A many-valued framework for coalgebraic logics over generalised metric spaces

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Motivation

Coalgebras encompass a wide variety of dynamical systems.

Their behaviour can be universally characterised using the theory of coalgebras.

However, in real life, the complexity of dynamical systems often makes bisimilarity a too strict concept.

Consequently, the focus should be on **quantitative** behaviour (e.g. ordered, fuzzy, or probabilistic behavior):

(bi)similarity **pseudometric** that measures how similar two systems are from the point of view of their behaviours

The **many-valued setting** can be used to properly captured such behavior.

What does many-valued mean?

This talk: quantale-enriched categories.

$\mathscr{V}$-enriched categories

- Let $\mathscr{V} = (V, \otimes, e, [-,-])$ be a commutative quantale.

Examples

- $\mathscr{V} = (\mathbb{2}, \wedge, 1)$
- $\mathscr{V} = ([0, \infty]^{\mathsf{op}}, +, 0)$
- $\mathscr{V} = ([0, 1], \otimes, 1)$, with $\otimes$ the Łukasiewicz product

- A (small) $\mathscr{V}$-category $\mathscr{A}$ consists of a set A endowed with a $\mathscr{V}$-valued relation $\mathscr{A} : A \times A \to \mathscr{V}$ such that

$$e \leq \mathscr{A}(a, a) \qquad \text{and} \qquad \mathscr{A}(a, b) \otimes \mathscr{A}(b, c) \leq \mathscr{A}(a, c)$$

Examples

[Lawvere'73]

- preordered sets
- quasi-metric spaces
- fuzzy preorders

$\mathscr{V}$-enriched categories

- Let $\mathscr{V} = (V, \otimes, e, [-,-])$ be a commutative quantale.

Examples

- $\mathscr{V} = (2, \wedge, 1)$
- $\mathscr{V} = ([0,\infty]^{\mathsf{op}}, +, 0)$
- $\mathscr{V} = ([0,1], \otimes, 1)$, with $\otimes$ the Łukasiewicz product

- A (small) $\mathscr{V}$-category $\mathscr{A}$ consists of a set A endowed with a $\mathscr{V}$-valued relation $\mathscr{A} : A \times A \to \mathscr{V}$ such that

$$a \le a \qquad \text{and} \qquad (a \le b) \wedge (b \le c) \Rightarrow (a \le c)$$

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- A (small) $\mathscr{V}$-category $\mathscr{A}$ consists of a set A endowed with a $\mathscr{V}$-valued relation $\mathscr{A} : A \times A \to \mathscr{V}$ such that

$$0 \geq \mathscr{A}(a,a) \quad \text{and} \quad \mathscr{A}(a,b) + \mathscr{A}(b,c) \geq \mathscr{A}(a,c)$$

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- A (small) $\mathscr{V}$-category $\mathscr{A}$ consists of a set A endowed with a $\mathscr{V}$-valued relation $\mathscr{A} : A \times A \to \mathscr{V}$ such that

$$1 \leq \mathscr{A}(a,a) \quad \text{and} \quad \mathscr{A}(a,b) \otimes \mathscr{A}(b,c) \leq \mathscr{A}(a,c)$$

Examples

[Lawvere'73]

- preordered sets
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$\mathscr{V}$-enriched categories

- A $\mathscr{V}$-functor $f : \mathscr{A} \to \mathscr{A}'$ consists of a map $f : A \to A'$ such that

$$\mathscr{A}(a, b) \leq \mathscr{A}'(fa, fb)$$

- $\mathscr{V}$-categories and $\mathscr{V}$-functors organise themselves into a complete and cocomplete symmetric monoidal closed (2-)category $\mathscr{V}$-cat.

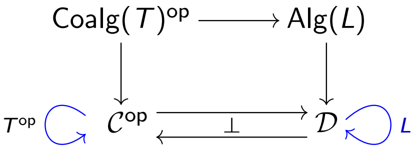
- Hence we may speak of $\mathscr{V}$-cat-enriched categories.

  The main example: $\mathscr{V}$-cat is enriched over itself.

What is this talk about?

Finding a logical connection for many-valued coalgebraic logics.

Coalgebras and their logics – the abstract recipe



The coalgebraic data:

- Category $\mathcal{C}$
- Functor $T : \mathcal{C} \to \mathcal{C}$
- T-coalgebra $c : X \to TX$
- T-coalgebra morphism

  $f : (X, c) \to (X', c')$

$$\begin{array}{ccc} X & \xrightarrow{c} & TX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Tf} \\ X' & \xrightarrow{c'} & TX' \end{array}$$

The logical data:

- Contravariant adjunction

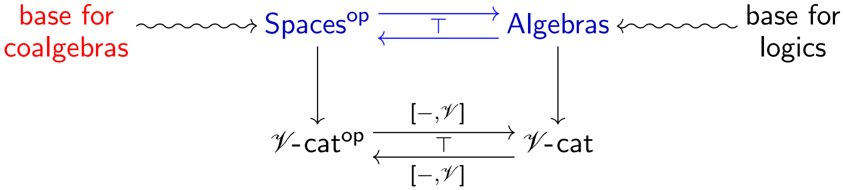
  $S \dashv P : \mathcal{D} \to \mathcal{C}^{op}$
- Functor $L : \mathcal{D} \to \mathcal{D}$
- Natural transformation

  $\delta : LP \to PT^{op}$

A many-valued logical connection

Purpose: to look for a contravariant adjunction (to be used in the future for logics) for coalgebras over quantale-enriched categories.

An ideal picture:



- For $\mathscr{V} = \mathbb{2}$, this is relatively well understood [B, Kurz & Velebil LMCS2015]
- What about for other quantale $\mathscr{V}$?

The coalgebraic story

On the structure of $\mathscr{V}$-categories
[B, Kurz & Velebil CALCO2015]

First, notice that each $\mathscr{V}$-category $\mathscr{A}$ determines the following data:

- A, the underlying set of objects of the $\mathscr{V}$-category $\mathscr{A}$
- $A_r = \{(a, b) \in A \times A \mid r \leq \mathscr{A}(a, b)\}$, the r-level set
- $d_0^r, d_1^r : A_r \to A$ the usual projection maps

The above data can be organised as to describe a diagram $F_{\mathscr{A}} : \mathsf{N} \to \mathsf{Set}$, hence a diagram of **discrete** $\mathscr{V}$-categories

$$\mathsf{N} \overset{F_{\mathscr{A}}}{\to} \mathsf{Set} \overset{D}{\to} \mathscr{V}\text{-}\mathsf{cat}$$

Then the (weighted) colimit of $DF_{\mathscr{A}}$ is the original $\mathscr{V}$-category $\mathscr{A}$.

On functors for $\mathscr{V}$-cat-coalgebras

[B, Kurz & Velebil CALCO2015]

In order to understand endofunctors (and their coalgebras) on $\mathscr{V}$-cat, look first at endofunctors on Set, then ask:

How to move from Set to $\mathscr{V}$-cat?

Fact: Functors $T : \mathsf{Set} \to \mathsf{Set}$ can be canonically extended to $\mathscr{V}$-cat-functors $T_{\mathscr{V}} : \mathscr{V}\text{-cat} \to \mathscr{V}\text{-cat}$.

Here canonically means $T_{\mathscr{V}} = \mathrm{Lan}_D(DT)$.

Call $T_{\mathscr{V}}$ the **$\mathscr{V}$-cat-ification** of T.

$$\begin{array}{ccc} \mathsf{Set} & \xrightarrow{D} & \mathscr{V}\text{-cat} \\ {\scriptstyle T}\downarrow & & \vdots\, {\scriptstyle T_{\mathscr{V}}} \\ \mathsf{Set} & \xrightarrow{D} & \mathscr{V}\text{-cat} \end{array}$$

How? The construction of the extension applies DT to the "$\mathscr{V}$-nerve" $F_{\mathscr{A}}$ of a $\mathscr{V}$-category $\mathscr{A}$, and then takes the appropriate "quotient" (colimit).

Then each T-coalgebra can be seen as a discrete version of a $T_{\mathscr{V}}$-coalgebra.

$$\begin{array}{ccc} \mathrm{Coalg}(T) & \dashrightarrow & \mathrm{Coalg}(T_{\mathscr{V}}) \\ \downarrow & & \downarrow \\ \mathsf{Set} & \xrightarrow{D} & \mathscr{V}\text{-cat} \end{array}$$

On functors for $\mathscr{V}$-cat-coalgebras

[B, Kurz & Velebil CALCO2015]

An easier recipe: if T preserves weak pullbacks, then its $\mathscr{V}$-cat-ification can be computed using Barr's relation lifting.

$$T_{\mathscr{V}}\mathscr{A}(\mathfrak{a},\mathfrak{b}) = \bigvee_r \{ r \mid (\mathfrak{a},\mathfrak{b}) \in \mathrm{Rel}_T(A_r) \}$$

Example: for $\mathscr{V}$ completely distributive, the $\mathscr{V}$-cat-ification of the powerset functor $\mathcal{P}$ gives the familiar Pompeiu-Hausdorff metric:

Let $\mathscr{A}$ be a $\mathscr{V}$-category. Then $\mathcal{P}_{\mathscr{V}}\mathscr{A}$ is the $\mathscr{V}$-category with objects $\mathcal{P}X$, and $\mathscr{V}$-homs

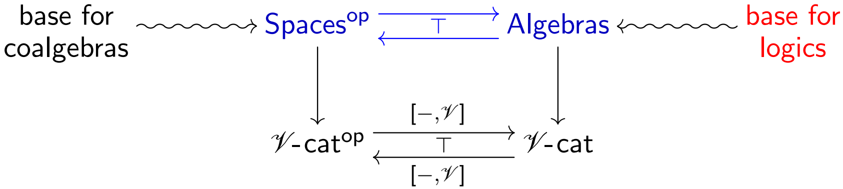
$$\mathcal{P}_{\mathscr{V}}\mathscr{A}(\mathfrak{a},\mathfrak{b}) = \big(\bigwedge_{a\in\mathfrak{a}}\bigvee_{b\in\mathfrak{b}}\mathscr{A}(a,b)\big)\bigwedge\big(\bigwedge_{b\in\mathfrak{b}}\bigvee_{a\in\mathfrak{a}}\mathscr{A}(a,b)\big)$$

Back to the logical connection

A many-valued logical connection

Purpose: to look for a contravariant adjunction (to be used in the future for logics) for coalgebras over quantale-enriched categories.

An ideal picture:



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- What about for other quantale $\mathscr{V}$?

A hint from positive coalgebraic logics

[B, Kurz & Velebil LMCS2015]

- The simplest case: the quantale $\mathbb{2}$

$$\mathsf{Poset}^{\mathsf{op}} \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathsf{DLat}$$

  - Posets: antisymmetric $\mathbb{2}$-enriched categories.
  - Distributive lattices: antisymmetric finitely complete and cocomplete $\mathbb{2}$-categories such that finite limits (meets) distribute over finite colimits(joins).

- Move from $\mathbb{2}$ to an arbitrary quantale $\mathscr{V}$ – a naive approach:
  - Replace posets by antisymmetric $\mathscr{V}$-categories.
  - Replace distributive lattices by finitely complete and cocomplete $\mathscr{V}$-categories such that finite **conical** limits distribute over finite **conical** colimits.

- **Does it work?** A minimal requirement: the quantale $\mathscr{V}$ itself should have a distributive lattice reduct.

The contravariant adjunction – step I

- Distributive lattice with adjoint pairs of $\mathscr{V}$-operators (dlao($\mathscr{V}$))
  - $(A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice.
  - A is endowed with a family of adjoint maps

$$\Diamond_r \dashv \Box_r : A \to A \; , \; r \in \mathscr{V}$$

    satisfying the following:
    - $\Diamond_1 a = a$
    - $\Diamond_{r \otimes r'} a = \Diamond_r \Diamond_{r'} a$
    - For each family $(r_i)_{i \in I}$ in $\mathscr{V}$ with $\bigvee_{i \in I} r_i = r$,

$$\Diamond_{r_i} a \leq \Diamond_r a \quad \forall \, i \in I$$

$$\Diamond_{r_i} a \leq b \; , \quad \forall \, i \in I \implies \Diamond_r a \leq b$$

- Morphisms of dlao($\mathscr{V}$) are those preserving all operations.
- Hence we obtain a category DLatAO($\mathscr{V}$) (more precisely a $\mathscr{V}$-cat-category)

The contravariant adjunction – step II

The dual of DLatAO($\mathscr{V}$) can be obtained by restricted Priestley duality:

- Objects are Priestley spaces $(X, \tau, \leq)$, endowed with a family of binary relations $(R_r)_{r \in \mathscr{V}}$ satisfying
  - $x' \leq x$ and $R_r(x, y)$ and $y \leq y'$ imply $R_r(x', y')$ (weakening)
  - $R_1 = \leq$
  - $R_r \circ R_{r'} = R_{r \otimes r'}$
  - $R_{\bigvee_{i \in I} r_i} = \bigcup_{i \in I} R_{r_i}$

  and several topological conditions. Call such $(X, \tau, \leq, (R_r)_{r \in \mathscr{V}})$ a relational Priestley space.
- Morphisms are monotone continuous maps $f : X \to Y$ such that
  - $R_r(x, y) \implies R_r(fx, fy)$
  - $R_r(u, fx) \iff (\exists x' \in X \,.\, u \leq fx' \text{ and } R_r(x', x))$
  - $R_r(fx, u) \iff (\exists x' \in X \,.\, R_r(x, x') \text{ and } fx' \leq u)$

The contravariant adjunction – step II

- Denote by $\mathsf{RelPriest}(\mathscr{V})$ the resulting category. Hence

$$\mathsf{RelPriest}(\mathscr{V})^{\mathsf{op}} \cong \mathsf{DLatAO}(\mathscr{V})$$

- Each relational Priestley space X becomes a $\mathscr{V}$-category by

$$\mathscr{X}(x,y) = \bigvee\{r \mid R_r(x,y)\}$$

- Assume that $\mathscr{V}$ is completely distributive and recall that each relational Priestley space is in particular compact Hausdorff.
- The $\mathscr{V}$-category structure and the compact Hausdorff structure on X are **compatible**, in the sense that the convergence map assigning to each ultrafilter on X its limit point is a $\mathscr{V}$-functor.
- In particular, each relational Priestley space is Cauchy complete.

Conclusion

$$\mathsf{RelPriest}(\mathscr{V})^{op} \cong \mathsf{DLatAO}(\mathscr{V})$$

$\mathsf{RelPriest}(\mathscr{V})^{op}$	$\cong$	$\mathsf{DLatAO}(\mathscr{V})$
$\rightsquigarrow$		$\rightsquigarrow$
compact Hausdorff $\mathscr{V}$-categories, hence antisymmetric and Cauchy complete		finitely complete and cocomplete $\mathscr{V}$-categories with distributivity condition

- We have obtained a duality between spaces and algebras, both carrying underlying $\mathscr{V}$-category structure.
- The duality above is yet unsatisfactory, as it is not obtained by "homming" into $\mathscr{V}$.
- In [Băbuș & Kurz CMCS2016], a duality between completely distributive $\mathscr{V}$-categories and atomic Cauchy complete $\mathscr{V}$-categories was provided. How are the two dualities related?

The logics story ... maybe at FROM2018?

Thank you for your attention!