Representable functions in Moisil logic*

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Gr. C. Moisil was the first who attempted an algebrization of the many-valued logic of Łukasiewicz's. In 1941, he introduced 3- and 4-valued *Łukasiewicz algebras*, and later generalized them to the n-valued [Moi41] and the ∞-valued case [Moi72]. Some years later, it was shown by A. Rose that this class of algebras is inadequate for the logic above. An alternative was devised by C. C. Chang, who introduced in 1958 [Cha58] the class of *MV-algebras*. Still, one may forcefully argue that Łukasiewicz algebras can still be considered algebras of logic, albeit for a different one, which is nowadays dubbed *Moisil logic*. The algebras themselves came to be known as *Łukasiewicz-Moisil algebras* or plainly *Moisil algebras* – a relatively short introduction is [Cig70], while an exhaustive monograph from the early 1990s is [BoiFilGeoRud91]. Later developments may be found in [Leu08] and [DiaLeu15].

Now, it is interesting to see what exactly does it mean that such algebras are or not adequate for some logic. We work here solely on the finite case, i.e. we fix an n and consider algebras of order $n+1$ with standard model consisting of "truth values" $0, \frac{1}{n}, \cdots, 1$. Since a classical result of Moisil's, paralleling Stone's celebrated result on Boolean algebras, states that any Moisil algebra is isomorphic to a subdirect product of the standard model, it is enough to focus on the structure of the standard model. This is exactly what Rose did: he showed that for $n \geq 4$ the elements $0, 1, \frac{n-1}{n}, 1$ form a Moisil subalgebra of the standard model which is not closed under Łukasiewicz implication, therefore that is not expressible in terms of the Moisil operations.

This raises the question: what functions from the rth power of the standard model to the standard model are represented by Moisil formulas? That is what we are going to answer here. In order to be able to express how we arrived at said answer, we shall firstly delve more deeply into how a Moisil algebra looks like.

A Moisil algebra is a de Morgan algebra endowed with n unary operations $\Delta_1, ..., \Delta_n$ called "nuances" or "Chrysippian endomorphisms" which are required to satisfy certain equational conditions. Generally (i.e. on the logic side), if φ is a formula, then $\Delta_i\varphi$ has the intuitive meaning that φ has the "truth value"

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greater than $\frac{n-i+1}{n}$. From this property, it may easily be seen that the nuances are progressively "contained", each one in the next. A natural question that we may consider is whether we may get rid altogether of these "classical" nuances and replace them with independent, mutually exclusive ones. Such operations $J_1, ..., J_n$ were introduced in [Cig82] and later used in [DiaLeu15] in order to obtain an alternative formulation and equational axiomatization of Moisil algebras. If we add the operation $J_0 := \Delta_1$, we obtain $n+1$ different mutually exclusive nuances, one for each truth value, satisfying the equation

$$J_i\left(\frac{j}{n}\right) = \delta_{ij}$$

on the standard model, where by δ_{ij} we mean the Kronecker delta.

With this tool now in our hands, we may derive our characterization of the representable functions. If $(a_1, ..., a_r)$ is a possible input of such a function f, then clearly $f(a_1, ..., a_r)$ is an element of a subalgebra generated by $\{a_1, ..., a_r\}$. But that subalgebra consists just of $\{0, 1, a_1, ..., a_r, 1-a_1, ..., 1-a_r\}$. So, if f is representable, then for any $(a_1, ..., a_r)$ we have that $f(a_1, ..., a_r) \in \{0, 1, a_1, ..., a_r, 1-a_1, ..., 1-a_r\}$. The challenge is to show that this condition is actually sufficient. We remember that in the Boolean case, any function is representable by a formula. The tool used to show this is the disjunctive normal form, i.e. from the truth table of the function we produce the disjunction between the possible cases that lead up to the value 1. This is also, in a way, what we are doing here: if we have a function f, we form the disjunction:

$$\bigvee_{(a_1,...,a_r)\in L_{n+1}^r} \big(J_{na_1}(x_1) \wedge \ldots \wedge J_{na_r}(x_r) \wedge s(a_1, ..., a_r, f(a_1, ..., a_r))\big),$$

where the J_i's serve to "separate" the lines of the truth table and the last term serves to identify the value on each such line. For example, if $n = 3$ and $r = 4$, then $s\left(1, 0, \frac{1}{3}, 0, \frac{2}{3}\right) = Nx_3$, where N is the de Morgan negation. The general construction, therefore, is given by:

$$s(a_1, ..., a_r, a) := \begin{cases} 1, & \text{if } a = 1; \\ 0, & \text{if } a = 0; \\ x_i, & \text{if } a \notin \{0,1\} \text{ and } i = \min\{j \mid a = a_j\}; \\ Nx_i, & \text{if } a \notin \{0, 1, a_1, ..., a_r\} \text{ and } i = \min\{j \mid a = 1 - a_j\}. \end{cases}$$

These ideas help us further in obtaining alternate proofs for the cardinality of the free Moisil algebra of order $n+1$ and for characterizing the representable functions of Łukasiewicz logic.

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