb{R}\longrightarrow\mathbb{C}$$

is

$$W_\psi:\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R},$$

$$W_\psi(q,p)=\frac{1}{\pi\hbar}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}2px}\,\overline{\psi(q-x)}\;\psi(q+x)\,dx$$

•The **continuous Gaussian** function

$$g_\kappa:\mathbb{R}\longrightarrow\mathbb{R},\qquad g_\kappa(q)=\mathrm{e}^{-\frac{\kappa}{2}q^2},$$

where $\kappa\in(0,\infty)$, satisfies the relation

$$F[g_\kappa]=\frac{1}{\sqrt{\kappa}}\,g_{\frac{1}{\kappa}}.$$

• The corresponding Wigner function is

$$W_{g_\kappa}(q,p)=2\sqrt{\tfrac{\pi}{\kappa}}\,\mathrm{e}^{-\kappa q^2}\,\mathrm{e}^{-\frac{1}{\kappa}p^2},$$

that is

$$W_{g_\kappa}(q,p)=2\sqrt{\tfrac{\pi}{\kappa}}\,g_{2\kappa}(q)\,g_{\frac{2}{\kappa}}(p).$$

• Since

$$W_{g_\kappa}(q,p)=2\sqrt{\tfrac{\pi}{\kappa}}\,\mathrm{e}^{-\frac{1}{2}(q\ \ p)\begin{pmatrix}2\kappa & 0\\ 0 & \frac{2}{\kappa}\end{pmatrix}\begin{pmatrix}q\\ p\end{pmatrix}}$$

the function

$$g_\kappa(q)=\mathrm{e}^{-\frac{\kappa}{2}q^2}$$

represents a Gaussian state of continuous variable with the covariance matrix

$$\sigma=\begin{pmatrix}2\kappa & 0\\ 0 & \frac{2}{\kappa}\end{pmatrix}>0.$$

Hilbert space $\mathbb{C}^d$ *(odd $d=2j+1$)*

• **Discrete Wigner function** of a pure state

$$\psi:\mathbb{Z}_d\longrightarrow\mathbb{C}$$

is

$$\mathbf{W}_\psi:\mathbb{Z}_d\times\mathbb{Z}_d\longrightarrow\mathbb{R},$$

$$\mathbf{W}_\psi(n,k)=\frac{1}{d}\sum_{m=-j}^{j}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}2km}\,\overline{\psi(n-m)}\;\psi(n+m)$$

•The **discrete Gaussian** function

$$\mathbf{g}_\kappa:\mathbb{Z}_d\longrightarrow\mathbb{R},\qquad \boxed{\mathbf{g}_\kappa(n)=\sum_{m=-\infty}^{\infty}\mathrm{e}^{-\frac{\kappa\pi}{d}(md+n)^2}},$$

where $\kappa\in(0,\infty)$, satisfies the relation

$$\mathbf{F}[\mathbf{g}_\kappa]=\frac{1}{\sqrt{\kappa}}\,\mathbf{g}_{\frac{1}{\kappa}}.$$

• The corresponding discrete Wigner function is

$$\begin{aligned}\mathbf{W}_{\mathbf{g}_\kappa}(n,k)=&\frac{1}{2\sqrt{2\kappa d}}\;\mathbf{g}_{2\kappa}(n)\,\mathbf{g}_{\frac{2}{\kappa}}(k)\\ &+\frac{1}{2\sqrt{2\kappa d}}\;\mathbf{g}_{2\kappa}(n)\,\tilde{\mathbf{g}}_{\frac{2}{\kappa}}(k)\\ &+\frac{1}{2\sqrt{2\kappa d}}\;\tilde{\mathbf{g}}_{2\kappa}(n)\,\mathbf{g}_{\frac{2}{\kappa}}(k)\\ &-\frac{1}{2\sqrt{2\kappa d}}\;\tilde{\mathbf{g}}_{2\kappa}(n)\,\tilde{\mathbf{g}}_{\frac{2}{\kappa}}(k).\end{aligned}$$

where the function

$$\tilde{\mathbf{g}}_\kappa:\mathbb{Z}_d\longrightarrow\mathbb{R},\qquad \tilde{\mathbf{g}}_\kappa(n)=\sum_{m=-\infty}^{\infty}\mathrm{e}^{-\frac{\kappa\pi}{d}((m+\frac{1}{2})d+n)^2},$$

can also be regarded as a discrete Gaussian.

• Proofs based on Fourier series and

$$\int_{-\infty}^{\infty}\varphi(x)\,dx=\sum_{m=-\infty}^{\infty}\int_{m\sqrt{2\pi d}}^{(m+1)\sqrt{2\pi d}}\varphi(x)\,dx,$$

$$\sum_{m=-\infty}^{\infty}\varphi(m)=\sum_{n=-j}^{j}\sum_{m=-\infty}^{\infty}\varphi(md+n),$$

$$\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\xi t}\,\mathrm{e}^{-at^2}\,dt=\sqrt{\tfrac{\pi}{a}}\;\mathrm{e}^{-\frac{\xi^2}{4a}},$$

$$\sum_{n,m=-\infty}^{\infty}N_{n,m}=\sum_{k,\ell=-\infty}^{\infty}N_{k+\ell,k-\ell}+\sum_{k,\ell=-\infty}^{\infty}N_{k+\ell+1,k-\ell}.$$

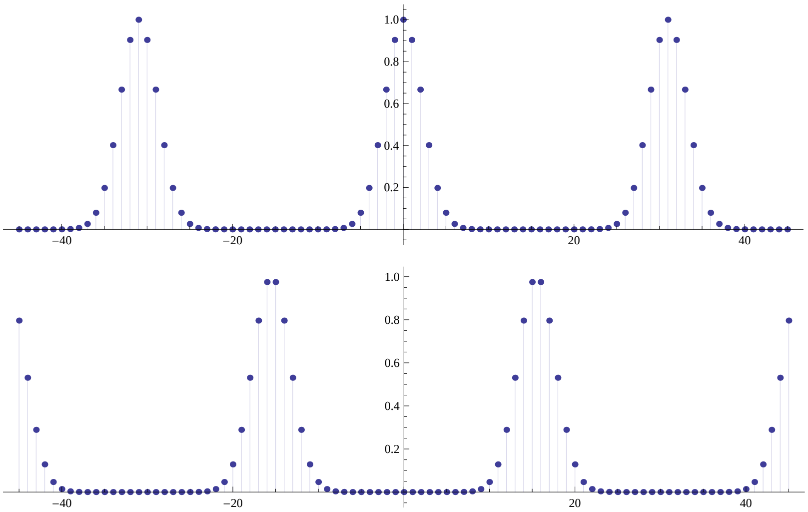


Figure 1: The discrete Gaussian functions $\mathbf{g}_1$ and $\tilde{\mathbf{g}}_1$ in the case $d=31$.

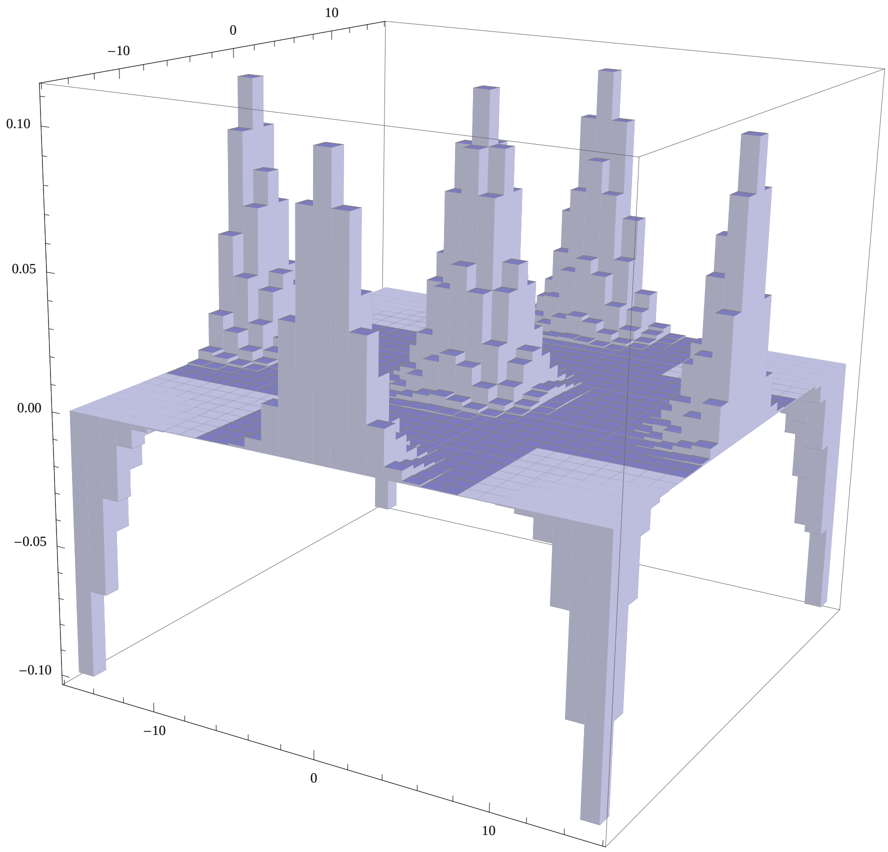


Figure 2: The discrete Wigner function $\mathbf{W}_{\mathbf{g}_1}$ in the case $d=31$.

$$\mathbf{W}_{\mathbf{g}_\kappa}(n,k) = \frac{1}{2\sqrt{2\kappa d}}\,\mathbf{g}_{2\kappa}(n)\,\mathbf{g}_{\frac{2}{\kappa}}(k) + \frac{1}{2\sqrt{2\kappa d}}\,\mathbf{g}_{2\kappa}(n)\,\tilde{\mathbf{g}}_{\frac{2}{\kappa}}(k)$$
$$+\frac{1}{2\sqrt{2\kappa d}}\,\tilde{\mathbf{g}}_{2\kappa}(n)\,\mathbf{g}_{\frac{2}{\kappa}}(k) - \frac{1}{2\sqrt{2\kappa d}}\,\tilde{\mathbf{g}}_{2\kappa}(n)\,\tilde{\mathbf{g}}_{\frac{2}{\kappa}}(k).$$

Finite frame quantization

[N. Cotfas, J.-P. Gazeau and A. Vourdas, J. Phys. A:Math. Theor. **44** (2011) 175303]

Hilbert space $L^2(\mathbb{R})$

• The **canonical coherent states**

$$|q,p\rangle = D(q,p)|0\rangle,$$

where $q,p\in\mathbb{R}$

$$D(q,p)=\mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}pq}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pQ}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}qP},$$

and

$$|0\rangle=\frac{|g_1\rangle}{||g_1||}=\frac{|g_1\rangle}{\sqrt[4]{\pi}},\qquad g_1(q)=\mathrm{e}^{-\frac{q^2}{2}},$$

satisfy the resolution of the identity

$$\mathbb{I}=\frac{1}{2\pi}\iint_{\mathbb{R}^2}|q,p\rangle\langle q,p|\;dqdp$$

and

$$\langle q|a,b\rangle=\frac{1}{\sqrt[4]{\pi}}\mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}ab}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}bq}\,\mathrm{e}^{-\frac{(q-a)^2}{2}}.$$

• **Canonical state quantization:**

To a function

$$f:\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{C}$$

we associate the operator (if it exists !)

$$A_f=\frac{1}{2\pi}\iint_{\mathbb{R}^2}f(q,p)\,|q,p\rangle\langle q,p|\;dqdp.$$

• Examples:

In the case $f(q,p)=q$, we get

$$A_q=\frac{1}{2\pi}\iint_{\mathbb{R}^2}q\,|q,p\rangle\langle q,p|\;dqdp=Q.$$

In the case $f(q,p)=p$, we get

$$A_p=\frac{1}{2\pi}\iint_{\mathbb{R}^2}p\,|q,p\rangle\langle q,p|\;dqdp=P.$$

• We have

$$P=F^{-1}QF.$$

Hilbert space $\mathbb{C}^d$ *(odd $d=2j+1$)*

• **The discrete coherent states**

$$|n,k\rangle=\mathbf{D}(n,k)|0\rangle,$$

where $n,k\in\{-j,-j+1,...,j-1,j\}$,

$$\mathbf{D}(n,k)=\mathrm{e}^{-\frac{\pi\mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k\mathbf{Q}}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}n\mathbf{P}}$$

and

$$|0\rangle=\frac{|\mathbf{g}_1\rangle}{||\mathbf{g}_1||},\qquad \mathbf{g}_1(n)=\sum_{m=-\infty}^{\infty}\mathrm{e}^{-\frac{\pi}{d}(md+n)^2},$$

satisfy the resolution of the identity

$$\mathbb{I}=\frac{1}{d}\sum_{n,k=-j}^{j}|n,k\rangle\langle n,k|$$

and

$$\langle j;m|n,k\rangle=\frac{1}{||\mathbf{g}_1||}\mathrm{e}^{-\frac{\pi\mathrm{i}}{d}nk}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}km}\,\mathbf{g}_1(m-n).$$

• **Finite frame quantization:**

To a function

$f:\{-j,-j+1,...,j\}\times\{-j,-j+1,...,j\}\to\mathbb{C}$

we associate the operator

$$\boxed{\mathbf{A}_f=\frac{1}{d}\sum_{n,k=-j}^{j}f(n,k)\,|n,k\rangle\langle n,k|}.$$

• Examples:

In the case $f(n,k)=n$, we get

$$\mathbf{A}_q=\frac{1}{d}\sum_{n,k=-j}^{j}n\,|n,k\rangle\langle n,k|\neq\mathbf{Q}.$$

In the case $f(n,k)=k$, we get

$$\mathbf{A}_p=\frac{1}{d}\sum_{n,k=-j}^{j}k\,|n,k\rangle\langle n,k|\neq\mathbf{P}.$$

• We have

$$\mathbf{A}_p=\mathbf{F}^{-1}\mathbf{A}_q\mathbf{F},\quad\text{similar to}\quad \mathbf{P}=\mathbf{F}^{-1}\mathbf{Q}\mathbf{F},$$

and

$$\mathbf{Q}|j;m\rangle=m\,|j;m\rangle,\qquad \mathbf{A}_q|j;m\rangle=\lambda_m\,|j;m\rangle,$$

where $\lambda_m=\frac{1}{||\mathbf{g}_1||}\sum_{n=-j}^{j}n(\mathbf{g}_1(m-n))^2.$

• By using $\mathbf{A}_q$, $\mathbf{A}_p$ instead of $\mathbf{Q}$, $\mathbf{P}$ we get an alternative version for the discrete phase space $\mathbb{Z}_d\times\mathbb{Z}_d$.

Finite Fractional Fourier Transform - an alternative definition

[N. Cotfas and D. Dragoman, J. Phys. A:Math. Theor. **46** (2013) 355301]

Hilbert space $L^2(\mathbb{R})$

- The harmonic oscillator Hamiltonian

$$
\begin{aligned}
H &= -\tfrac{1}{2}\tfrac{d^2}{dq^2} + \tfrac{1}{2}q^2 \\
&= -\tfrac{1}{2}\left(F^{-1}q^2F + q^2\right) \\
&= -\tfrac{1}{2}\left(\tfrac{d^2}{dq^2} + F\tfrac{d^2}{dq^2}F^{-1}\right) \\
&= -\tfrac{1}{2} + \tfrac{1}{2\pi}\iint_{\mathbb{R}^2} dq\, dp\, \tfrac{q^2+p^2}{2}\, |q,p\rangle\langle q,p|.
\end{aligned}
$$

commutes with Fourier transform,

$$HF = FH, \quad \text{that is} \quad F^{-1}HF = H$$

and the **Hermite-Gauss functions**

$$\Psi_n(q) = \frac{1}{\sqrt{2^n\, n!\, \sqrt{\pi}}}\, H_n(q)\, \mathrm{e}^{-\frac{q^2}{2}},$$

denoted by $|n\rangle$, are common eigenfunctions:

$$\langle n|m\rangle = \delta_{nm}, \qquad \mathbb{I} = \sum_{n=0}^{\infty} |n\rangle\langle n|,$$

$$H|n\rangle = (n+\tfrac{1}{2})|n\rangle, \qquad H = \sum_{n=0}^{\infty} (n+\tfrac{1}{2})|n\rangle\langle n|,$$

$$F|n\rangle = (-\mathrm{i})^n|n\rangle, \qquad F = \sum_{n=0}^{\infty} (-\mathrm{i})^n|n\rangle\langle n|,$$

- **Fractional Fourier** transform

$$F^\alpha = \sum_{n=0}^{\infty} (-\mathrm{i})^{n\alpha}|n\rangle\langle n|.$$

- We have

$$F = P_0 - \mathrm{i}P_1 - P_2 + \mathrm{i}P_3 = \sum_{m=0}^{3} (-\mathrm{i})^m\, P_m,$$

where

$$
\begin{aligned}
P_0 &= \tfrac{1}{4}(\mathbb{I} + F + F^2 + F^3) \\
P_2 &= \tfrac{1}{4}(\mathbb{I} - F + F^2 - F^3) \\
P_1 &= \tfrac{1}{4}(\mathbb{I} + \mathrm{i}F - F^2 - \mathrm{i}F^3) \\
P_3 &= \tfrac{1}{4}(\mathbb{I} - \mathrm{i}F - F^2 + \mathrm{i}F^3).
\end{aligned}
$$

Hilbert space $\mathbb{C}^d$ *(odd $d=2j+1$)*

- The Hamiltonian

$$\mathbf{H} = -\tfrac{1}{2}\left(\mathcal{D}^2 + \mathbf{F}\mathcal{D}^2\mathbf{F}^{-1}\right)$$

where D^2 is the finite difference operator

$$\mathcal{D}^2\psi(n) = \psi(n+1) - 2\psi(n) + \psi(n-1)$$

is Fourier invariant

$$\mathbf{HF} = \mathbf{FH},$$

and its eigenfunctions (**Harper functions**) $\mathbf{h}_n$ considered in the increasing order of sign alternations, satisfy the relation

$$\mathbf{F}\mathbf{h}_n = (-\mathrm{i})^n\mathbf{h}_n.$$

Consequently

$$\mathbf{F} = \sum_{n=0}^{d-1} (-\mathrm{i})^n|\mathbf{h}_n\rangle\langle\mathbf{h}_n|.$$

- **Finite fractional Fourier** transform

$$\boxed{\mathbf{F}^\alpha = \sum_{n=0}^{d-1} (-\mathrm{i})^{n\alpha}|\mathbf{h}_n\rangle\langle\mathbf{h}_n|}.$$

- Since $\mathbf{F}|n,k\rangle = |k,-n\rangle$, the operator

$$\mathbf{H}_f = -\tfrac{1}{2} + \tfrac{1}{d}\sum_{n,k=-j}^{j} \tfrac{n^2+k^2}{2}\, |n,k\rangle\langle n,k|$$

defined by using finite frame quantization is Fourier invariant

$$\mathbf{H}_f\mathbf{F} = \mathbf{F}\mathbf{H}_f,$$

and its eigenfunctions $\mathbf{f}_n$, considered in the increasing order of sign alternations, satisfy the relation

$$\mathbf{F}\mathbf{f}_n = (-\mathrm{i})^n\mathbf{f}_n.$$

- New **finite fractional Fourier** transform

$$\boxed{\mathbf{F}^\alpha = \sum_{n=0}^{d-1} (-\mathrm{i})^{n\alpha}|\mathbf{f}_n\rangle\langle\mathbf{f}_n|}.$$

A discrete Wigner function [W.K. Wootters, *Ann. Phys.* **176**(1987)1-21]

Hilbert space $L^2(\mathbb{R})$

- The Gaussian function
$$\psi_c(q) = \frac{1}{\sqrt{2\pi\hbar b}} \mathrm{e}^{-\frac{\mathrm{i}a}{2\hbar b}(q-\frac{c}{a})^2}$$
is an eigenfunction of $aQ + bP$:
$$(aQ + bP)\psi_c = c\,\psi_c.$$

- To each $(q,p)\in\mathbb{R}\times\mathbb{R}$ we associate the **parity operator** $\Pi(q,p)$ defined as
$$\Pi(q,p)\psi(a) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}2p(q-a)}\,\psi(2q-a),$$
$$\langle a|\Pi(q,p)|b\rangle = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}2p(q-a)}\,\delta_b(2q-a) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}p(a-b)}\,\delta(2q-a-b).$$

The operators $\Pi(q,p)$ have the properties:

a) $\mathrm{tr}\,\Pi(q,p) = 1$

b) $\mathrm{tr}(\Pi(q,p)\,\Pi(a,b)) = 2\pi\hbar\,\delta(q-a)\,\delta(p-b)$

c) $\{\,\Pi(q,p)\mid(q,p)\in\mathbb{R}\times\mathbb{R}\}$ is a complete set

d) $\int_{c_1}^{c_2}|\psi_c\rangle\langle\psi_c|dc = \frac{1}{2\pi\hbar}\iint\limits_{c_1\leq aq+bp\leq c_2}\Pi(q,p)\,dqdp$

- To each density operator $\hat\rho : L^2(\mathbb{R})\longrightarrow L^2(\mathbb{R})$, we associate the **Wigner function**
$$W_{\hat\rho}:\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R},$$
$$W_{\hat\rho}(q,p) = \frac{1}{2\pi\hbar}\mathrm{tr}(\hat\rho\,\Pi(q,p))$$
satisfying the relation
$$\hat\rho = \iint\limits_{\mathbb{R}^2} W_{\hat\rho}(q,p)\,\Pi(q,p)\,dqdp.$$

It has the properties (among many others):

1) $\iint\limits_{\mathbb{R}^2} W_{\hat\rho}(q,p)\,dqdp = 1$

2) $\mathrm{tr}(\hat\rho_1\,\hat\rho_2) = \iint\limits_{\mathbb{R}^2} W_{\hat\rho_1}(q,p)\,W_{\hat\rho_2}(q,p)\,dqdp$

3) probability that $aQ+bP$ will yield a value between c_1 and c_2 in the state $\hat\rho$ is
$$\mathrm{tr}\left(\hat\rho\int_{c_1}^{c_2}|\psi_c\rangle\langle\psi_c|dc\right) = \iint\limits_{c_1\leq aq+bp\leq c_2} W_{\hat\rho}(q,p)\,dqdp.$$

Hilbert space $\mathbb{C}^d$ *(prime* $d=2j+1$*)*

- To each $(n,k)\in\mathbb{Z}_d\times\mathbb{Z}_d$ we associate the **parity operator** $\Pi(n,k)$ defined as
$$\Pi(n,k)\psi(m) = \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}2k(n-m)}\,\psi(2n-m),$$
$$\langle j;a|\Pi(n,k)|j;b\rangle = \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}2k(n-a)}\,\delta_b(2n-a) = \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k(a-b)}\,\delta_{2n,a+b}.$$

The operators $\Pi(n,k)$ have the properties:

a) $\mathrm{tr}\,\Pi(n,k) = 1$

b) $\mathrm{tr}(\Pi(n_1,k_1)\,\Pi(n_2,k_2)) = d\,\delta_{n_1n_2}\,\delta_{k_1k_2}$

c) $\{\,\Pi(n,k)\mid(n,k)\in\mathbb{Z}_d\times\mathbb{Z}_d\}$ complete set

d) For each $a,b\in\mathbb{Z}_d$, the operators
$$P_{\lambda_c} = \frac{1}{d}\sum_{(n,k)\in\lambda_c}\Pi(n,k)$$
corresponding to the d lines
$$\lambda_c = \{\,(n,k)\in\mathbb{Z}_d\times\mathbb{Z}_d\mid an+bk=c\,\}$$
are mutually orthogonal projectors, and
$$\sum_{c\in\mathbb{Z}_d}P_{\lambda_c} = \mathbb{I}.$$

- To each density operator $\hat\rho:\mathbb{C}^d\longrightarrow\mathbb{C}^d$, we associate the **Wigner function**
$$W_{\hat\rho}:\mathbb{Z}_d\times\mathbb{Z}_d\longrightarrow\mathbb{R},$$
$$\boxed{W_{\hat\rho}(n,k) = \frac{1}{d}\,\mathrm{tr}(\hat\rho\,\Pi(n,k))}$$
satisfying the relation
$$\hat\rho = \sum_{n,k\in\mathbb{Z}_d}W_{\hat\rho}(n,k)\,\Pi(n,k).$$

It has the properties:

1) $\sum\limits_{n,k\in\mathbb{Z}_d}W_{\hat\rho}(n,k) = 1$

2) $\mathrm{tr}(\hat\rho_1\,\hat\rho_2) = \sum\limits_{n,k\in\mathbb{Z}_d}W_{\hat\rho_1}(n,k)\,W_{\hat\rho_2}(n,k)$

3) probability that $a\mathbf{Q}+b\mathbf{P}$ will yield the eigenvalue c in the state $\hat\rho$ is
$$\mathrm{tr}\,(\hat\rho P_{\lambda_c}) = \sum_{(n,k)\in\lambda_c}W_{\hat\rho}(n,k).$$

Discrete Wigner, Husimi and Glauber-Sudarshan functions

[M.A. Marchiolli and M. Ruzzi, Journal of Russian Laser Research **32** (2011) 381]

Hilbert space $L^2(\mathbb{R})$

• The **canonical coherent states**

$$|q,p\rangle = D(q,p)|0\rangle,$$

where

$$D(q,p) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(qP-pQ)} = \mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}pq}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pQ}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}qP} = \mathrm{e}^{\frac{\mathrm{i}}{2\hbar}pq}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}qP}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pQ},$$

and

$$|0\rangle = \frac{|g_1\rangle}{||g_1||} = \frac{|g_1\rangle}{\sqrt[4]{\pi}}, \qquad g_1(q) = \mathrm{e}^{-\frac{q^2}{2}},$$

satisfy the resolution of the identity

$$\mathbb{I} = \frac{1}{2\pi}\iint_{\mathbb{R}^2} |q,p\rangle\langle q,p|\; dqdp.$$

• The **parity operators** satisfy:

$$\Pi(q,p) = \frac{1}{2\pi\hbar}\iint_{\mathbb{R}^2} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(qb-pa)}D(a,b)\; dadb.$$

• **Cahill-Glauber approach:**

By using the mapping kernel

$$T^{(s)}(q,p) = \frac{1}{2\pi\hbar}\iint_{\mathbb{R}^2} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(qb-pa)}D(a,b)\,(\langle 0|a,b\rangle)^{-s}\; dadb,$$

where the parameter $s\in\mathbb{C}$ satisfies $|s|\le 1$, a linear operator O can be written as

$$O = \frac{1}{2\pi\hbar}\iint_{\mathbb{R}^2} \mathcal{O}^{(s)}(q,p)\,T^{(-s)}(q,p)\; dqdp,$$

where $\mathcal{O}^{(s)}:\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{C}$,

$$\mathcal{O}^{(s)}(q,p) = \mathrm{tr}(O\,T^{(s)}(q,p)).$$

Particularly,
$\mathcal{O}^{(0)}(q,p)$ is the **Wigner function**,
$\mathcal{O}^{(-1)}(q,p)$ is the **Husimi function**,
$\mathcal{O}^{(1)}(q,p)$ is the **Glauber-Sudarshan function**.

Hilbert space $\mathbb{C}^d$ *(odd $d=2j+1$)*

• **The discrete coherent states**

$$|n,k\rangle = \mathbf{D}(n,k)|0\rangle,$$

where $n,k\in\{-j,-j+1,\ldots,j-1,j\}$,

$$\mathbf{D}(n,k) = \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}\{2^{-1}nk\}}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k\mathbf{Q}}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}n\mathbf{P}} = \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}\{2^{-1}nk\}}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}n\mathbf{P}}\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{d}k\mathbf{Q}},$$

with $\{2^{-1}nk\}\in\{-j,-j+1,\ldots,j-1,j\}$ the element satisfying the relation

$$2\{2^{-1}nk\}\equiv nk \pmod d$$

and

$$|0\rangle = \frac{|\mathbf{g}_1\rangle}{||\mathbf{g}_1||}, \qquad \mathbf{g}_1(n) = \sum_{m=-\infty}^{\infty}\mathrm{e}^{-\frac{\pi}{d}(md+n)^2}.$$

• The **discrete parity operators** satisfy:

$$\Pi(n,k) = \frac{1}{d}\sum_{m,\ell=-j}^{j}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}(n\ell-km)}\mathbf{D}(m,\ell).$$

• **Discrete Cahill-Glauber approach:**

By using the mapping kernel

$$\mathbf{T}^{(s)}(n,k) = \frac{1}{d}\sum_{m,\ell=-j}^{j}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}(n\ell-km)}\mathbf{D}(m,\ell)\,(\langle 0|m,\ell\rangle)^{-s}$$

where the parameter $s\in\mathbb{C}$ satisfies $|s|\le 1$, a linear operator $\mathbf{O}$ can be written as

$$\mathbf{O} = \frac{1}{d}\sum_{m,\ell=-j}^{j}\mathcal{O}^{(s)}(q,p)\,\mathbf{T}^{(-s)}(q,p),$$

where $\mathcal{O}^{(s)}:\mathbb{Z}_d\times\mathbb{Z}_d\longrightarrow\mathbb{C}$,

$$\boxed{\mathcal{O}^{(s)}(n,k) = \mathrm{tr}(\mathbf{O}\,\mathbf{T}^{(s)}(n,k))}.$$

Particularly,
$\mathcal{O}^{(0)}(n,k)$ is the **discrete Wigner function**,
$\mathcal{O}^{(-1)}(n,k)$ is the **discrete Husimi function**,
$\mathcal{O}^{(1)}(n,k)$ is the **discrete Glauber-Sudarshan function**.

Wigner functions for arbitrary finite-dimensional systems

[T. Tilma, M.J. Everitt, J.H. Samson *et al.*, *Phys. Rev. Lett.* **117** (18) Oct. 2016]

Hilbert space $L^2(\mathbb{R})$

- Representation of Weyl-Heisenberg group

$$D(q,p) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(qP-pQ)} = \mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}pq}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pQ}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}qP} = \mathrm{e}^{\frac{\mathrm{i}}{2\hbar}pq}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}qP}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pQ}$$

- Parity operators

$$\Pi(q,p) = D(q,p)\,\Pi\,D(q,p)^+ = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}pQ}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}qP}\,\Pi\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}qP}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}pQ},$$

where

$$\Pi\psi(q) = \psi(-q),$$

- Phase space is the plane

$$\mathbb{R}^2 = \{\ \Omega = (q,p)\ |\ \ q,p \in \mathbb{R}\}$$

with measure $d\Omega = dqdp$.

- To each self-adjoint operator $\hat{\rho}$ we associate the **Wigner function**

$$W_{\hat{\rho}} : \mathbb{R}^2 \longrightarrow \mathbb{R},$$

$$W_{\hat{\varrho}}(\Omega) = \mathrm{tr}(\hat{\varrho}\,\Delta(\Omega)),$$

where

$$\Delta(\Omega) = \tfrac{1}{\pi\hbar}\,\Pi(\Omega).$$

It has the properties:

1) we can reconstruct $\hat{\varrho}$ from $W_{\hat{\varrho}}$;

2) $\Omega \mapsto W_{\hat{\varrho}}(\Omega)$ is a real function;

3) $\iint\limits_{\mathbb{R}^2} W_{\hat{\varrho}}(\Omega)\,d\Omega = \mathrm{tr}\,\hat{\varrho}$ and $\iint\limits_{\mathbb{R}^2} \Delta(\Omega)\,d\Omega = \mathbb{I}$;

4) $\iint\limits_{\mathbb{R}^2} W_{\hat{\varrho}_1}(\Omega)\,W_{\hat{\varrho}_2}(\Omega)\,d\Omega = \mathrm{tr}\,(\hat{\varrho}_1\hat{\varrho}_2)$;

5) $\hat{\varrho}$ invariant under unitary U, then so is $W_{\hat{\varrho}}$.

Hilbert space $\mathbb{C}^2$ (case of a qubit)

- Two-dimensional representation of SU(2)

$$D^{\frac{1}{2}}(\theta,\phi,\Phi) = \mathrm{e}^{\mathrm{i}\phi\sigma_z}\mathrm{e}^{\mathrm{i}\theta\sigma_y}\mathrm{e}^{\mathrm{i}\Phi\sigma_z}$$

(described in terms of Euler angles).

- "Parity" operators

$$\Pi(\theta,\phi) = D^{\frac{1}{2}}(\theta,\phi,\Phi)\,\Pi\,D^{\frac{1}{2}}(\theta,\phi,\Phi)^+ = \mathrm{e}^{\mathrm{i}\phi\sigma_z}\mathrm{e}^{\mathrm{i}\theta\sigma_y}\,\Pi\,\mathrm{e}^{-\mathrm{i}\theta\sigma_y}\mathrm{e}^{-\mathrm{i}\phi\sigma_z},$$

where $\Pi = \mathbb{I}_2 - \sqrt{3}\sigma_z.$

- Phase space is the sphere

$$S_2 = \{\ \Omega = (\theta,\phi)\ |\ \ \theta \in [0,\pi],\ \ \phi \in [0,2\pi)\ \}$$

with Haar measure $d\Omega = \sin\theta\,d\theta d\phi$.

- To each self-adjoint operator $\hat{\rho}$ we associate the **Wigner function**

$$W_{\hat{\rho}} : S_2 \longrightarrow \mathbb{R},$$

$$\boxed{W_{\hat{\varrho}}(\theta,\phi) = \mathrm{tr}(\hat{\varrho}\,\Delta(\theta,\phi))}$$

where

$$\Delta(\theta,\phi) = \tfrac{1}{2}\Pi(\theta,\phi).$$

It has some similar properties

A Wigner function defined by using $\mathbb{R}^3$ as a phase space

[N.M. Atakishiyev, S.M. Chumakov, K.B. Wolf, *J. Math. Phys.* **39** (1998) 6247]

Hilbert space $L^2(\mathbb{R})$

- Representation of Weyl-Heisenberg group

$$D(q,p)=\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(qP-pQ)}=\mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}pq}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pQ}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}qP}$$
$$=\mathrm{e}^{\frac{\mathrm{i}}{2\hbar}pq}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}qP}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}pQ}.$$

- Parity operators

$$\Pi(q,p)=D(q,p)\,\Pi\,D(q,p)^+$$

where

$$\Pi\psi(q)=\psi(-q),$$

satisfy the relation

$$\Pi(q,p)=\frac{1}{2\pi\hbar}\iint_{\mathbb{R}^2} dxdy\ \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(qy-px)}D(x,y).$$

- To each self-adjoint operator $\hat{\rho}$ we associate the **Wigner function**

$$W_{\hat{\rho}}:\mathbb{R}^2\longrightarrow\mathbb{R},$$
$$W_{\hat{\varrho}}(q,p)=\frac{1}{\pi\hbar}\,\mathrm{tr}(\hat{\varrho}\,\Pi(q,p)).$$

Hilbert space $\mathbb{C}^{2j+1}$

- $(2j+1)$-dimensional representation of SU(2)

$$D^j[\mathbf{y}]:\mathbb{C}^{2j+1}\longrightarrow\mathbb{C}^{2j+1}$$

in the polar parametrization

$$g[\mathbf{y}]=\mathrm{e}^{-\mathrm{i}\mathbf{y}\cdot\mathbf{J}}=\mathrm{e}^{-\mathrm{i}\eta(v_1J_1+v_2J_2+v_3J_3)}$$

where $\mathbf{y}=\eta(v_1,v_2,v_3)$ with

$0\le\eta<4\pi$ is the rotation angle,

$(v_1,v_2,v_3)=(\sin\theta\sin\phi,\sin\theta\cos\phi,\cos\theta)$ is the rotation axis,

and

$$dg[\mathbf{y}]=\frac{1}{2}\ \sin^2\frac{\eta}{2}\ \sin\theta\ d\eta\,d\theta\,d\phi$$

is the invariant measure.

- For each $\mathbf{x}\in\mathbb{R}^3$ we define

$$\Pi(\mathbf{x})=\int_{SU(2)}dg[\mathbf{y}]\ \mathrm{e}^{\mathrm{i}\mathbf{x}\cdot\mathbf{y}}D^j[\mathbf{y}]$$

- To each self-adjoint operator

$$\hat{\rho}:\mathbb{C}^{2j+1}\longrightarrow\mathbb{C}^{2j+1}$$

we associate the **Wigner function**

$$W_{\hat{\rho}}:\mathbb{R}^3\longrightarrow\mathbb{R},$$
$$\boxed{W_{\hat{\varrho}}(\mathbf{x})=\mathrm{tr}(\hat{\varrho}\,\Pi(\mathbf{x}))}.$$

The space $\mathbb{C}^{2j+1}$ regarded as a space of tempered distributions

[J. Kijowski *et al.*, Wigner function of a qubit, arXiv:1512.02867v1, 9 Dec 2015]

Hilbert space $L^2(\mathbb{R})$

- **Periodic distributions with periodic Fourier transform**.

If $\psi : \mathcal{S}(\mathbb{R}) \longrightarrow \mathbb{C}$ is periodic, that is

$$\psi = T_a\psi,$$

where $T_a\psi$ is the distribution

$$\langle T_a\psi, \varphi\rangle = \langle \psi, \varphi(q-a)\rangle,$$

then

$$\begin{aligned}\langle F[\psi], \varphi\rangle &= \langle \psi, F[\varphi]\rangle \\ &= \langle T_a\psi, F[\varphi]\rangle \\ &= \langle \psi, F[\varphi](q-a)\rangle \\ &= \langle \psi, F[\mathrm{e}^{\frac{\mathrm{i}}{\hbar}ap}\varphi]\rangle \\ &= \langle F[\psi], \mathrm{e}^{\frac{\mathrm{i}}{\hbar}ap}\varphi\rangle \\ &= \langle \mathrm{e}^{\frac{\mathrm{i}}{\hbar}ap}\, F[\psi], \varphi\rangle,\end{aligned}$$

that is

$$(1-\mathrm{e}^{\frac{\mathrm{i}}{\hbar}ap})\; F[\psi] = 0.$$

Consequently, $F[\psi]$ has the form

$$F[\psi] = \sum_{n\in\mathbb{Z}} c_n\, \delta_{n\frac{\hbar}{a}} \qquad (c_n \in \mathbb{C}).$$

$$\boxed{\left.\begin{array}{r}\psi \text{ periodic}\\ F[\psi] \text{ periodic}\end{array}\right\} \Rightarrow \left\{\begin{array}{l} F[\psi] \text{ has the form:} \\ F[\psi] = \sum_{n=-j}^{j} \psi_n \sum_{\ell=-\infty}^{\infty} \delta_{(n+\ell d)\frac{\hbar}{a}}.\end{array}\right.}$$

(to be compared with Zak basis states).

- The **standard Wigner function**

$$W_\psi(q,p) = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}2px}\, \overline{\psi(q-x)}\; \psi(q+x)\, dx$$

has the marginal properties

$$\int_{-\infty}^{\infty} W_\psi(q,p)\, dp = |\psi(q)|^2,$$
$$\int_{-\infty}^{\infty} W_\psi(q,p)\, dq = |F[\psi](p)|^2.$$

Hilbert space $\mathbb{C}^d \quad (d=2j+1)$

- $\mathbb{C}^d$ can be regarded as a space of functions :

$$\mathbb{C}^d = \{\, \psi : \{-j, -j+1, ..., j-1, j\} \longrightarrow \mathbb{C}\,\}.$$

$\mathbb{C}^d$ can be regarded as a space of distributions by identifying the function

$$\psi : \{-j, -j+1, ..., j-1, j\} \longrightarrow \mathbb{C} : n \mapsto \psi_n$$

with the periodic distribution

$$\psi(x) = \sum_{n=-j}^{j} \psi_n \sum_{\ell=-\infty}^{\infty} \delta(x-(n+\ell d)\hbar),$$

that is

$$\psi = \sum_{n=-j}^{j} \psi_n \sum_{\ell=-\infty}^{\infty} \delta_{(n+\ell d)\hbar}.$$

- By relaxing the periodicity (*"not the wave function but the physical state must be periodic"*), the authors identify ψ with

$$\psi(x) = \sum_{\ell=-\infty}^{\infty} \sum_{n=-j}^{j} \psi_n\, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\varphi_0 x}\; \delta(x-(n+\ell d)\hbar),$$

(φ_0 is a constant) that is, with the distribution

$$\boxed{\psi = \sum_{n=-j}^{j} \psi_n \sum_{\ell=-\infty}^{\infty} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\varphi_0(n+\ell d)}\; \delta_{(n+\ell d)\hbar}}.$$

- The embedding $\mathbb{C}^d \subset \mathcal{S}'(\mathbb{R})$ allows us to use the definitions of Fourier transform and Wigner function from the continuous case.
- **Phase space** is the 2-dimensional sphere of radius $\sqrt{j}$ equipped with the symplectic structure

$$\omega = -j\, \sin\vartheta\; d\vartheta \wedge d\varphi = d\varphi \wedge d\xi,$$

where

$\xi = j(1-\cos\vartheta) - \frac{\hbar}{2}$ plays the role of "position"

φ plays the role of "momentum".

- **Discrete Wigner distribution**

$$W_\psi(\xi,\varphi) = \frac{1}{2d} \sum_{\ell,m=-\infty}^{\infty} \sum_{n=-j}^{j} \bar{\psi}_n\, \psi_{m-n}\, \mathrm{e}^{-\frac{\pi\mathrm{i}}{d}\ell(2n-\ell)}\; \delta(\xi - \tfrac{\hbar}{2}m)\; \delta(\varphi - \varphi_0 - \tfrac{\pi}{d}\ell)$$

has some interesting properties.

Finite fractional Fourier-Kravchuk transform

Hilbert space $L^2(\mathbb{R})$

- By orthogonalizing

$$1,\quad q,\quad q^2,\quad q^3,\ \ldots$$

with respect to the scalar product

$$\langle\varphi,\psi\rangle = \int_{-\infty}^{\infty} \mathrm{e}^{-q^2}\,\overline{\varphi(q)}\,\psi(q)\,dq$$

we get the **Hermite polynomials**

$$H_0,\quad H_1,\quad H_2,\quad H_3,\quad \ldots$$

where

$$H_n(q) = (-1)^n\,\mathrm{e}^{q^2}\,\tfrac{d^n}{dq^n}\left(\mathrm{e}^{-q^2}\right).$$

They satisfy the relation

$$\int_{-\infty}^{\infty} \mathrm{e}^{-q^2}\,H_n(q)\,H_k(q)\,dq = 2^n\,n!\,\sqrt{\pi}\,\delta_{nk}.$$

- The **Hermite-Gauss** functions

$$\Psi_0,\quad \Psi_1,\quad \Psi_2,\quad \Psi_3,\quad \ldots$$

where

$$\Psi_n(q) = \frac{1}{\sqrt{2^n\,n!\,\sqrt{\pi}}}\,H_n(q)\,\mathrm{e}^{-\frac{q^2}{2}}$$

satisfy the orthogonality relation

$$\int_{-\infty}^{\infty} \Psi_n(q)\,\Psi_m(q)\,dq = \delta_{nm},$$

that is

$$\langle\Psi_n|\Psi_m\rangle = \delta_{nm},$$

and the completitude relation

$$\mathbb{I} = \sum_{n=0}^{\infty} |\Psi_n\rangle\langle\Psi_n|.$$

- **Fourier transform**:

$$F = \sum_{n=0}^{\infty}(-\mathrm{i})^n|\Psi_n\rangle\langle\Psi_n|.$$

- **Fractional Fourier transform** :

$$F^\alpha = \sum_{n=0}^{\infty}(-\mathrm{i})^{n\alpha}|\Psi_n\rangle\langle\Psi_n|.$$

Hilbert space $\mathbb{C}^{2j+1}$

- By orthogonalizing

$$1,\quad k,\quad k^2,\quad k^3,\ \ldots$$

with respect to the scalar product

$$\langle\varphi,\psi\rangle = \sum_{n=-j}^{j} C_{2j}^{j+k}\,\overline{\varphi(k)}\,\psi(k)$$

we get the **Kravchuk polynomials**

$$K_{-j},\quad K_{-j+1},\quad \ldots\quad K_{j-1},\quad K_j,$$

where

$$K_n(k) = \sum_{m=0}^{j+n}(-1)^m\,C_{j+k}^m\,C_{j-k}^{j+n-m}.$$

They satisfy the relation

$$\frac{1}{2^{2j}}\sum_{k=-j}^{j} C_{2j}^{j+k}\,K_n(k)\,K_m(k) = C_{2j}^{j+n}\,\delta_{nm}.$$

- The **Kravchuk** functions

$$\mathfrak{K}_{-j},\quad \mathfrak{K}_{-j+1},\quad \ldots\quad \mathfrak{K}_{j-1},\quad \mathfrak{K}_j,$$

where

$$\mathfrak{K}_n(k) = \frac{1}{2^j}\sqrt{\frac{C_{2j}^{j+k}}{C_{2j}^{j+n}}}\,K_n(k)$$

satisfy the orthogonality relation

$$\sum_{k=-j}^{j} \mathfrak{K}_n(k)\,\mathfrak{K}_m(k) = \delta_{nm},$$

that is

$$\langle\mathfrak{K}_n|\mathfrak{K}_m\rangle = \delta_{nm},$$

and the completitude relation

$$\mathbb{I} = \sum_{n=-j}^{j} |\mathfrak{K}_n\rangle\langle\mathfrak{K}_n|.$$

In addition,

$$\mathfrak{K}_n(k) = \mathfrak{K}_k(n).$$

- **Finite Fourier-Kravchuk transform**:

$$\mathbf{F}_K = \sum_{n=-j}^{j}(-\mathrm{i})^{j+n}|\mathfrak{K}_n\rangle\langle\mathfrak{K}_n|.$$

- **Finite fractional Fourier-Kravchuk** :

$$\boxed{\mathbf{F}_K^\alpha = \sum_{n=-j}^{j}(-\mathrm{i})^{(j+n)\alpha}|\mathfrak{K}_n\rangle\langle\mathfrak{K}_n|}.$$

Finite Wigner-Kravchuk function

[T. Hakioğlu and K. Wolf, J. Phys. A: Math. Gen. **33** (2000) 3313]

Hilbert space $L^2(\mathbb{R})$

- **Fractional Fourier transform** :

$$F^\alpha = \sum_{n=0}^{\infty} (-\mathrm{i})^{n\alpha} |\Psi_n\rangle\langle\Psi_n|$$

satisfies the relations:

$$F^4 = \mathbb{I}, \qquad F^2 = \Pi$$

where $\Pi\psi(q) = \psi(-q)$.

- **Elementary translation operators**:

$$U = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}Q}$$
$$V = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}P} = FUF^{-1}$$

- **Displacement operators**

$$\begin{aligned} D(q,p) &= \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(qP-pQ)} \\ &= \mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}pq}\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}pQ}\, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}qP} = \mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}pq}\, U^p\, V^q \\ &= \mathrm{e}^{\frac{\mathrm{i}}{2\hbar}pq}\, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}qP}\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}pQ} = \mathrm{e}^{\frac{\mathrm{i}}{2\hbar}pq}\, V^q\, U^p. \end{aligned}$$

- **Parity operators**

$$\begin{aligned} \Pi(q,p) &= D(q,p)\,\Pi\,D(q,p)^+ \\ &= \frac{1}{2\pi\hbar}\iint_{\mathbb{R}^2} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(qb-pa)} D(a,b)\,da db. \end{aligned}$$

- **Wigner function**

$$W_\psi(q,p) = \frac{1}{\pi\hbar}\langle\psi|\Pi(q,p)|\psi\rangle,$$
$$W_{\hat\varrho}(q,p) = \frac{1}{\pi\hbar}\,\mathrm{tr}(\hat\varrho\,\Pi(q,p)).$$

Hilbert space $\mathbb{C}^{2j+1}$

- **Fractional Fourier-Kravchuk transform** :

$$\mathbf{F}_K^\alpha = \sum_{n=-j}^{j} (-\mathrm{i})^{(j+n)\alpha} |\mathfrak{K}_n\rangle\langle\mathfrak{K}_n|$$

satisfies the relations:

$$\mathbf{F}_K^4 = \mathbb{I}, \qquad \mathbf{F}_K^2 = \Pi$$

where $\Pi\psi(n) = \psi(-n)$.

- **Kravchuk translation operators**:

$$\boxed{\begin{aligned} \mathbf{U}_K &= \mathbf{F}_K^{\frac{4}{d}} \\ \mathbf{V}_K &= \mathbf{F}\mathbf{U}_K\mathbf{F}^{-1}. \end{aligned}}$$

- **Kravchuk displacement operators**

$$\mathbf{D}_K(n,k) = \mathrm{e}^{-\frac{\pi\mathrm{i}}{d}}\, \mathbf{U}_K^n\, \mathbf{V}_K^k.$$

- **Kravchuk parity operators**

$$\Pi_K(n,k) = \frac{1}{d^2}\sum_{\ell=-j}^{j}\sum_{m=-j}^{j} \mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}(nm-k\ell)}\mathbf{D}_K(\ell,m).$$

- **Finite Wigner-Kravchuk function**

$$\boxed{\mathbf{W}_K[\psi](n,k) = \langle\psi|\Pi_K(n,k)|\psi\rangle}$$

corresponding to a function ψ,

$$\boxed{\mathbf{W}_K[\hat\varrho](n,k) = \mathrm{tr}(\hat\varrho\,\Pi_K(n,k))}$$

corresponding to a self-adjoint operator $\hat\varrho$.

Bases and frames in the continuous case

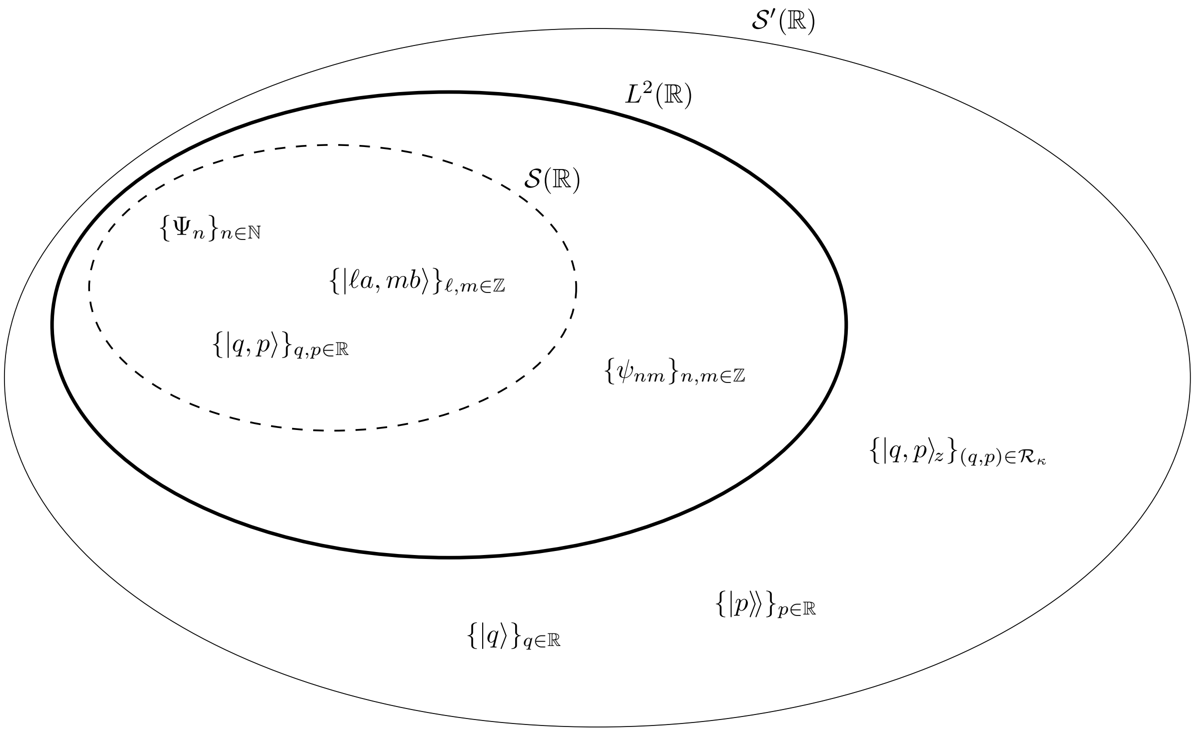


Figure 3: Bases and frames ($\mathcal{R}_\kappa = [-\frac{\kappa}{2},\frac{\kappa}{2})\times[-\frac{\pi}{\kappa},\frac{\pi}{\kappa})$).

- Hermite-Gauss functions: $\Psi_n(q) = \frac{1}{\sqrt{2^n\, n!\,\sqrt{\pi}}}\, H_n(q)\, \mathrm{e}^{-\frac{q^2}{2}},$

  Wavelets: $\psi_{nm}(x) = \mathrm{e}^{\frac{2\pi \mathrm{i}}{\kappa}mx}\, \psi(x-n\kappa),\ \psi(x) = \begin{cases} \frac{1}{\sqrt{\kappa}} & \text{for } \ x\in[-\frac{\kappa}{2},\frac{\kappa}{2}), \\ 0 & \text{for } \ x\notin[-\frac{\kappa}{2},\frac{\kappa}{2}), \end{cases}$

  Eigenbasis of Q: $|q\rangle \equiv \delta_q$

  Eigenbasis of P: $|p\rangle\!\rangle \equiv \frac{1}{\sqrt{2\pi\hbar}}\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}pq},$

  Coherent states: $|q,p\rangle = D(q,p)|0\rangle$

  Zak basis states: $|q,p\rangle_z = \sqrt{\frac{\kappa}{2\pi}}\sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}n\kappa p}\ |q+n\kappa\rangle$

- $\langle\Psi_n|\Psi_m\rangle = \delta_{nm},$

  $\langle\psi_{nm}|\psi_{k\ell}\rangle = \delta_{nk}\delta_{m\ell},$

  $\langle a|b\rangle = \delta(a-b),$

  $\langle\!\langle a|b\rangle\!\rangle = \delta(a-b),$

  $\langle q,p|a,b\rangle = \ldots$ (non-orthogonal)

  ${}_z\langle q,p|a,b\rangle_z = \delta(q-a)\,\delta(p-b).$

- $\mathbb{I} = \sum_{n=0}^{\infty} |\Psi_n\rangle\langle\Psi_n|,$

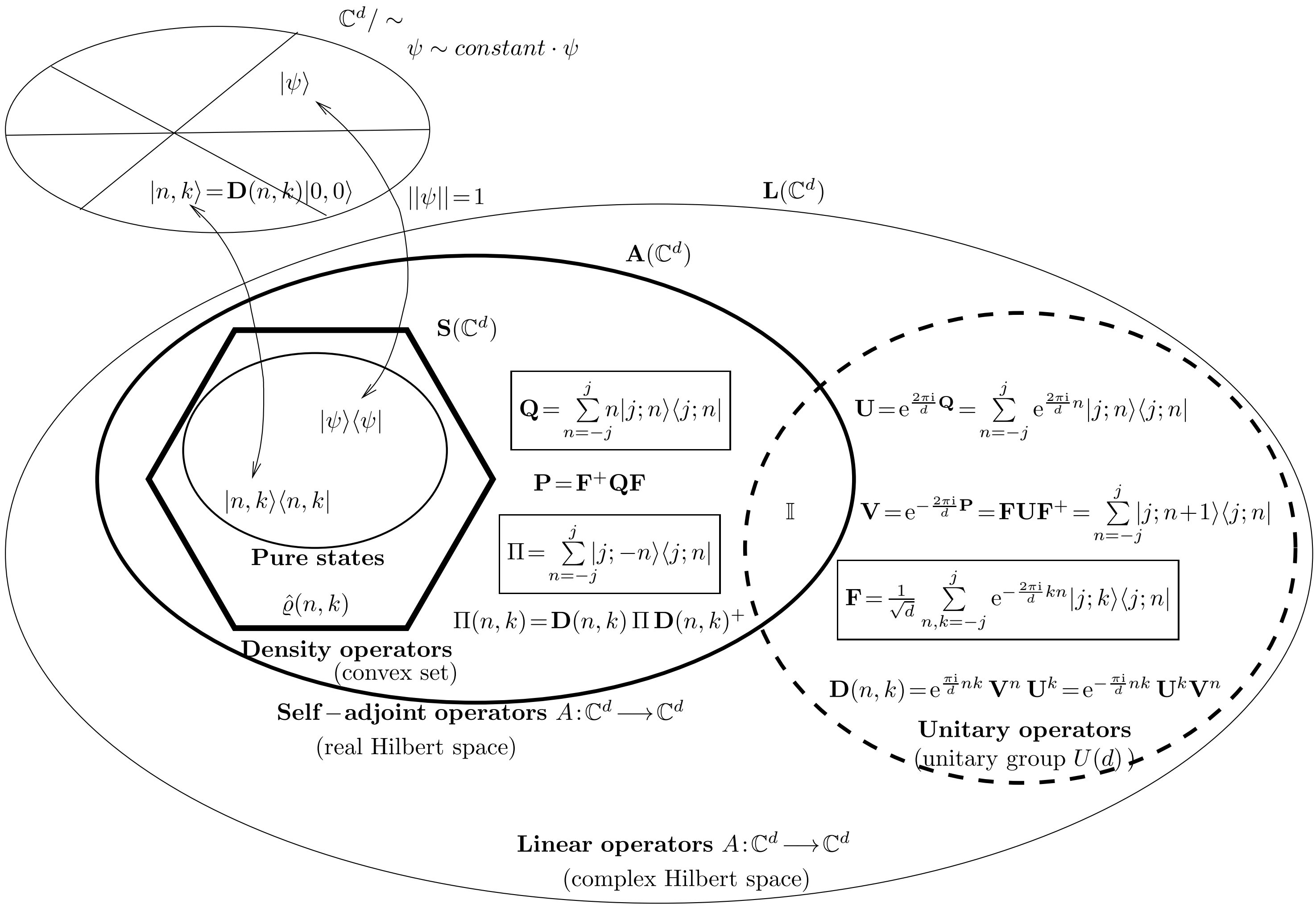
  $\mathbb{I} = \sum_{n=-\infty}^{\infty}\ \sum_{m=-\infty}^{\infty} |\psi_{nm}\rangle\langle\psi_{nm}|,$

  $\mathbb{I} = \int_{-\infty}^{\infty} dq\, |q\rangle\langle q|,$

  $\mathbb{I} = \int_{-\infty}^{\infty} dp\, |p\rangle\!\rangle\langle\!\langle p|,$

  $\mathbb{I} = \frac{1}{2\pi}\iint_{\mathbb{R}^2} |q,p\rangle\langle q,p|\ dqdp,$

  $\mathbb{I} = \iint_{[-\frac{\kappa}{2},\frac{\kappa}{2})\times[-\frac{\pi}{\kappa},\frac{\pi}{\kappa})} dqdp\, |q,p\rangle_z\, {}_z\langle q,p|.$



- $\mathbf{L}(\mathbb{C}^d)$ = space of all the linear operators $A : \mathbb{C}^d \longrightarrow \mathbb{C}^d$ with the scalar product
$$\langle A, B\rangle = \mathrm{tr}(A^+B)$$
is a **complex Hilbert space**, and $\left\{\frac{1}{\sqrt{d}}\mathbf{D}(n,k)\right\}$ is an orthonormal basis.
- $\mathbf{A}(\mathbb{C}^d)$ = space of the self-adjoint operators $A : \mathbb{C}^d \longrightarrow \mathbb{C}^d$ with the scalar product
$$\langle A, B\rangle = \mathrm{tr}(AB)$$
is a **real Hilbert space**, and $\left\{\frac{1}{\sqrt{d}}\Pi(n,k)\right\}$ is an orthonormal basis.
- $\mathbf{S}(\mathbb{C}^d)$ = set of all the **density operators** $\hat{\varrho} : \mathbb{C}^d \longrightarrow \mathbb{C}^d$ is a convex set.
- Given $|0,0\rangle \in \mathbb{C}^d$, the vectors $|n,k\rangle = \mathbf{D}(n,k)|0,0\rangle$
form a system of **coherent states**:
$$\mathbb{I} = \frac{1}{d}\sum_{n,k} |n,k\rangle\langle n,k|, \qquad \mathbf{D}(m,\ell)\,|n,k\rangle\langle n,k|\,\mathbf{D}(m,\ell)^+ = |n{+}m, k{+}\ell\rangle\langle n{+}m, k{+}\ell|.$$
- Given a density operator $\hat{\varrho}(0,0)$, the operators
$$\hat{\varrho}(n,k) = \mathbf{D}(n,k)\hat{\varrho}(0,0)\mathbf{D}(n,k)^+$$
form a system of **coherent density operators**:
$$\mathbb{I} = \frac{1}{d}\sum_{n,k} \hat{\varrho}(n,k), \qquad \mathbf{D}(m,\ell)\,\hat{\varrho}(n,k)\,\mathbf{D}(m,\ell)^+ = \hat{\varrho}(n+m, k+\ell).$$