COMPLEX ANALYSIS - Problems

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R Fundamental definitions and results

R0 $(x_1+y_1\mathrm{i})(x_2+y_2\mathrm{i})=(x_1x_2-y_1y_2)+(x_1y_2+x_2y_1)\mathrm{i}$

$\overline{x+y\mathrm{i}}=x-y\mathrm{i}, \quad |x+y\mathrm{i}|=\sqrt{x^2+y^2}, \quad d(z_1,z_2)=|z_1-z_2|$

R1 $\frac{1}{1-z}=1+z+z^2+z^3+\ldots$ for $|z|<1$

R2 $\mathrm{e}^z=1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\ldots$ for any z

R3 In the neighbourhood of a pole z_0 of order k

$$f(z)=\frac{a_{-k}}{(z-z_0)^k}+\ldots+\frac{a_{-1}}{(z-z_0)}+a_0+a_1(z-z_0)+\ldots$$

and $\mathbf{Res}_{z_0}f=a_{-1}=\frac{1}{(k-1)!}\lim\limits_{z\to z_0}\left((z-z_0)^k f(z)\right)^{(k-1)}$

R4 Complex integral of $f:D\longrightarrow\mathbb{C}$ along $\gamma:[a,b]\longrightarrow D$ is

$$\int_\gamma f(z)\,dz=\int_a^b f(\gamma(t))\,\gamma'(t)dt$$

R4 **Theorem of Residues**. For an open set $D\subseteq\mathbb{C}$:

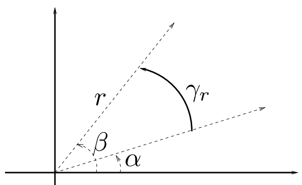
$f:D\longrightarrow\mathbb{C}$ holomorphic function
S set of isolated singular points
γ path homotopic to zero in $D\cup S$
$\Rightarrow$ $\int_\gamma f(z)dz = 2\pi\mathrm{i}\sum\limits_{z\in S} n(z,\gamma)\mathbf{Res}_z f$

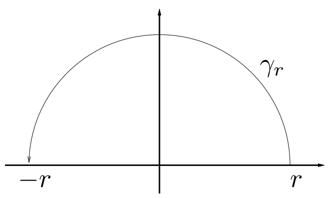
R4 **Lemma 1.** For $\gamma_r:[\alpha,\beta]\to\mathbb{C},\ \gamma_r(t)=r\,\mathrm{e}^{\mathrm{i}t}$

$$\lim_{z\to\infty} z\,f(z)=0 \ \Rightarrow\ \lim_{r\to\infty}\int_{\gamma_r} f(z)\,dz=0$$

R5 **Lemma 2 (Jordan).** For $\gamma_r:[0,\pi]\to\mathbb{C},\ \gamma_r(t)=r\,\mathrm{e}^{\mathrm{i}t}$

$$\lim_{z\to\infty} f(z)=0 \ \Rightarrow\ \lim_{r\to\infty}\int_{\gamma_r} f(z)\,\mathrm{e}^{\mathrm{i}z}\,dz=0$$





A Problems

1 $(1-2\mathrm{i})^3=?,\quad \frac{1-\mathrm{i}}{3+4\mathrm{i}}=?,\quad \mathrm{e}^{\frac{5\pi\mathrm{i}}{6}}=?,\quad d(1+\mathrm{i},2-\mathrm{i})=?,$

$(z\,\sin z)'=?,\quad (\mathrm{e}^{z^2})'=?,\quad \lim\limits_{n\to\infty}\left(\frac{n}{n+1}+\frac{1}{n}\mathrm{i}\right)=?.$

2 Expand in a power series of z the functions

$f_1(z)=\mathrm{e}^{z^2},\qquad f_2(z)=z\,\mathrm{e}^{\frac{1}{z}},\qquad f_3(z)=\frac{1}{z^3-z^4}$

Solution.

$f_1(z)=1+\frac{z^2}{1!}+\frac{z^4}{2!}+\frac{z^6}{3!}+\ldots$ for any z.

$f_2(z)=z\left(1+\frac{1}{1!}\frac{1}{z}+\frac{1}{2!}\frac{1}{z^2}+\frac{1}{3!}\frac{1}{z^3}+\ldots\right)=\ldots+\frac{1}{3!}\frac{1}{z^2}+\frac{1}{2!}\frac{1}{z}+1+z$ for $|z|>0$.

$f_3(z)=\frac{1}{z^3}\left(1+z+z^2+z^3+\ldots\right)=\frac{1}{z^3}+\frac{1}{z^2}+\frac{1}{z}+1+z+\ldots$ for $0<|z|<1$.

3 Compute the residues of $f(z)=\frac{1}{z^3(z^2+1)^2}$

Hint. 0 is a pole of order 3 and $\pm\mathrm{i}$ are poles of order 2.

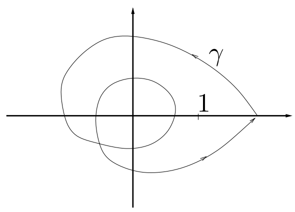
B Problems

1 Prove that $f:\mathbb{C}\longrightarrow\mathbb{C},\quad f(z)=1+\bar{z}$ is not $\mathbb{C}$-differentiable.

Hint. Use the Cauchy-Riemann theorem.

2 For $\gamma:[0,\pi]\longrightarrow\mathbb{C},\ \gamma(t)=\mathrm{e}^{\mathrm{i}t}$, compute $\int_\gamma(\bar{z}+\mathrm{i})\,dz$.

Hint. Use the definiton of the complex line integral.



3 Compute the integral $\int_\gamma \frac{1}{z^2(1-z)}\,dz$ where γ is the path from the figure above.

Solution. By using the theorem of residues, we get

$$\int_\gamma \frac{1}{z^2(1-z)}\,dz=2\pi\mathrm{i}\,(2\,\mathbf{Rez}_0 f+\mathbf{Rez}_1 f)=2\pi\mathrm{i}.$$

4 Compute the residue of $f(z)=z\,\mathrm{e}^{\frac{1}{z}}$ at the singular point $z_0=0$.

Solution. $f(z)=\ldots+\frac{1}{3!}\frac{1}{z^2}+\frac{1}{2!}\frac{1}{z}+1+z$ for any z $\Rightarrow$ $\mathbf{Res}_0 f=\frac{1}{2}$.

C Problems

1 Compute the residues of $f(z)=z^2\mathrm{e}^{\frac{1}{z-1}}$ at the singular point $z_0=1$.

Hint. $f(z)=\left((z-1)^2+2(z-1)+1\right)\left(1+\frac{1}{1!}\frac{1}{z-1}+\frac{1}{2!}\frac{1}{(z-1)^2}+\frac{1}{3!}\frac{1}{(z-1)^3}+\ldots\right)$.

2 Compute the integrals $I_1=\int_0^{2\pi}\frac{1}{2+\cos t}\,dt$ and $I_2=\int_0^{2\pi}\frac{1}{2+\sin t}\,dt$.

Solution. By using the path $\gamma:[0,2\pi]\longrightarrow\mathbb{C},\ \gamma(t)=\mathrm{e}^{\mathrm{i}t}$, we get

$$I_1=\int_0^{2\pi}\frac{1}{2+\frac{\mathrm{e}^{\mathrm{i}t}+\mathrm{e}^{-\mathrm{i}t}}{2}}dt=\int_0^{2\pi}\frac{1}{\mathrm{i}\mathrm{e}^{\mathrm{i}t}}\frac{2}{4+\mathrm{e}^{\mathrm{i}t}+\mathrm{e}^{-\mathrm{i}t}}(\mathrm{e}^{\mathrm{i}t})'\,dt$$

$$=-\mathrm{i}\int_\gamma\frac{1}{z}\frac{2}{4+z+\frac{1}{z}}\,dz=-\mathrm{i}\int_\gamma\frac{2}{z^2+4z+1}dz=2\pi\mathbf{Res}_{-2+\sqrt{3}}\ \frac{2}{z^2+4z+1}=\frac{2\pi}{\sqrt{3}}.$$

3 Compute the integrals $\int_0^\infty\frac{x^2}{(x^2+1)(x^2+4)}dx$, $\int_0^\infty\frac{x^2}{(x^2+4)(x^2+9)}dx$.

Solution. The integral is convergent: at infinity $\frac{x^2}{(x^2+1)(x^2+4)}\sim\frac{1}{x^2}$.

By using $\gamma_r:[0,\pi]\longrightarrow\mathbb{C},\ \gamma_r(t)=r\,\mathrm{e}^{\mathrm{i}t}$ and $f(z)=\frac{z^2}{(z^2+1)(z^2+4)}$, we get

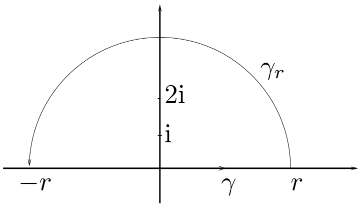
$$\int_{\gamma_r}f(z)dz+\int_{-r}^r f(x)dx=2\pi\mathrm{i}\,(\mathbf{Rez}_\mathrm{i}f+\mathbf{Rez}_{2\mathrm{i}}f),\quad\text{whence}$$

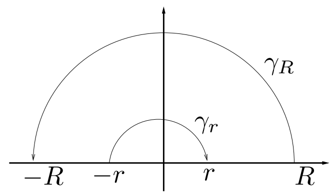
$$\lim_{r\to\infty}\int_{\gamma_r}f(z)dz+\int_{-\infty}^\infty f(x)dx=2\pi\mathrm{i}\,(\mathbf{Rez}_\mathrm{i}f+\mathbf{Rez}_{2\mathrm{i}}f).$$

Since $|z\,f(z)|=\frac{|z^3|}{|z^2+1|\cdot|z^2+4|}=\frac{|z^3|}{|z^2-(-1)|\cdot|z^2-(-4)|}\le\frac{|z|^3}{|\,|z|^2-1|\cdot|\,|z|^2-4|}$,

we have $\lim_{z\to\infty}z\,f(z)=0$ and in view of lemma 1, $\lim\limits_{r\to\infty}\int_{\gamma_r}f(z)dz=0$.

Consequently, $\int_0^\infty f(x)dx=\pi\mathrm{i}\,(\mathbf{Rez}_\mathrm{i}f+\mathbf{Rez}_{2\mathrm{i}}f)=\frac{\pi}{6}$.





4 Prove that $\int_0^\infty\frac{\sin x}{x}=\frac{\pi}{2}$ (Poisson's integral).

Solution. Let $0<r<R$ and $\gamma_R,\gamma_r:[0,\pi]\to\mathbb{C},\ \gamma_R(t)=R\,\mathrm{e}^{\mathrm{i}t},\ \gamma_r(t)=r\,\mathrm{e}^{\mathrm{i}(\pi-t)}$.

From the residue theorem (right hand side of fig.) it follows the relation

$\int_{\gamma_R}\frac{\mathrm{e}^{\mathrm{i}z}}{z}\,dz+\int_{-R}^{-r}\frac{\mathrm{e}^{\mathrm{i}x}}{x}\,dx+\int_{\gamma_r}\frac{\mathrm{e}^{\mathrm{i}z}}{z}\,dz+\int_r^R\frac{\mathrm{e}^{\mathrm{i}x}}{x}\,dx=0$, that is

$$\int_{\gamma_R}\frac{\mathrm{e}^{\mathrm{i}z}}{z}\,dz+\int_{\gamma_r}\frac{\mathrm{e}^{\mathrm{i}z}}{z}\,dz+\int_r^R\frac{\mathrm{e}^{\mathrm{i}x}-\mathrm{e}^{-\mathrm{i}x}}{x}\,dx=0,$$

$$\int_{\gamma_R}\frac{\mathrm{e}^{\mathrm{i}z}}{z}\,dz+\int_{\gamma_r}\frac{1}{z}\,dz+\int_{\gamma_r}\frac{\mathrm{e}^{\mathrm{i}z}-1}{z}\,dz+2\mathrm{i}\int_r^R\frac{\sin x}{x}\,dx=0.$$

If g is an antiderivative of $f(z)=\frac{\mathrm{e}^{\mathrm{i}z}-1}{z}$, by using $\int_{\gamma_r}\frac{1}{z}\,dz=-\pi\mathrm{i}$, we get

$$\int_{\gamma_R}\frac{\mathrm{e}^{\mathrm{i}z}}{z}\,dz-\pi\mathrm{i}+(g(r)-g(-r))+2\mathrm{i}\int_r^R\frac{\sin x}{x}\,dx=0.$$

But, in view of Jordan's lemma $\lim_{R\to\infty}\int_{\gamma_R}\frac{\mathrm{e}^{\mathrm{i}z}}{z}=0$,

and for $R\to\infty$ and $r\to 0$ we get $2\mathrm{i}\int_0^\infty\frac{\sin x}{x}\,dx=\pi\mathrm{i}$.

5 Find the holomorphic function $f:\mathbb{C}\to\mathbb{C}$ with $\mathfrak{Im}f(x+y\mathrm{i})=2xy-2x$, $f(\mathrm{i})=0$.

6 For $z_0\neq 0$, expand $f:\mathbb{C}^*\to\mathbb{C},\ f(z)=\frac{1}{z}$ in a power series of $(z-z_0)$.

Solution. We have

$$f(z)=\frac{1}{z_0+z-z_0}=\frac{1}{z_0}\frac{1}{1-\frac{z-z_0}{-z_0}}=\frac{1}{z_0}\left(1-\frac{1}{z_0}(z-z_0)+\frac{1}{z_0^2}(z-z_0)^2-\ldots\right),$$

for $|z-z_0|<|z_0|$;

$$f(z)=\frac{1}{z_0+z-z_0}=\frac{1}{z-z_0}\frac{1}{1-\frac{-z_0}{z-z_0}}=\frac{1}{z-z_0}\left(1-\frac{z_0}{(z-z_0)}+\frac{z_0^2}{(z-z_0)^2}-\ldots\right),$$

for $|z-z_0|>|z_0|$.

7 Expand in a power series of $(z-\mathrm{i})$ the functions:

$f_1(z)=\frac{1}{z},\qquad f_2(z)=\frac{1}{z-1},\qquad f_3(z)=\frac{1}{z}+\frac{1}{z-1}.$