A Multi-Valued Framework for Coalgebraic Logics over Generalised Metric Spaces

Adriana Balan*

Department of Mathematical Methods and Models
University Politehnica of Bucharest, Romania
adriana.balan@mathem.pub.ro

It is by now generally acknowledged that the theory of universal coalgebra incorporates a wide variety of dynamic systems [18]. The classical study of their behaviour is based on qualitative reasoning – that is, Boolean, meaning that two systems (the systems' states) are bisimilar or not. This is best formalised within modal logics and the associated proof techniques. But in recent years there has been a growing interest in studying systems' behaviour in terms of quantity. There are situations where one behaviour is smaller than (or, is simulated by) another behaviour, or there is a measurable distance between behaviours in terms of real numbers, as it was done in [17,21]. It follows that a convenient framework for developing a theory that parallels the one of coalgebras over sets is given by coalgebras over $\mathscr{V}$-cat, the category of (small) enriched categories over a quantale $\mathscr{V}$. It has been recognised that $\mathscr{V}$-cat can be perceived as a category of "quasi-metric" spaces and nonexpanding maps, which subsumes both ordered sets and monotone mappings, and (generalised) metric spaces [12]. Several versions of many-valued modal logics for transition systems have been subsequently developed to incorporate such quantitative reasoning (e.g. [4,7,8,15,20]).

Recall that classical modal logics naturally fits in the uniform framework of (Boolean) coalgebraic logics [5], and that the latter can *abstractly* be described by the following diagram

$$T^{\mathsf{op}} \circlearrowright \mathsf{Set}^{\mathsf{op}} \underset{S}{\overset{P}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{BA} \circlearrowleft L$$

where the functor P maps a set to the Boolean algebra of its subsets, while S associates to a Boolean algebra the set of its ultrafilters. There is a pair of endofunctors on top of this contravariant adjunction: T on sets giving the type of transition systems (coalgebras), and L on Boolean algebras whose algebras should be seen as the modal algebras of the logic of T-coalgebras. They are connected by a natural transformation $\delta : LP \to PT^{\mathsf{op}}$ which assigns to each modal operator its interpretation as a subset of acceptable successor states.

Therefore it seems natural to elaborate a similar uniform strong framework for reasoning about transition structures over $\mathscr{V}$-cat. The present talk will focus on the results obtained so far in this line of research.

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Coalgebras over $\mathscr{V}$-cat. Formally, transition structures over (generalised) metric spaces are precisely the coalgebras for an endofunctor on $\mathscr{V}$-cat. But how to obtain such a functor? We have the plethora of Set-endofunctors as examples, hence the obviuous solution is to adapt them to quantale-enriched categories. Therefore, we shall explain how to extend functors $T : \mathsf{Set} \to \mathsf{Set}$ (and more generally Set-functors which naturally carry a $\mathscr{V}$-metric structure) to $\mathscr{V}$-cat-functors $T^{\mathscr{V}} : \mathscr{V}\text{-}\mathsf{cat} \to \mathscr{V}\text{-}\mathsf{cat}$. This construction, called "$\mathscr{V}$-cat-ification", has been obtained by the author in [1], in collaboration with A. Kurz and J. Velebil. Using the density of the discrete functor $D : \mathsf{Set} \to \mathscr{V}\text{-}\mathsf{cat}$, we apply T to the "$\mathscr{V}$-nerve" of a $\mathscr{V}$-category, and then take an appropriate "quotient" in $\mathscr{V}$-cat. If T preserves weak pullbacks, then the $T^{\mathscr{V}}$ above can be obtained using Barr's relation lifting in a form of "lowest-cost paths" (see also [21, Ch. 4.3] and [10]). For example, the extension $\mathcal{P}^{\mathscr{V}}$ of the powerset functor $\mathcal{P}$ yields the familiar Pompeiu-Hausdorff metric, if the quantale is completely distributive.

Multivalued logical connection. The next step, following the well-established tradition in coalgebraic logics explained earlier (see also [14]), is to seek for a contravariant adjunction on top of which multi-valued coalgebraic logics is to be considered. This adjunction involves on one side a category of algebras Alg, and on the other side, a category of spaces Sp, obtained by conveniently restricting the adjunction $[-\mathscr{V}] \dashv [-,\mathscr{V}] : \mathscr{V}\text{-}\mathsf{cat}^{\mathsf{op}} \to \mathscr{V}\text{-}\mathsf{cat}$. Moreover, we look for $\mathscr{V}$ to live in both categories and to act as a "dualising" object, in the sense of [16]. A requirement for Alg is to be a variety[1] in the "world of $\mathscr{V}$-categories", at least monadic over $\mathscr{V}$-cat.

In *classical modal/coalgebraic logics* (no enrichment), this is achieved by taking Sp to be Set, and Alg to be the category of Boolean algebras, as detailed earlier. One step further, the case of the simplest two-elements quantale $\mathscr{V} = \mathbb{2}$ targets *positive modal/coalgebraic logics* (see [6,2] for a detailed exposition), from an order-enriched point of view, by choosing Sp to be the category of posets and monotone maps, and Alg to be the category of bounded distributive lattices DLat.[2] These in turn, are part of an adjunction $S' \dashv P' : \mathsf{Poset}^{\mathsf{op}} \to \mathsf{DLat}$ with P' taking uppersets and S' taking prime filters, or, equivalently, $P'X = \mathsf{Poset}(X, \mathbb{2})$ and $S'A = \mathsf{DLat}(A, \mathbb{2})$, where $\mathbb{2}$ plays the role of a "dualising object", being considered, depending on the context, either as a poset or as a distributive lattice.

In view of the above, the logical connection between Sp and Alg for $\mathscr{V}$-cat that we shall propose in the sequel starts from an adaptation of the Priestley duality [9]. The algebras of the logic will be built on distributive lattices endowed with a family of *adjoint pairs of operators* (shortly dlao($\mathscr{V}$)) indexed by the elements of the quantale $(r * - \dashv r \pitchfork - : A \to A)_{r \in \mathscr{V}}$, satisfying

$$1 * a = a \qquad\qquad (r \otimes r') * a = r * (r' * a)$$
$$0 * a = 0 \qquad\qquad (r \vee r') * a = (r * a) \vee (r' * a)$$

[1] This variety is not expected to be finitary unless $\mathscr{V}$ itself is finite.

[2] Which is a finitary *ordered* variety [3].

The morphisms between dlao($\mathscr{V}$) are distributive lattice maps preserving the adjoint operators. The resulting category, denoted $\mathsf{DLatAO}(\mathscr{V})$, is an algebraic category and will play in the sequel the role of Alg.[3] Each dlao($\mathscr{V}$) A becomes a $\mathscr{V}$-category [13] with $\mathscr{V}$-homs $A(a,a') = \bigvee\{r \in \mathscr{V} \mid r * a \leq a'\} = \bigvee\{r \in \mathscr{V} \mid a \leq r \pitchfork a'\}$, which is finitely complete and cocomplete [19], and each dlao($\mathscr{V}$)-morphism is also a $\mathscr{V}$-functor. Consequently, $\mathsf{DLatAO}(\mathscr{V})$ is a subcategory of $\mathscr{V}\text{-}\mathsf{cat}$, monadic if the quantale $\mathscr{V}$ satisfies certain conditions.

By adapting the arguments in [9], we see that the dual category to $\mathsf{DLatAO}(\mathscr{V})$ has as objects Priestley spaces $(X, \tau, \leq)$ endowed with a family of binary relations $(R_r)_{r\in\mathscr{V}}$, satisfying, besides the topological conditions given in [9], the following requirements: R_1 is the order relation on X, $R_r \circ R_{r'} = R_{r\otimes r'}$ and $R_{r\vee r'} = R_r \vee R_{r'}$, for all $r, r' \in \mathscr{V}$. The morphisms are the continuous bounded maps from [9]. Denote by $\mathsf{RelPriest}(\mathscr{V})$ the resulting category. Then the dual equivalence

$$\mathsf{RelPriest}(\mathscr{V})^{\mathsf{op}} \cong \mathsf{DLatAO}(\mathscr{V})$$

is obtained by restricting the usual Priestley duality.

To gain more insight on $\mathsf{RelPriest}(\mathscr{V})$, we shall use the monoidal topology approach of [10]. If the quantale is completely distributive, then each relational Priestley space $(X, \tau, \leq, (R_r)_{r\in\mathscr{V}})$ turns to be a compact $\mathscr{V}$-topological space [11], in particular a Cauchy complete $\mathscr{V}$-category. Hence the forgetful functor $\mathsf{RelPriest}(\mathscr{V}) \to \mathscr{V}\text{-}\mathsf{cat}$ assigns to each $\mathscr{V}$-relational Priestley space an antisymmetric and Cauchy complete $\mathscr{V}$-category, called a $\mathscr{V}$-poset. Let $\mathsf{Poset}(\mathscr{V})$ denote the category of $\mathscr{V}$-posets and $\mathscr{V}$-monotone maps (i.e. $\mathscr{V}$-functors). Then we obtain a contravariant adjunction $S^{\mathscr{V}} \dashv P^{\mathscr{V}}$ mapping each dlao($\mathscr{V}$) to the $\mathscr{V}$-poset of its lattice prime filters, respectively each $\mathscr{V}$-poset $\mathscr{X}$ to the (complete and cocomplete) dlao($\mathscr{V}$) of $\mathscr{V}$-upper sets (that is, $\mathscr{V}$-presheaves) $[\mathscr{X}, \mathscr{V}]$.

$$T^{\mathscr{V}\mathsf{op}} \circlearrowright \mathsf{Poset}(\mathscr{V})^{\mathsf{op}} \underset{S^{\mathscr{V}}}{\overset{P^{\mathscr{V}}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{DLatAO}(\mathscr{V}) \circlearrowleft L^{\mathscr{V}}$$

For each functor $T^{\mathscr{V}}$ on $\mathscr{V}\text{-}\mathsf{cat}$, an abstract logics will be a pair $(L^{\mathscr{V}}, \delta^{\mathscr{V}})$, where $L^{\mathscr{V}} : \mathsf{DLatAO}(\mathscr{V}) \to \mathsf{DLatAO}(\mathscr{V})$ is an endofunctor giving the syntax of the logics (the multi-valued modalities), and $\delta^{\mathscr{V}} : P^{\mathscr{V}} T^{\mathscr{V}\mathsf{op}} \to L^{\mathscr{V}} P^{\mathscr{V}})$ a natural transformation inducing the (one-step) semantics. This multi-valued coalgebraic logics is under current investigation, as well as important properties it might have, like completeness and expressiveness.

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[3] Observe that for $\mathscr{V} = \mathbb{2}$ the operators are trivial and we regain DLat.

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