LINEAR ALGEBRA - Solved Exercises 5

(Real and complex finite-dimensional vector spaces)

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A PARTIAL EVALUATION A

Exercise 1

Let $\mathcal{W}_1=\{\,x=(x_1,x_2,x_3)\in\mathbb{R}^3 \mid x_1-x_2-x_3=0\,\}$
$\mathcal{W}_2=\mathrm{span}\{\,(1,-1,0),(2,-2,0),(1,1,1),\ (3,1,2)\,\}$.

A1 Find the dimension of $\mathcal{W}_1$.

A2 Find the dimension of $\mathcal{W}_2$.

A3 Find the dimension of $\mathcal{W}_1\cap\mathcal{W}_2$.

Exercise 2

A4 Find the matrices of the Pauli operators
$\sigma_1:\mathbb{C}^2\longrightarrow\mathbb{C}^2,\quad \sigma_1(x_1,x_2)=(x_2,x_1)$,
$\sigma_2:\mathbb{C}^2\longrightarrow\mathbb{C}^2,\quad \sigma_2(x_1,x_2)=(-\mathrm{i}x_2,\mathrm{i}x_1)$,
$\sigma_3:\mathbb{C}^2\longrightarrow\mathbb{C}^2,\quad \sigma_3(x_1,x_2)=(x_1,-x_2)$
in the canonical basis.

A5 Compute
$\sigma_1^2,\quad \sigma_2^2,\quad \sigma_3^2,\quad \sigma_1\sigma_2-\mathrm{i}\sigma_3,\quad \sigma_2\sigma_3-\mathrm{i}\sigma_1,\quad \sigma_3\sigma_1-\mathrm{i}\sigma_2.$

Exercise 3

Let $A:\mathbb{R}^2\longrightarrow\mathbb{R}^2,\quad A(x_1,x_2)=(2x_1,x_1+x_2)$
$B:\mathbb{R}^3\to\mathbb{R}^3,\ B(x_1,x_2,x_3)=(2x_1,x_2+x_3,x_2-x_3)$.

A6 Find eigenvalues of A and the corresp. eigenspaces.

A7 Find eigenvalues of B and the corresp. eigenspaces.

Exercise 4

A8 In $\mathbb{C}^2$, find the change of basis matrix
from $\mathcal{B}=\{v_1=(1,0),\ v_2=(0,1)\}$
to $\mathcal{B}'=\{v_1'=(1,-\mathrm{i}),\ v_2'=(\mathrm{i},0)\}$,
and its inverse.

A9 Find the coordinates of $x=(1,\mathrm{i})$ with respect to the two bases.

Exercise 5

A10 Find the operator $A:\mathbb{C}^3\longrightarrow\mathbb{C}^3$ satisfying
$A(1,0,0)=(\mathrm{i},0,0)$,
$A(0,1,\mathrm{i})=(0,1,\mathrm{i})$,
$A(0,1,-\mathrm{i})=(0,-1,\mathrm{i})$.

A11 Indicate two 2-dimensional subspaces $\mathcal{V}_1,\mathcal{V}_2\subset\mathbb{R}^3$ such that $\mathcal{V}_1\cap\mathcal{V}_2=\{\,(\alpha,2\alpha,3\alpha)\mid\alpha\in\mathbb{R}\,\}$.

A12 Prove that the linear operator $B:\mathbb{R}^4\longrightarrow\mathbb{R}^4$,
$B(x_1,x_2,x_3,x_4)=(x_1,x_3+x_4,x_2+x_4,x_2+x_3)$
is diagonalizable.

B PARTIAL EVALUATION B

Exercise 1

B1 In $\mathbb{R}^3$, compute $\langle(1,2,3),(1,0,-1)\rangle$.

B2 In $\mathbb{C}^2$, compute $\langle(1-\mathrm{i},2+3\mathrm{i}),(1+2\mathrm{i},3-\mathrm{i})\rangle$.

B3 In $\mathbb{R}^3$, find the orthogonal projection $\mathcal{P}_w x$ of $x=(1,3,-2)$ on $w=(1,1,0)$.

B4 In $\mathbb{R}^3$, find an orthonormal basis in the subspace $\mathcal{H}=\{\,x=(x_1,x_2,x_3)\mid x_1-2x_2+x_3=0\,\}$.

B5 Describe the orthogonal complement $\mathcal{H}^\perp$ of $\mathcal{H}$ by indicating an orthonormal basis.

Exercise 2

B6 Find the orthogonal projectors $P,P^\perp:\mathbb{R}^3\to\mathbb{R}^3$ corresponding to $\mathcal{H}$ and $\mathcal{H}^\perp$ from Exercise 1.

B7 Prove that P and $P^\perp$ are self-adjoint.

B8 Prove that $P+P^\perp=\mathbb{I}$, where $\mathbb{I}:\mathbb{R}^3\to\mathbb{R}^3,\ \mathbb{I}x=x$.

B9 Prove that $P^2=P$ and $(P^\perp)^2=P^\perp$.

B10 Prove that $PP^\perp=0$, where $0:\mathbb{R}^3\to\mathbb{R}^3,\ 0x=0$.

Exercise 3

B11 Prove that
$A:\mathbb{C}^2\longrightarrow\mathbb{C}^2,\quad A(x_1,x_2)=(x_1+2x_2,2x_1-x_2)$
is self-adjoint operator.

B12 Find the eigenvalues of A.

B13 Find an orthonormal eigenbasis of A.

B14 Describe the spectral resolution of A, and check it.

Exercise 4

B15 Prove that
$U:\mathbb{C}^2\to\mathbb{C}^2,\ U(x_1,x_2)=\left(\frac{1}{\sqrt{2}}x_1+\frac{1}{\sqrt{2}}x_2,\frac{1}{\sqrt{2}}x_1-\frac{1}{\sqrt{2}}x_2\right)$
is a unitary operator.

B16 Find the eigenvalues of U.

B17 Find an orthonormal eigenbasis of U.

B18 Describe the spectral resolution of U, and check it.

Exercise 5

B19 Prove that
$\varrho:\mathbb{C}^2\longrightarrow\mathbb{C}^2,\quad \varrho(x_1,x_2)=\left(\tfrac{1}{2}x_1-\tfrac{1}{4}x_2,-\tfrac{1}{4}x_1+\tfrac{1}{2}x_2\right)$
is a *density operator*, that is, ϱ is self-adjoint, its eigenvalues are greater than or equal to 0, and $\mathrm{tr}\,\varrho=1$.

Some definitions, notations and results

1 **Definition**. In a vector space over the field $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$,

$$\boxed{\mathrm{span}\{v_1,v_2,...,v_n\}\overset{\text{def}}{=}\{\alpha_1v_1+\alpha_2v_2+...+\alpha_nv_n\mid\alpha_k\in\mathbb{K}\}}$$

2 **Theorem**. In the case of vector space over $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$,

λ is an eigenvalue of a linear operator $A:\mathcal{V}\to\mathcal{V}$ $\Leftrightarrow$
- $\lambda\in\mathbb{K}$ and
- λ is a root of the equation

$$\begin{vmatrix} a_1^1-\lambda & a_2^1 & \cdots & a_n^1\\ a_1^2 & a_2^2-\lambda & \cdots & a_n^2\\ \vdots & \vdots & \ddots & \vdots\\ a_1^n & a_2^n & \cdots & a_n^n-\lambda\end{vmatrix}=0$$

3 The *eigenspace* corresponding to λ is $\boxed{\{\,x\mid Ax=\lambda x\,\}}$.

4 The **usual inner product** (also called scalar product) :
In $\mathbb{R}^3$: $\langle(x_1,x_2,x_3),(y_1,y_2,y_3)\rangle=x_1y_1+x_2y_2+x_3y_3$
In $\mathbb{C}^2$: $\langle(x_1,x_2),(y_1,y_2)\rangle=\bar{x}_1y_1+\bar{x}_2y_2$.

5 **Orthogonal projection** of x on $w\neq0$ is $P_wx=\frac{\langle w,x\rangle}{\langle w,w\rangle}w$.

6 **Theorem** In a complex Hilbert space $\mathcal{H}$:
$A:\mathcal{H}\to\mathcal{H}$ is *self-adjoint* $\Leftrightarrow$ its matrix in an orthonormal basis is Hermitian ($A^\dagger=A$).
$A:\mathcal{H}\to\mathcal{H}$ is *unitary* $\Leftrightarrow$ its matrix in an orthonormal basis is unitary ($A^\dagger A=\mathbb{I}$).

7 **Dirac's notation** in $\mathbb{C}^2$. If to each $x=(x_1,x_2)\in\mathbb{C}^2$, we associate the matrices $|x\rangle=\begin{pmatrix}x_1\\x_2\end{pmatrix}$, $\langle x|=(\,\bar{x}_1\ \ \bar{x}_2\,)$,
then $\langle x|y\rangle=(\,\bar{x}_1\ \ \bar{x}_2\,)\begin{pmatrix}y_1\\y_2\end{pmatrix}=\bar{x}_1y_1+\bar{x}_2y_2=\langle x,y\rangle$.
For $a=(a_1,a_2)$ and $b=(b_1,b_2)$, the formula $A=|a\rangle\langle b|$ defines the operator $A:\mathbb{C}^2\longrightarrow\mathbb{C}^2$, $A|x\rangle=|a\rangle\langle b|x\rangle$ with the matrix $A=|a\rangle\langle b|=\begin{pmatrix}a_1\\a_2\end{pmatrix}(\,\bar{b}_1\ \ \bar{b}_2\,)=\begin{pmatrix}a_1\bar{b}_1 & a_1\bar{b}_2\\ a_2\bar{b}_1 & a_2\bar{b}_2\end{pmatrix}$.

SOLUTIONS for PARTIAL EVALUATION A

$\boxed{A1}$ $\mathcal{W}_1=\{(x_1,x_2,x_1-x_2)\mid x_1,x_2\in\mathbb{R}\}$
$=\{x_1(1,0,1)+x_2(0,1,-1)|x_1,x_2\in\mathbb{R}\}=\mathrm{span}\{(1,0,1),(0,1,-1)\}$.
Since $(1,0,1)$ and $(0,1,-1)$ are linearly indep., it follows that $\{(1,0,1),(0,1,-1)\}$ is a basis and $\dim\mathcal{W}_1=2$.

$\boxed{A2}$ $\left.\begin{array}{l}(2,-2,0)=2(1,-1,0)\\(3,1,2)=(1,-1,0)+2(1,1,1)\end{array}\right\}\Rightarrow\ \{(1,-1,0),(1,1,1)\}$ is a basis in $\mathcal{W}_2\ \Rightarrow\ \dim\mathcal{W}_2=2$.

$\boxed{A3}$ $\mathcal{W}_2=\{\alpha(1,-1,0)+\beta(1,1,1)\mid\alpha,\beta\in\mathbb{R}\}$
$=\{(\alpha+\beta,-\alpha+\beta,\beta)\mid\alpha,\beta\in\mathbb{R}\}$. But
$(\alpha+\beta,-\alpha+\beta,\beta)\in\mathcal{W}_1\ \Leftrightarrow\ (\alpha+\beta)-(-\alpha+\beta)-\beta=0$
$\Leftrightarrow\ \beta=2\alpha$. Consequently,
$\mathcal{W}_1\cap\mathcal{W}_2=\{(3\alpha,\alpha,2\alpha)\mid\alpha\in\mathbb{R}\}\Rightarrow\{(3,1,2)\}$ is a basis.

$\boxed{A4}$ $\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad\sigma_2=\begin{pmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{pmatrix},\quad\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$

$\boxed{A5}$ $\sigma_1^2=\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad\sigma_2^2=\sigma_3^2=\begin{pmatrix}1&0\\0&1\end{pmatrix}.$

$\sigma_1\sigma_2-\mathrm{i}\sigma_3=\sigma_2\sigma_3-\mathrm{i}\sigma_1=\sigma_3\sigma_1-\mathrm{i}\sigma_2=\begin{pmatrix}0&0\\0&0\end{pmatrix}.$

$\boxed{A6}$ In the canonical basis $\{(1,0),(0,1)\}$, the matrix of A is
$A=\begin{pmatrix}2&0\\1&1\end{pmatrix}.\qquad\begin{vmatrix}2-\lambda&0\\1&1-\lambda\end{vmatrix}=0\Rightarrow\begin{array}{l}\lambda_1=1,\\\lambda_2=2.\end{array}$
$\{(x_1,x_2)\mid A(x_1,x_2)=1\,(x_1,x_2)\}=\{(0,\alpha)\mid\alpha\in\mathbb{R}\}$
$\{(x_1,x_2)\mid A(x_1,x_2)=2\,(x_1,x_2)\}=\{(\alpha,\alpha)\mid\alpha\in\mathbb{R}\}$

$\boxed{A7}$ In the canonical basis $\{(1,0,0),(0,1,0),(0,0,1)\}$,
$B=\begin{pmatrix}2&0&0\\0&1&1\\0&1&-1\end{pmatrix},\quad\begin{vmatrix}2-\lambda&0&0\\0&1-\lambda&1\\0&1&-1-\lambda\end{vmatrix}=0\Rightarrow\begin{array}{l}\lambda_1=2,\\\lambda_{2,3}=\pm\sqrt{2}.\end{array}$
$\{x=(x_1,x_2,x_3)\mid Bx=2\,x\}=\{(\alpha,0,0)\mid\alpha\in\mathbb{R}\}$
$\{x\mid Bx=\pm\sqrt{2}\,x\}=\{(0,\alpha,(\pm\sqrt{2}-1)\alpha)\mid\alpha\in\mathbb{R}\}$

$\boxed{A8}$ $\begin{cases}v_1'=\alpha_1^1\,v_1+\alpha_1^2\,v_2,\\v_2'=\alpha_2^1\,v_1+\alpha_2^2\,v_2,\end{cases}\Rightarrow\quad S=\begin{pmatrix}\alpha_1^1&\alpha_1^2\\\alpha_2^1&\alpha_2^2\end{pmatrix}=\begin{pmatrix}1&\mathrm{i}\\-\mathrm{i}&0\end{pmatrix}.$

Transpose of S is $S^T=\begin{pmatrix}1&-\mathrm{i}\\\mathrm{i}&0\end{pmatrix}$ and $\det S=-1$.

$S^{-1}=\frac{1}{\det S}\begin{pmatrix}(-1)^{1+1}\,0&(-1)^{1+2}\,\mathrm{i}\\(-1)^{2+1}\,(-\mathrm{i})&(-1)^{2+2}1\end{pmatrix}=\begin{pmatrix}0&\mathrm{i}\\-\mathrm{i}&-1\end{pmatrix}.$

$\boxed{A9}$ $x=(1,\mathrm{i})=x_1(1,0)+x_2(0,1)\quad\Rightarrow\quad x_1=1,\ \ x_2=\mathrm{i}.$
$x=(1,\mathrm{i})=x_1'(1,-\mathrm{i})+x_2'(\mathrm{i},0)\quad\Rightarrow\quad x_1'=-1,\ \ x_2'=-2\mathrm{i}.$

$\boxed{A10}$ $\mathcal{B}'=\{v_1'=(1,0,0),v_2'=(0,1,\mathrm{i}),v_3'=(0,1,-\mathrm{i})\}$
is a basis of $\mathbb{C}^3$ containing only eigenvectors of A,
$Av_1'=\mathrm{i}\,v_1',\quad Av_2'=1\,v_2',\quad Av_3'=-1\,v_3'.$
Matrix of A in the canonical basis, obtained from
$S^{-1}AS=\begin{pmatrix}\mathrm{i}&0&0\\0&1&0\\0&0&-1\end{pmatrix},\quad$ where $\ S=\begin{pmatrix}1&1&0\\0&1&1\\0&\mathrm{i}&-\mathrm{i}\end{pmatrix},$
is
$A=S\begin{pmatrix}\mathrm{i}&0&0\\0&1&0\\0&0&-1\end{pmatrix}S^{-1}=\begin{pmatrix}\mathrm{i}&0&0\\0&0&-\mathrm{i}\\0&\mathrm{i}&0\end{pmatrix},$
that is $\ A:\mathbb{C}^3\longrightarrow\mathbb{C}^3,\quad A(x_1,x_2,x_3)=(\mathrm{i}x_1,-\mathrm{i}x_3,\mathrm{i}x_2).$

$\boxed{A11}$ $\mathcal{V}_1=\mathrm{span}\{(1,2,3),(1,0,0)\},\ \ \mathcal{V}_2=\mathrm{span}\{(1,2,3),(0,1,0)\}.$

$\boxed{A12}$ $\begin{vmatrix}1-\lambda&0&0&0\\0&-\lambda&1&1\\0&1&-\lambda&1\\0&1&1&-\lambda\end{vmatrix}=0\quad\Rightarrow\quad\begin{array}{l}\lambda_1=1,\\\lambda_2=2,\\\lambda_3=-1,\\\lambda_4=-1,\end{array}$
$\{x\in\mathbb{R}^4\mid Bx=1\,x\}=\{(\alpha,0,0,0)\mid\alpha\in\mathbb{R}\},$
$\{x\in\mathbb{R}^4\mid Bx=2\,x\}=\{(0,\alpha,\alpha,\alpha)\mid\alpha\in\mathbb{R}\},$
$\{x\in\mathbb{R}^4\mid Bx=-1\,x\}=\{(0,\alpha,\beta,-\alpha-\beta)\mid\alpha,\beta\in\mathbb{R}\}.$
$\mathcal{B}'=\{(1,0,0,0),(0,1,1,1),(0,1,0,-1),(0,0,1,-1)\}$
is a basis of $\mathbb{R}^4$ containing only eigenvectors of B.

SOLUTIONS for PARTIAL EVALUATION B

$\boxed{B1}$ $\langle(1,2,3),(1,0,-1)\rangle=1\cdot1+2\cdot0+3\cdot(-1)=-2$

$\boxed{B2}$ $\langle(1-\mathrm{i},2+3\mathrm{i}),(1+2\mathrm{i},3-\mathrm{i})\rangle=(\overline{1-\mathrm{i}})(1+2\mathrm{i})+(\overline{2+3\mathrm{i}})(3-\mathrm{i})$
$=(1+\mathrm{i})(1+2\mathrm{i})+(2-3\mathrm{i})(3-\mathrm{i})=2-8\mathrm{i}.$

$\boxed{B3}$ $P_wx=\frac{\langle w,x\rangle}{\langle w,w\rangle}w=\frac{\langle(1,1,0),(1,3,-2)\rangle}{\langle(1,1,0),(1,1,0)\rangle}(1,1,0)=(2,2,0)$

$\boxed{B4}$ $\mathcal{H}=\{(x_1,x_2,-x_1+2x_2)\mid x_1,\ x_2\in\mathbb{R}\}$
$=\{x_1(1,0,-1)+x_2(0,1,2)\mid x_1,\ x_2\in\mathbb{R}\}\ \Rightarrow$
$\{v_1=(1,0,-1),\ v_2=(0,1,2)\}$ is a basis in $\mathcal{H}$.
By using Gram-Schmidt, we obtain the orthogonal basis $\{w_1=(1,0,-1),\ w_2=v_2-P_{w_1}v_2=(1,1,1)\}$. Consequently,
$\left\{u_1=\frac{w_1}{||w_1||}=\left(\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right),u_2=\frac{w_2}{||w_2||}=\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)\right\}$
is an orthonormal basis of $\mathcal{H}$.

$\boxed{B5}$ $\mathcal{H}^\perp=\left\{x\ \middle|\ \begin{array}{l}\langle x,v_1\rangle=0\\\langle x,v_2\rangle=0\end{array}\right\}=\left\{(x_1,x_2,x_3)\ \middle|\ \begin{array}{l}x_1-x_3=0\\x_2+2x_3=0\end{array}\right\}$
$=\{(x_1,-2x_1,x_1)\mid x_1\in\mathbb{R}\}=\{x_1(1,-2,1)\mid x_1\in\mathbb{R}\}.$
Consequently, $\ \left\{u_3=\frac{(1,-2,1)}{||(1,-2,1)||}=\left(\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}}\right)\right\}$
is an orthonormal basis of $\mathcal{H}^\perp$.

$\boxed{B6}$ $P=|u_1\rangle\langle u_1|+|u_2\rangle\langle u_2|=\frac{1}{6}\begin{pmatrix}5&2&-1\\2&2&2\\-1&2&5\end{pmatrix},\qquad(*)$

$P^\perp=|u_3\rangle\langle u_3|=\frac{1}{6}\begin{pmatrix}1&-2&1\\-2&4&-2\\1&-2&1\end{pmatrix}.\qquad(**)$

$\boxed{B7}$ $(|x\rangle\langle y|)^\dagger=|y\rangle\langle x|\ \Rightarrow\ P^\dagger=P$ and $(P^\perp)^\dagger=P^\perp$.

$\boxed{B8}$ $\begin{array}{c}\{u_1,u_2,u_3\}\text{ orthonormal basis of }\mathbb{R}^3\\\Downarrow\\|u_1\rangle\langle u_1|+|u_2\rangle\langle u_2|+|u_3\rangle\langle u_3|=\mathbb{I}\end{array}$ Also $\begin{array}{c}P+P^\perp=\mathbb{I}.\\\text{See }(*),(**).\end{array}$

$\boxed{B9-10}$ $\langle u_j|u_k\rangle=\delta_{jk}\ \Rightarrow P^2=P,\ \ (P^\perp)^2=P^\perp,\quad PP^\perp=0.$
These relations also follow from (*) and (**).

$\boxed{B11}$ In the canonical basis $\{(1,0),(0,1)\}$, the matrix of A is
$A=\begin{pmatrix}1&2\\2&-1\end{pmatrix}=\bar{A}^T=A^\dagger.\quad\boxed{6}\ \Rightarrow\ A$ is self-adjoint.

$\boxed{B12}$ $\begin{vmatrix}1-\lambda&2\\2&-1-\lambda\end{vmatrix}=0\Rightarrow\begin{array}{l}\lambda_1=\sqrt{5},\\\lambda_2=-\sqrt{5}.\end{array}$

$\boxed{B13}$ $\{x=(x_1,x_2)\mid Ax=\pm\sqrt{5}\,x\}=\{\alpha(2,\pm\sqrt{5}-1)\mid\alpha\in\mathbb{R}\}.$
$\{v_1=(2,\sqrt{5}-1),v_2=(2,-\sqrt{5}-1)\}$ satisfy $\langle v_1,v_2\rangle=0.$
$\left\{u_1=\frac{(2,\sqrt{5}-1)}{\sqrt{10-2\sqrt{5}}},u_2=\frac{(2,-\sqrt{5}-1)}{\sqrt{10+2\sqrt{5}}}\right\}$ satisfy $\langle u_j,u_k\rangle=\delta_{jk}.$

$\boxed{B14}$ Spectral resolution is $\quad A=\lambda_1|u_1\rangle\langle u_1|+\lambda_2|u_2\rangle\langle u_2|.$
We have $A|u_1\rangle=(\lambda_1|u_1\rangle\langle u_1|+\lambda_2|u_2\rangle\langle u_2|)|u_1\rangle=\lambda_1|u_1\rangle$
$A|u_2\rangle=(\lambda_1|u_1\rangle\langle u_1|+\lambda_2|u_2\rangle\langle u_2|)|u_2\rangle=\lambda_2|u_2\rangle.$

$\boxed{B15}$ The matrix of U in the canonical basis $\{(1,0),(0,1)\}$
$U=\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\end{pmatrix}$ satisfies $UU^\dagger=\begin{pmatrix}1&0\\0&1\end{pmatrix}$. We use $\boxed{6}$.

$\boxed{B16}$ $\begin{vmatrix}\frac{1}{\sqrt{2}}-\lambda&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}-\lambda\end{vmatrix}=0\Rightarrow\begin{array}{l}\lambda_1=1,\\\lambda_2=-1.\end{array}$

$\boxed{B17}$ $\{x=(x_1,x_2)\mid Ux=\pm1\,x\}=\{\alpha(1,\pm\sqrt{2}-1)\mid\alpha\in\mathbb{R}\}.$
$\{v_1=(1,\sqrt{2}-1),v_2=(1,-\sqrt{2}-1)\}$ satisfy $\langle v_1,v_2\rangle=0.$
$\left\{u_1=\frac{(1,\sqrt{2}-1)}{\sqrt{4-2\sqrt{2}}},u_2=\frac{(1,-\sqrt{2}-1)}{\sqrt{4+2\sqrt{2}}}\right\}$ satisfy $\langle u_j,u_k\rangle=\delta_{jk}.$

$\boxed{B18}$ Spectral resolution is $\quad U=\lambda_1|u_1\rangle\langle u_1|+\lambda_2|u_2\rangle\langle u_2|.$
We have $U|u_1\rangle=(\lambda_1|u_1\rangle\langle u_1|+\lambda_2|u_2\rangle\langle u_2|)|u_1\rangle=\lambda_1|u_1\rangle$
$U|u_2\rangle=(\lambda_1|u_1\rangle\langle u_1|+\lambda_2|u_2\rangle\langle u_2|)|u_2\rangle=\lambda_2|u_2\rangle.$

$\boxed{B19}$ In the canonical basis $\{(1,0),(0,1)\}$, the matrix of ϱ is
$\varrho=\begin{pmatrix}\frac{1}{2}&-\frac{1}{4}\\-\frac{1}{4}&\frac{1}{2}\end{pmatrix}.\quad\begin{vmatrix}\frac{1}{2}-\lambda&-\frac{1}{4}\\-\frac{1}{4}&\frac{1}{2}-\lambda\end{vmatrix}=0\Rightarrow\begin{array}{l}\lambda_1=\frac{1}{4}\geq0,\\\lambda_2=\frac{3}{4}\geq0.\end{array}$
$\mathrm{tr}\,\varrho=\frac{1}{2}+\frac{1}{2}=1,\quad$ and $\quad\varrho^\dagger=\bar{\varrho}^T=\varrho\ \Rightarrow\ \varrho$ is self-adjoint.